HEREDITARILY STRUCTURALLY COMPLETE INTERMEDIATE LOGICS: CITKIN'S THEOREM VIA ESAKIA DUALITY

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Abstract. A deductive system is said to be structurally complete if its admissible rules are derivable. In addition, it is called hereditarily structurally complete if all its finitary extensions are structurally complete. Citkin (1978) proved that an intermediate logic is hereditarily structurally complete if and only if the variety of Heyting algebras associated with it omits five finite algebras. Despite its importance in the theory of admissible rules, a direct proof of Citkin's theorem is not widely accessible. In this paper we offer a self-contained proof of Citkin's theorem, based on Esakia duality and the method of subframe formulas. As a corollary, we obtain a short proof of Citkin's 2019 characterization of hereditarily structurally complete positive logics.

1. Introduction

A rule ρ is said to be *admissible* in a deductive system $\vdash$ if the set of tautologies of $\vdash$ is closed under the applications of ρ. On the other hand, a rule ρ is called *derivable* in $\vdash$ if ρ belongs to the consequence relation of the system.[1] Clearly, every derivable rule is admissible. While the converse holds for classical propositional calculus **CPC**, it fails for many non-classical systems, including intuitionistic propositional calculus **IPC**.

This motivated the study of criteria for admissibility in modal and intermediate logics, undertaken by Rybakov and others [57]. As a consequence, the problem of finding *bases* for admissible rules was solved for **IPC** by Iemhoff [35, 36, 37], building on the work of Ghilardi on unification [30, 31], and independently by Rozière [54]. Later on, similar results have been obtained for modal and Łukasiewicz logics by Jeřábek [39, 40, 41].

A classical problem in the theory of admissible rules asks to determine which deductive systems are *structurally complete*, i.e., share with **CPC** the property that all admissible rules are derivable. Addressing this question, Prucnal showed that all finitary extensions of the $\langle \rightarrow \rangle$-fragment of **IPC** are structurally complete [50]. Notably, his argument extends immediately to the $\langle \wedge, \rightarrow \rangle$-fragment of **IPC** [51]. Subsequently, a similar result was obtained by Dzik and Wrónski, who proved that all finitary extensions of Gödel-Dummet logic are structurally complete [24].

These investigations suggested that, while a full characterization of structurally complete intermediate logics could be out of reach, still it might be possible to describe intermediate logics that are structurally complete in a *hereditary* way, i.e., not only they are structurally complete, but so are all their finitary extensions. This conjecture was confirmed by Citkin [16], who proved that an intermediate logic is hereditarily structurally complete if and only if the variety of Heyting algebras associated with it omits five finite

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[1] Formal definitions are detailed in Section 2.

algebras. Since then, the relation between structural completeness and its hereditary version in intermediate logics has been further investigated in [18].

Despite being one of the important milestones in the theory of admissible rules, Citkin's proof has never been published in English—the only detailed proof is in Russian [17]. Yet another source for it is a generalization to axiomatic extensions of the modal system **K4** by Rybakov [56, 57] which, in turn, is not self-contained. Accordingly, the goal of this paper is to provide a relatively simple and mostly self-contained proof of Citkin's theorem in the hope to make it more widely available (Theorem 7.3). Our proof strategy is similar to that of Citkin [17], however, we mostly use the methods of Esakia duality and of subframe formulas for Heyting algebras, whereas Citkin's original proof is purely algebraic. Apart from its simplicity, our approach has the advantage of yielding a very short proof (see Section 8) of Citkin's recent characterization of hereditarily structurally complete positive logics, i.e., $\langle \wedge, \vee, \to \rangle$-fragments of intermediate logics [19].

The paper is organized as follows. In Section 2 we introduce the main definitions of the paper. We also discuss our main proof strategy: The problem of characterizing hereditarily structurally complete intermediate logics is equivalent to that of describing primitive varieties of Heyting algebras. In the rest of the paper we focus on the latter problem. In Section 3 we review the main tool of the paper, Esakia's duality for Heyting algebras. Building on Esakia duality, in Section 4 the description of finitely generated free Heyting algebras by means of universal models is recalled. In Section 5 we introduce Citkin's five finite algebras $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$, and show that these are omitted by any primitive variety of Heyting algebras (Lemma 5.1), thus proving one direction of Citkin's theorem. To prove the other direction, we shift the focus to varieties of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$, which are investigated in Section 6 by means of subframe formulas. In particular, we show that these varieties are locally finite and we describe the structure of their finite subdirectly irreducible members (Theorem 6.13). Section 7 completes the proof Citkin's theorem (Theorem 6.13 and Corollary 7.5). The obtained results and techniques are employed, in Section 8, to derive a new proof of Citkin's description of hereditarily structurally complete positive logics (Corollary 8.3). We conclude the paper by Section 9 where we review some important properties of hereditarily structurally complete intermediate and positive logics.

2. Hereditary structural completeness

Let Fm be the set of formulas in countably many variables of some fixed, but arbitrary, algebraic language. A *deductive system* is a consequence relation $\vdash$,[2] defined over the set of formulas Fm, that is substitution-invariant in the following sense: for every substitution σ and set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash \varphi \text{, then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Let $\vdash$ be a deductive system. A deductive system $\vdash'$ is said to be an *extension* of $\vdash$ if for every set $\Gamma \cup \{\varphi\}$,

$$\text{if } \Gamma \vdash \varphi \text{, then } \Gamma \vdash' \varphi.$$

[2] In the literature, intermediate logics are usually identified with sets of formulas, as opposed to consequence relations [15]. However, we opted for this presentation since when dealing with the distinction between admissible and derivable rules, it is convenient to identify every intermediate logic with the consequence relation associated with it.

Moreover, $\vdash$ is said to be *finitary* if for every set $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash \varphi\text{, then there exists a finite set } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash \varphi.$$

A *rule* is an expression of the form $\Gamma \rhd \varphi$ where Γ is a finite subset of Fm. Let $\vdash$ be a deductive system. A rule $\Gamma \rhd \varphi$ is said to be *admissible* in $\vdash$ if for all substitutions σ:

$$\text{if } \emptyset \vdash \sigma(\gamma) \text{ for all } \gamma \in \Gamma\text{, then } \emptyset \vdash \sigma(\varphi).$$

Similarly, a rule $\Gamma \rhd \varphi$ is said to be *derivable* in $\vdash$ if $\Gamma \vdash \varphi$. Accordingly, we say that

1. $\vdash$ is *structurally complete* if every rule that is admissible in $\vdash$ is also derivable in $\vdash$.
2. $\vdash$ is *hereditarily structurally complete* if every finitary extension of $\vdash$ is SC.

For further variants of structural completeness, we refer the reader to [23, 46, 47, 60].

Under certain assumptions, hereditary structural completeness can be formulated in purely algebraic terms [3, 46, 52]. To explain how this could be done, it is convenient to recall some basic definitions from universal algebra [4, 14]. We denote by $\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\text{U}}$ the class operators of closure under isomorphism, homomorphic images, subalgebras, direct products, and ultraproducts, respectively. We assume direct products and ultraproducts of empty families of algebras are trivial algebras. A *variety* is a class of algebras axiomatized by equations or, equivalently, a class of algebras closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$. A *quasi-variety* is a class of algebras axiomatized by quasi-equations or, equivalently, a class of algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\text{U}}$. Given a class of algebras K, we denote by $\mathbb{V}(\mathsf{K})$ and $\mathbb{Q}(\mathsf{K})$, respectively, the least variety and quasi-variety containing K. It is well known that $\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K})$ and $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\text{U}}(\mathsf{K})$. A class $\mathsf{M} \subseteq \mathsf{K}$ is a *subvariety* (resp. *subquasi-variety*) of K if M is a variety (resp. a quasi-variety). A variety K is said to be *primitive* if every subquasi-variety M of K is a variety.

When a finitary deductive system $\vdash$ is *algebraized* by a variety K in the sense of [13], the lattice of axiomatic extensions of $\vdash$ is dually isomorphic to that of subvarieties of K. In addition, an axiomatic extension $\vdash'$ of $\vdash$ is hereditarily structurally complete if and only if the subariety of K corresponding to $\vdash'$ is primitive [52, Thm. 6.12(2)], see also [3, Prop. 2.4]. Consequently, in this case the task of characterizing hereditarily structurally complete axiomatic extensions of $\vdash$ is equivalent to that of characterizing primitive subvarieties of K.

A special instance of this phenomenon is given by *intermediate logics*, i.e., axiomatic extensions of intuitionistic propositional logic **IPC**. This is because **IPC** is algebraized by the variety of *Heyting algebras*, i.e., algebras of the form $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ where $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1 such that for every $a, b, c \in A$,

$$a \wedge b \leqslant c \Longleftrightarrow a \leqslant b \to c.$$

Thus the task of characterizing hereditarily structurally complete intermediate logics can be rephrased in purely algebraic terms as that of describing primitive varieties of Heyting algebras. This is what we do in the rest of the paper.

To this end, we rely on some basic observation. Let K be a variety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be *weakly projective in* K if for every $\boldsymbol{B} \in \mathsf{K}$, if $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$, then $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$.[3]

[3]This concept should not be confused with the *stronger* classical notion of projectivity. Also, observe that our terminology differs from that of [3], where weakly projective algebras are called *primitive*, and primitive varieties are called *deductive*.

Moreover, an algebra $\boldsymbol{A}$ is said to be *finitely subdirectly irreducible*, FSI for short, when the identity relation is meet-irreducible in the congruence lattice of $\boldsymbol{A}$.

Lemma 2.1. *Let* K *be a primitive variety of finite type. The finite nontrivial FSI members of* K *are weakly projective in* K.

Proof. Consider a finite nontrivial FSI algebra $\boldsymbol{A} \in \mathsf{K}$. Then let $\boldsymbol{B} \in \mathsf{K}$ be such that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$. Since K is primitive, all its subquasi-varieties are varieties, whence $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B}) \subseteq \mathbb{V}(\boldsymbol{B}) = \mathbb{Q}(\boldsymbol{B})$. Now, it is well known that all FSI members of $\mathbb{Q}(\boldsymbol{B})$ belong to $\mathbb{ISP}_{\mathrm{U}}(\boldsymbol{B})$ [20, Lem. 1.5]. Thus $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\boldsymbol{B})$. Since $\boldsymbol{A}$ is finite and nontrivial, and the type of K is finite, this yields $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$. We conclude that $\boldsymbol{A}$ is weakly projective in K. $\boxtimes$

A variety is said to be *locally finite* when its finitely generated members are finite. We also rely on the following observation [33, Prop. 5.1.24], see also [32].

Theorem 2.2. *A locally finite variety* K *is primitive if and only if its finite nontrivial FSI members are weakly projective in* K.

3. Esakia duality

The study of Heyting algebras is simplified by their topological representation, known as *Esakia duality* [25, 26], which we will briefly recall here. Given a poset $\langle X; \leqslant \rangle$ and a set $U \subseteq X$, the smallest upset and downset containing U are denoted respectively by ${\uparrow}U$ and ${\downarrow}U$. In case $U = \{x\}$, we shall write ${\uparrow}x$ and ${\downarrow}x$ instead of ${\uparrow}\{x\}$ and ${\downarrow}\{x\}$, respectively. Then an *Esakia space* $\boldsymbol{X} = \langle X; \tau, \leqslant \rangle$ comprises a zero-dimensional compact Hausdorff space$\langle X; \tau \rangle$ and a poset $\langle X; \leqslant \rangle$ such that

(i) ${\uparrow}x$ is closed for all $x \in X$, and
(ii) ${\downarrow}U$ is clopen, for every clopen $U \subseteq X$.

Observe that the topology of *finite* Esakia spaces is necessarily discrete (because they are Hausdorff), and that finite posets endowed with the discrete topology are Esakia spaces. We will make a systematic use of this observation, since most Esakia spaces considered in this paper will be finite.

For Esakia spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$, an *Esakia morphism* $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ is a continuous order-preserving map $f \colon X \to Y$ such that for all $x \in X$ and $y \in Y$,

$$\text{if } f(x) \leqslant y, \text{ then there is } z \in X \text{ such that } x \leqslant z \text{ and } f(z) = y. \tag{1}$$

Esakia duality states that the category ESP of Esakia spaces endowed with Esakia morphisms is dually equivalent to the category HA of Heyting algebras and Heyting algebra homomorphisms [26, Thm. 3.4.4].

The dual equivalence functors are defined as follows. Given a Heyting algbera $\boldsymbol{A}$, we denote the set of its (non-empty proper) prime filters of $\boldsymbol{A}$ by $\mathrm{Pr}\boldsymbol{A}$, and set

$$\gamma^{\boldsymbol{A}}(a) := \{F \in \mathrm{Pr}\boldsymbol{A} \colon a \in F\} \tag{2}$$

for every $a \in A$. It turns out that the structure $\boldsymbol{A}_* := \langle \mathrm{Pr}\boldsymbol{A}; \tau, \subseteq \rangle$ is an Esakia space, where τ is the topology on $\mathrm{Pr}\boldsymbol{A}$ with subbasis $\{\gamma^{\boldsymbol{A}}(a) \colon a \in A\} \cup \{\gamma^{\boldsymbol{A}}(a)^c \colon a \in A\}$. Moreover, for every Heyting algebra homomorphism $f \colon \boldsymbol{A} \to \boldsymbol{B}$, let $f_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ be the Esakia morphism defined by the rule $F \mapsto f^{-1}[F]$.

Conversely, let $\boldsymbol{X}$ be an Esakia space. We denote by $\mathrm{Cup}\boldsymbol{X}$ the set of clopen upsets of $\boldsymbol{X}$. Then the structure $\boldsymbol{X}^* := \langle \mathrm{Cup}\boldsymbol{X}; \cap, \cup, \to, \emptyset, X \rangle$, where $U \to V := X \smallsetminus {\downarrow}(U \smallsetminus V)$, is a

Heyting algebra. Moreover, for every Esakia morphism $f\colon \boldsymbol{X} \to \boldsymbol{Y}$, let $f^*\colon \boldsymbol{Y}^* \to \boldsymbol{X}^*$ be the homomorphism of Heyting algebras given by the rule $U \mapsto f^{-1}[U]$.

Esakia duality is witnessed by the pair of contravariant functors

$$(-)_*\colon \mathsf{HA} \longleftrightarrow \mathsf{ESP}\colon (-)^*.$$

Observe that the dual equivalence functors preserve *finiteness*.

Let $\boldsymbol{X}$ be an Esakia space. An *Esakia subspace* (E-subspace for short) of $\boldsymbol{X}$ is a closed upset of $\boldsymbol{X}$, equipped with the subspace topology and the restriction of the order. For every $x \in X$, the upset $\uparrow x$ endowed with the subspace topology is easily seen to be an E-subspace of $\boldsymbol{X}$.

A *bisimulation equivalence* on $\boldsymbol{X}$ is an equivalence relation R on X such that for every $x, y, z \in X$,

(i) if $\langle x, y\rangle \in R$ and $x \leqslant z$, then there is $w \in \uparrow y$ such that $\langle z, w\rangle \in R$, and
(ii) if $\langle x, y\rangle \notin R$, then there is a clopen U such that $x \in U$ and $y \notin U$, which in addition is a union of equivalence classes of R.

In this case, we denote by $\boldsymbol{X}/R$ the Esakia space consisting of the quotient space of $\boldsymbol{X}$ with respect to R, equipped with the partial order $\leqslant^{\boldsymbol{X}/R}$ defined as follows for every $x, y \in X$:

$$x/R \leqslant^{\boldsymbol{X}/R} y/R \iff \text{there are } x', y' \in X \text{ such that}$$
$$\langle x, x'\rangle, \langle y, y'\rangle \in R \text{ and } x' \leqslant^{\boldsymbol{X}} y'.$$

The map $x \mapsto x/R$ for every $x \in X$ is an Esakia morphism from $\boldsymbol{X}$ to $\boldsymbol{X}/R$, and the kernel of f is a bisimulation equivalence on $\boldsymbol{X}$ for every Esakia morphism $f\colon \boldsymbol{X} \to \boldsymbol{Y}$. If, moreover, f is surjective, then $\boldsymbol{X}/\ker f \cong \boldsymbol{Y}$.

Remark 3.1. Observe that condition (i) in the definition of a bisimulation equivalence is equivalent to the requirement that for every $x, y, z \in X$ such that $\langle x, y\rangle \in R$, $\langle x, z\rangle \notin R$, $x \neq y$, and $x \leqslant z$, there is $y \leqslant w \in X$ such that $\langle z, w\rangle \in R$. We rely on this observation without further notice. ⊠

The disjoint union $\boldsymbol{X}_1 \uplus \cdots \uplus \boldsymbol{X}_n$ of finitely many Esakia spaces $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is their order-disjoint and topologically disjoint union, which is also an Esakia space.

Lemma 3.2. *Let $\boldsymbol{A}$ be a Heyting algebra.*

(i) *$\boldsymbol{A}$ is FSI if and only if its top element is* prime *(i.e., if $x \vee y = 1$ then $x = 1$ or $y = 1$), or, equivalently, the poset underlying $\boldsymbol{A}_*$ is* rooted *(i.e., it has a least element).*
(ii) *There is a dual lattice isomorphism σ from the congruence lattice of $\boldsymbol{A}$ to that of E-subspaces of $\boldsymbol{A}_*$, such that $(\boldsymbol{A}/\theta)_* \cong \sigma(\theta)$ for any congruence θ of $\boldsymbol{A}$, and for any E-subspace $\boldsymbol{Y}$ of $\boldsymbol{A}_*$, we have $\boldsymbol{Y}^* \cong \boldsymbol{A}/\sigma^{-1}(\boldsymbol{Y})$.*
(iii) *There is a dual lattice isomorphism ρ from the lattice of subalgebras of $\boldsymbol{A}$ to that of bisimulation equivalences on $\boldsymbol{A}_*$, such that if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ then $\boldsymbol{B}_* \cong \boldsymbol{A}_*/\rho(\boldsymbol{B})$, and if R is a bisimulation equivalence on $\boldsymbol{A}_*$ then $(\boldsymbol{A}_*/R)^* \cong \rho^{-1}(R)$.*
(iv) *The disjoint union of finitely many Esakia spaces $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is isomorphic to the dual of the direct product of the Heyting algebras $\boldsymbol{X}_1^*, \ldots, \boldsymbol{X}_n^*$.*

The statement of (i) is well known (see for instance [5, Thm. 2.9]). Condition (ii) is [26, Thm. 3.4.16], while, condition (iii) was established in [25] (alternatively, see [8, Lem. 3.4]). The proof of (iv) is as for Boolean algebras, cf. [14, Lem. IV.4.8].

Remark 3.3. Proofs in this paper would often require the reader to check whether there exists a surjective Esakia morphism between two given finite Esakia spaces. To simplify this task, we shall recall a general criterion. Let $\boldsymbol{X}$ be a finite Esakia space and $x, y \in X$.

1. Suppose that y is the only immediate successor of x. Then let R be the least equivalence relation on X such that $\langle x, y\rangle \in R$. Observe that R is a bisimulation equivalence on $\boldsymbol{X}$. The natural map $f \colon \boldsymbol{X} \to \boldsymbol{X}/R$ is called an *α-reduction*.
2. Suppose that the set of immediate successors of x ans y coincide. Then the least equivalence relation R on X such that $\langle x, y\rangle \in R$ is a a bisimulation equivalence on $\boldsymbol{X}$, and the natural map $f \colon \boldsymbol{X} \to \boldsymbol{X}/R$ is called a *β-reduction*.

Now, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be finite Esakia spaces. In [9, Lem. 3.1.7] it is shown that there exists a surjective Esakia morphism $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ if and only if there exists a finite sequence $f_1, \dots, f_n$ of α or β-reductions $f_i \colon \boldsymbol{Z}_i \to \boldsymbol{Z}_{i+1}$ such that $\boldsymbol{Z}_1 = \boldsymbol{X}$ and $\boldsymbol{Z}_{n+1} \cong \boldsymbol{Y}$. In other words, in order to determine whether the exists a surjective Esakia morphism from $\boldsymbol{X}$ to $\boldsymbol{Y}$, it suffices to check whether $\boldsymbol{X}$ can be "transformed" into $\boldsymbol{Y}$ by means of α and β-reductions. ⊠

4. Universal models

Even if finitely generated free Heyting algebras are not fully understood, major insights in their dual structure were provided by [2, 34, 55, 58], see also [10, 21, 27, 29]. Our presentation is reminiscent of [9] and [15]. Given $1 \leqslant n \in \omega$ and a poset $\langle X; \leqslant\rangle$, and element $x \in X$ is said to have *depth* n if the upset ${\uparrow}x$ contains at least one chain of length n, and no chain of length $n+1$. Moreover, a finite sequence of zeros and ones is said to be a *colour*. Given two colours of the same length $\mathbf{a} = \langle a_1, \dots, a_n\rangle$ and $\mathbf{c} = \langle c_1, \dots, c_n\rangle$, we set

$$\begin{aligned}
&\mathbf{a} \leqslant \mathbf{c} \iff a_i \leqslant c_i \text{ for every } i = 1, \dots, n, \text{ and}\\
&\mathbf{a} < \mathbf{c} \iff \mathbf{a} \leqslant \mathbf{c} \text{ and } a_i < c_i \text{ for some } i = 1, \dots, n.
\end{aligned}$$

Accordingly, when we write $\mathbf{a} \leqslant \mathbf{c}$ or $\mathbf{a} < \mathbf{c}$, it should be understood that the colours $\mathbf{a}$ and $\mathbf{c}$ have the same length.

For every $n \in \omega$, we shall define a poset $\mathbb{U}(n) = \langle U(n); \leqslant\rangle$ as the union of a chain of posets $\{\mathbb{D}_m \colon 1 \leqslant m \in \omega\}$. To this end, observe that there are exactly 2^n distinct colours of length n. Then let D_1 be a set of 2^n elements painted with distinct colours of length n, and $\mathbb{D}_1 = \langle D_1; \leqslant_1\rangle$ the poset obtained equipping D_1 with the discrete partial order. Moreover, if $\mathbb{D}_m$ has already been defined, then let $\mathbb{D}_{m+1}$ be the poset obtained extending $\mathbb{D}_m$ in accordance to the following rules:

(i) For every point x of $\mathbb{D}_m$ of depth m and of colour $\mathbf{a}$, and every colour $\mathbf{c} < \mathbf{a}$, we add to $\mathbb{D}_m$ a unique point y painted with $\mathbf{c}$ such that

$$\uparrow^{\mathbb{D}_{m+1}} y = \{y\} \cup \uparrow^{\mathbb{D}_m} x;$$

(ii) For every antichain Z in $\mathbb{D}_m$ such that $|Z| \geqslant 2$ containing at least one point of depth m, and every colour $\mathbf{c}$ such that $\mathbf{c} \leqslant \mathbf{a}$ for every colour of some element in Z, we add to $\mathbb{D}_m$ a unique point y painted with $\mathbf{c}$ such that

$$\uparrow^{\mathbb{D}_{m+1}} y = \{y\} \cup \uparrow^{\mathbb{D}_m} Z.$$

It is clear that $\mathbb{D}_m$ is a subposet of $\mathbb{D}_{m+1}$ for every $1 \leqslant m \in \omega$, whence it makes sense to define $\mathbb{U}(n)$ as the union of the chain $\{\mathbb{D}_m : 1 \leqslant m \in \omega\}$. The importance of the poset $\mathbb{U}(n)$ is captured by the following observation:

Theorem 4.1. *Let $n \in \omega$, and let $\boldsymbol{F}(n)$ be the free n-generated Heyting algebra.*

(i) *$\mathbb{U}(n)$ is isomorphic to the topology-free reduct of the subposet of $\boldsymbol{F}(n)_*$ consisting of the elements of finite depth.*
(ii) *If $x \in \boldsymbol{F}(n)_*$, then either x has finite depth or for every $1 \leqslant n \in \omega$ there is an element $y \in \boldsymbol{Fm}(n)_*$ of depth n such that $x \leqslant y$.*
(iii) *For all $m \in \omega$, the poset $\mathbb{U}(n)$ has only finitely many points of depth $\leqslant m$.*

The statements of (i) and (ii) are [9, Thms. 3.2.9 and 3.1.10(4)], while (iii) follows immediately from the definition of $\mathbb{U}(n)$.

Corollary 4.2. *Let $n \in \omega$, and let $\boldsymbol{F}(n)$ be the free n-generated Heyting algebra. If $\boldsymbol{X}$ is an infinite E-subspace of $\boldsymbol{Fm}(n)_*$, then $\boldsymbol{X}$ contains an element of depth m for every $1 \leqslant m \in \omega$.*

Proof. Consider an infinite E-subspace $\boldsymbol{X}$ of $\boldsymbol{Fm}(n)_*$ and suppose, with a view to contradiction, that $\boldsymbol{X}$ does not contain any element of depth m for some $1 \leqslant m \in \omega$. We have two cases: either $\boldsymbol{X}$ contains an element of infinite depth or not. If $\boldsymbol{X}$ contains an element of infinite depth, then we obtain a contradiction because of condition (ii) of Theorem 4.1. Then all elements of $\boldsymbol{X}$ must have finite depth and, therefore, depth $< m$. As $\boldsymbol{X}$ is infinite, this means that $\boldsymbol{X}$ has infinitely many elements of depth $< m$. Moreover, since $\boldsymbol{X}$ is an E-subspace of $\boldsymbol{Fm}(n)_*$, the same holds for $\boldsymbol{Fm}(n)_*$. But this contradicts conditions (i) and (iii) of Theorem 4.1. Thus we have arrived at a contradiction. ⊠

In the rest of the paper we will rely on the following observation:

Theorem 4.3. *Let K be a variety of Heyting algebras. Then K is locally finite if and only if K has, up to isomorphism, only finitely many finite n-generated FSI members, for every $n \in \omega$.*

Proof. The "only" if part is straightforward. To prove the "if" part, we reason by contrapostion: suppose that K is not locally finite. Then there is some $n \in \omega$ and an n-generated infinite algebra $\boldsymbol{A} \in \mathsf{K}$. Clearly $\boldsymbol{A}$ is a homomorphic image of the free n-generated Heyting algebra $\boldsymbol{F}(n)$, whence $\boldsymbol{A}_*$ can be identified with an E-subspace of $\boldsymbol{F}(n)_*$ in the light of condition (ii) of Lemma 3.2. Moreover, the fact that $\boldsymbol{A}$ is infinite guarantees that so is $\boldsymbol{A}_*$. As a consequence, we can apply Corollary 4.2, obtaining that for every $1 \leqslant m \in \omega$ there is an element $x_m \in \boldsymbol{A}_*$ of depth m.

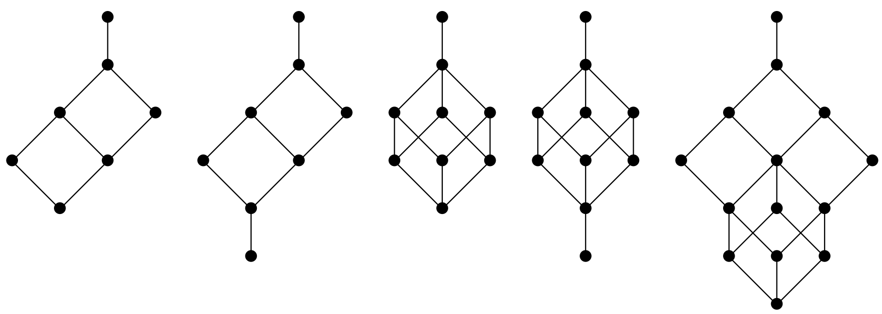
Now, the E-subspace $\uparrow^{\boldsymbol{A}_*} x_m$ of $\boldsymbol{A}_*$ is isomorphic to an FSI homomorphic image $\boldsymbol{A}_m := (\uparrow^{\boldsymbol{A}_*} x_m)^*$ of $\boldsymbol{A}$ by conditions (i) and (ii) of Lemma 3.2, whence $\boldsymbol{A}_m \in \mathbb{H}(\boldsymbol{A}) \subseteq \mathsf{K}$. Moreover, by conditions (i) and (iii) of Theorem 4.1 the upset $\uparrow^{\boldsymbol{A}_*} x_m$ is finite and, therefore, so is $\boldsymbol{A}_m$. Thus $\{\boldsymbol{A}_m : 1 \leqslant m \in \omega\}$ is a sequence of finite n-generated FSI members of K.

Moreover, observe that the size of the spaces $\{\uparrow^{\boldsymbol{A}_*} x_m : 1 \leqslant m \in \omega\}$ is not bounded by any natural number, as each x_m has depth m. As a consequence, also the cardinality of the algebras $\{\boldsymbol{A}_m : 1 \leqslant m \in \omega\}$ cannot be bounded by any natural. Since the algebras $\boldsymbol{A}_m$ are finite, we conclude that there must an infinite subset $\mathsf{C} \subseteq \{\boldsymbol{A}_m : 1 \leqslant m \in \omega\}$ of pairwise nonisomorphic algebras. Thus C is an infinite set of pairwise nonisomorphic finite n-generated FSI members of K. ⊠

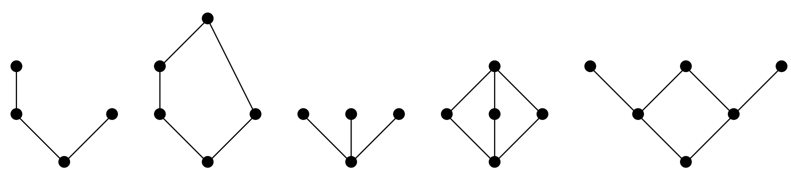
5. Citkin's five algebras

Consider the following FSI Heyting algebras:

C_1 C_2 C_3 C_4 C_5



Their dual Esakia spaces are the following rooted posets endowed with the discrete topology:

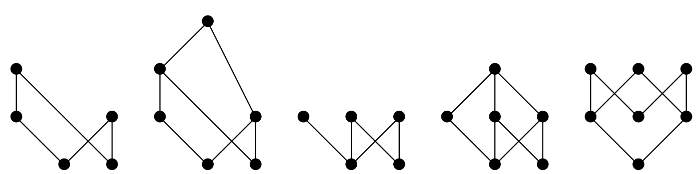


C_{1*} C_{2*} C_{3*} C_{4*} C_{5*}

The following result relates primitive varieties with the algebras $C_1, \dots, C_5$.

Lemma 5.1. *Primitive varieties of Heyting algebras omit* $C_1, \dots, C_5$.

Proof. Suppose, with a view to contradiction, that K is a primitive variety of Heyting algebras omitting $C_1, \dots, C_5$. Consider the following Esakia spaces $X_1, \dots, X_5$ endowed with the discrete topology:



X_1 X_2 X_3 X_4 X_5

First observe that each C_{i*} is an E-subspace of X_i, whence by Lemma 3.2(ii)

$$C_i \in \mathbb{H}(X_i^*) \text{ for every } i = 1, \dots, 5. \tag{3}$$

Moreover, by inspection one sees that for each C_{i*} there is a bisimulation equivalence R_i on the disjoint union $C_{i*} \uplus C_{i*}$ such that X_i is isomorphic to $(C_{i*} \uplus C_{i*})/R_i$. By Lemma 3.2(iii, iv) this implies

$$X_i^* \in \mathbb{IS}(C_i \times C_i) \text{ for every } i = 1, \dots, 5. \tag{4}$$

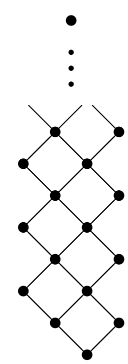
On the other hand, it is not hard to check that there is no surjective Esakia morphism from X_i to C_{i*}. By Lemma 3.2(iii) this implies

$$C_i \notin \mathbb{IS}(X_i^*) \text{ for every } i = 1, \dots, 5. \tag{5}$$

Now, by assumption there is some $i = 1, \ldots, 5$ such that $\boldsymbol{C}_i \in \mathsf{K}$. By (4) also $\boldsymbol{X}_i^* \in \mathsf{K}$. Moreover, by (3) and (5) we have $\boldsymbol{C}_i \in \mathbb{H}(\boldsymbol{X}_i^*)$ and $\boldsymbol{C}_i \notin \mathbb{IS}(\boldsymbol{X}_i^*)$. As a consequence, we conclude that $\boldsymbol{C}_i$ is not weakly projective in K. Since $\boldsymbol{C}_i$ is a finite nontrivial FSI member of K and K is primitive, this contradicts Lemma 2.1. Hence we reached a contradiction. ⊠

6. A structure theorem

In this section we give a description of the structure of varieties of Heyting algebras omitting $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_5$ (Theorem 6.13). To this end, recall that the *Rieger-Nishimura* lattice $\boldsymbol{RN}$ (depicted below) is the free one-generated Heyting algebra [49, 53]. As a consequence, $\mathbb{H}(\boldsymbol{RN})$ is the class of all one-generated Heyting algebras.



The Rieger-Nishimura lattice $\boldsymbol{RN}$.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras. The *sum* $\boldsymbol{A} + \boldsymbol{B}$ is the Heyting algebra obtained by pasting $\boldsymbol{B}$ *below* $\boldsymbol{A}$, gluing the top element of $\boldsymbol{B}$ to the bottom element of $\boldsymbol{A}$. As $+$ is clearly associative, there is no ambiguity in writing $\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n$ for the descending chain of finitely many Heyting algebras $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$, each glued to the previous one.

Then the *Kuznetsov-Gerčiu* variety is defined as follows:

$$\mathsf{KG} := \mathbb{V}(\{\boldsymbol{A}_1 + \cdots + \boldsymbol{A}_n \colon \boldsymbol{A}_1, \ldots, \boldsymbol{A}_n \in \mathbb{H}(\boldsymbol{RN}) \text{ and } 0 < n \in \omega\}). \tag{6}$$

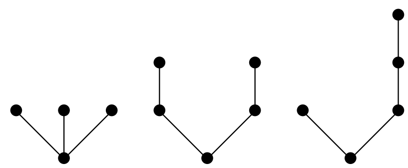
The variety KG was introduced in the study of finite axiomatizability, and of the finite model property in varieties of Heyting algebras [28, 43] (see also [6, 9, 48]). We shall see that varieties of Heyting algebras omitting $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_5$ are subvarieties of KG (Theorem 6.13).

To this end, it is convenient to recall some basic concept. In [61], every finite rooted Esakia space $\boldsymbol{Z}$ is associated with a formula $\beta(\boldsymbol{Z})$ in the language of Heyting algebras, called the *subframe formula* of $\boldsymbol{Z}$ (see also [7, 11, 15]). For the present purpose, the way in which subframe formulas are concretely defined is immaterial and, to explain their importance, it is sufficient to recall the definition of the following concept. An Esakia space $\boldsymbol{Y} = \langle Y; \tau^{\boldsymbol{Y}}, \leqslant^{\boldsymbol{Y}} \rangle$ is called a *subspace* of an Esakia space $\boldsymbol{X} = \langle X; \tau^{\boldsymbol{X}}, \leqslant^{\boldsymbol{X}} \rangle$, if $\langle Y; \tau^{\boldsymbol{Y}} \rangle$ is a subspace of $\langle X; \tau^{\boldsymbol{X}} \rangle$, the order $\leqslant^{\boldsymbol{Y}}$ is the restriction of $\leqslant^{\boldsymbol{X}}$ to Y^2, and for every clopen U of $\boldsymbol{Y}$, the downset generated by U with respect to $\leqslant^{\boldsymbol{X}}$ is clopen in $\boldsymbol{X}$. The following result clarifies the role of subframe formulas [11, Thm. 3.13]:

Theorem 6.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Z}$ be Esakia spaces such that $\boldsymbol{Z}$ is finite and rooted. Then $\boldsymbol{X}^* \vDash \beta(\boldsymbol{Z}) \approx 1$ if and only if $\boldsymbol{Z}$ is not the image of an Esakia morphism, whose domain is a subspace of $\boldsymbol{X}$.*

Remark 6.2. Recall that finite Esakia spaces coincide with finite posets endowed with the discrete topology. Thus if $\boldsymbol{X}$ is a finite Esakia space, then the above theorem specializes as follows: $\boldsymbol{X}^* \vDash \beta(\boldsymbol{Z}) \approx 1$ if and only if $\boldsymbol{Z}$ is not the image of an Esakia morphism, whose domain is a subposet of $\boldsymbol{X}$. ⊠

For the present purpose, the interest in subframe formulas is that they provide a convenient axiomatization of KG. To explain how this is obtained, consider the discrete rooted Esakia spaces $\boldsymbol{P}_1, \boldsymbol{P}_2$, and $\boldsymbol{P}_3$ whose underlying posets are depicted below:



$\boldsymbol{P}_1$ $\boldsymbol{P}_2$ $\boldsymbol{P}_3$

The proof of the following result can be found in [9, Thm. 4.3.4] (see also [6, 42]):

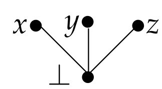
Theorem 6.3. KG *is the variety of Heyting algebras axiomatized by the equations*

$$\beta(\boldsymbol{P}_1) \approx 1 \qquad \beta(\boldsymbol{P}_2) \approx 1 \qquad \beta(\boldsymbol{P}_3) \approx 1.$$

Given a positive integer n, a poset $\langle X; \leqslant \rangle$ has *width* $\leqslant n$ if there is no $x \in X$ such that ${\uparrow}x$ contains an antichain of $n+1$ elements. Accordingly, a Heyting algebra $\boldsymbol{A}$ is said to have *width* $\leqslant n$ when so does the poset underlying $\boldsymbol{A}_*$.

Lemma 6.4. *Let* K *be a variety of Heyting algebras omitting* $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$. *Every finite member of* K *has width* $\leqslant 2$ *and, therefore, satisfies* $\beta(\boldsymbol{P}_1) \approx 1$.

Proof. Suppose, with a view to contradiction, that there is a finite $\boldsymbol{A} \in \mathsf{K}$ of width > 2. Then $\boldsymbol{A}_*$ contains a subposet isomorphic to $\boldsymbol{P}_1$. We label its elements as follows:



As $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$ and $\boldsymbol{A}_*$ is finite, by Lemma 3.2(ii) we can assume without loss of generality that the following holds:

Fact 6.5.
(i) $\bot$ is the minimum of $\boldsymbol{A}_*$ and the unique common lower bound of x, y, z.
(ii) $\{x, y, z\}$ in $\boldsymbol{A}_*$ is the unique three-element antichain in ${\downarrow}\{x, y, z\}$.

Then consider the following relation on $\boldsymbol{A}_*$:

$$R := \{\langle u, v \rangle \in A_* \times A_* \colon \text{either } u = v \text{ or } u, v \in \boldsymbol{A}_* \smallsetminus {\downarrow}\{x, y, z\}\}.$$

Bearing in mind that $\boldsymbol{A}_*$ is finite and, therefore, endowed with the discrete topology, it is easy to see that R is a bisimulation equivalence on $\boldsymbol{A}_*$. Accordingly, we consider the Esakia space $\boldsymbol{X} := \boldsymbol{A}_*/R$. In the light of Lemma 3.2(iii), we obtain $\boldsymbol{X}^* \in \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$. Observe that the relation R is the identity relation on ${\downarrow}\{x, y, z\}$ and identifies everything in $A_* \smallsetminus {\downarrow}\{x, y, z\}$. Thus the assumption that R is the identity on $\boldsymbol{A}_*$ means that A_* contains at most one element $\top$ not in ${\downarrow}\{x, y, z\}$. Denoting by $\boldsymbol{Y}$ the subposet ${\downarrow}\{x, y, z\}$ of $\boldsymbol{A}_*$, we obtain the following:

Fact 6.6. There is an Esakia space $\boldsymbol{X}$ such that $\boldsymbol{X}^* \in \mathsf{K}$ and one of the following holds:

(i) The poset underlying $\boldsymbol{X}$ is $\boldsymbol{Y}$;
(ii) $\boldsymbol{Y}$ is a subposet of $\boldsymbol{X}$ and $X = \{\top\} \cup Y$, where $\top$ strictly above exactly two elements between x, y, z; or
(iii) $\boldsymbol{Y}$ is a subposet of $\boldsymbol{X}$ and $X = \{\top\} \cup Y$, where $\top$ is the maximum of $\boldsymbol{X}$.

Proof. Suppose that conditions (i) and (ii) fail. Then, in particular, $A_* \neq {\downarrow}\{x, y, z\}$, otherwise $\boldsymbol{A}_*$ would satisfy condition (i). Consequently, $A_* = \{\top\} \cup Y$ where $\top \notin Y$.

We shall see that $\top$ is comparable with some element among x, y, and z. Suppose the contrary, with a view to contradiction. Then the least equivalence relation S on A_* that identifies $\top$ with x is easily seen to be a bisimulation equivalence on $\boldsymbol{A}_*$. Moreover, the poset underlying $\boldsymbol{A}_*/S$ is isomorphic to $\boldsymbol{Y}$. As by Lemma 3.2(iii), $\boldsymbol{A}_*/S \in \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$, taking $\boldsymbol{X} := \boldsymbol{A}_*/S$ we would obtain that condition (i) holds, which is false. Thus we conclude that $\top$ is comparable with some element among x, y, z, as desired. We can assume without loss of generality that this element is x. Since $\top \notin Y$, this implies $x < \top$.

An argument analogous to the one described above shows that the assumption that $y \nleqslant \top$ and $z \nleqslant \top$ leads to a contradiction. Then we can assume without loss of generality that $y \leqslant \top$ and, therefore, $y < \top$ (as $\top \notin Y$). Finally, if $z \nleqslant \top$, then condition (ii) holds, contradicting the assumption. Then we conclude that $z \leqslant \top$, whence $\top$ is the maximum of $\boldsymbol{A}_*$. Thus taking $\boldsymbol{X} := \boldsymbol{A}_*$, we obtain that condition (iii) holds, as desired. ⊠

Fact 6.7. The following relation is a bisimulation equivalence on $\boldsymbol{X}$:

$$S := \{\langle u, v\rangle \in X \times X \colon \{x, y, z\} \cap {\uparrow}u = \{x, y, z\} \cap {\uparrow}v\}.$$

Proof. First observe that S is an equivalence relation. Then it only remains to show that $\boldsymbol{Y}$ satisfies conditions (i) and (ii) in the definition of a bisimulation equivalence. Since $\boldsymbol{Y}$ is finite, its topology is discrete, whence condition (ii) is obviously satisfied. To prove condition (i), consider three elements $t, u, v \in Y$ such that $\langle t, u\rangle \in S$, $\langle t, v\rangle \notin S$, $t \neq u$, and $t \leqslant v$. We need to find some $u \leqslant w \in Y$ such that $\langle v, w\rangle \in S$. Clearly

$$\{x, y, z\} \cap {\uparrow}v \in \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}. \tag{7}$$

First consider the case in which $\{x, y, z\} \cap {\uparrow}v = \emptyset$. From Fact 6.6 it follows that $v = \top$ and $X = \{\top\} \cup Y$ where $\top \notin Y$. If condition (iii) of Fact 6.6 holds, then, by taking $w := \top$, we are done. Now suppose that condition (iii) of Fact 6.6 fails. Together with the fact that $X = \{\top\} \cup Y$ and $\top \notin Y$, this implies that condition (ii) of Fact 6.6 holds. Thus we can assume without loss of generality that $x, y < \top$ and $z \nleqslant \top$. Since $t \neq v = \top$, clearly $t \in Y$. Now, if $t \in {\downarrow}\{x, y\}$, then also $u \in {\downarrow}\{x, y\}$ (as $\langle t, u\rangle \in S$). Consequently, $u \leqslant \top = v$ and, by taking $w := v$, we are done. Next we consider the case where $t \notin {\downarrow}\{x, y\}$. We shall see that this case leads to a contradiction. To this end, observe that in this case $t \leqslant z$, as $t \in Y$ and $t \notin {\downarrow}\{x, y\}$. Moreover, since $t \leqslant v = \top$ and $z \nleqslant \top$, we obtain $t < z$. But the fact that $t \nleqslant x, y$ and $t < z$ implies that $\{x, y, z\}$ is a three-element antichain in $\boldsymbol{Y}$ different from $\{x, y, z\}$, contradicting Fact 6.5(ii).

If $\{x, y, z\} \cap {\uparrow}v = \{x\}$, then $\langle v, x\rangle \in S$. Moreover, from $t \leqslant v \leqslant x$ and $\langle t, u\rangle \in S$ it follows $u \leqslant x$. Thus, by setting $w := x$, we are done. A similar argument works if $v \cap {\uparrow}\{x, y, z\}$ is $\{y\}$ or $\{z\}$ (take respectively $w := y$ and $w := z$).

By (7) it only remains to consider the case where

$$\{x, y, z\} \cap {\uparrow}v \in \{\{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}. \tag{8}$$

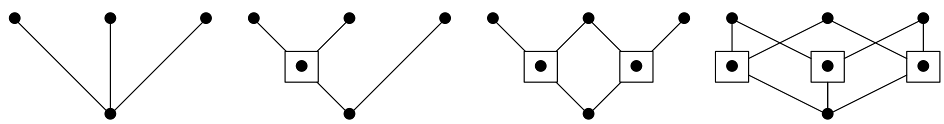
We shall show that this case leads to a contradiction. To this end, observe that

$$\{x, y, z\} \cap {\uparrow} v \subsetneq \{x, y, z\} \cap {\uparrow} t,$$

since $\langle t, v \rangle \notin S$ and $t \leqslant v$. Together with (8), this guarantees that $t \leqslant x, y, z$, whence also $u \leqslant x, y, z$ as $\langle u, t \rangle \in S$. By Fact 6.5(i), we have that $\bot$ is the unique common lower bound of x, y, z, whence $t = \bot = u$, contradicting the fact that $t \neq u$. Thus we conclude that S is a bisimulation equivalence on $\boldsymbol{X}$. ⊠

Recall that $\boldsymbol{X}^* \in \mathsf{K}$ and that S is a bisimulation equivalence on $\boldsymbol{X}$ by Facts 6.6 and 6.7. Thus by Lemma 3.2(iii), we have that $(\boldsymbol{X}/S)^* \in \mathbb{IS}(\boldsymbol{X}^*) \subseteq \mathsf{K}$. Accordingly, we can assume without loss of generality that S is the identity relation on $\boldsymbol{X}$.

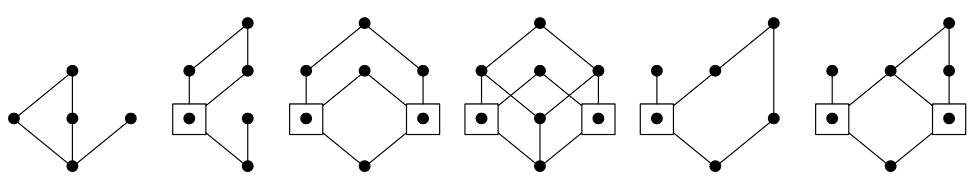
Bearing this in mind, if case (i) of Fact 6.6 holds, then the poset underlying $\boldsymbol{X}$ is one of the rooted posets depicted below (in which the elements other than $\bot, x, y$, and z are marked with squares):



$\boldsymbol{Z}_1$ $\boldsymbol{Z}_2$ $\boldsymbol{Z}_3$ $\boldsymbol{Z}_4$

Observe that $\boldsymbol{Z}_1 \cong \boldsymbol{C}_{3*}$ and $\boldsymbol{Z}_3 \cong \boldsymbol{C}_{5*}$. Moreover, there are bisimulation equivalences T and T', respectively on $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_4$, such that $\boldsymbol{Z}_2/T \cong \boldsymbol{C}_{1*}$ and $\boldsymbol{Z}_4/T' \cong \boldsymbol{C}_{4*}$. By Lemma 3.2(iii), this implies that $\mathbb{IS}(\boldsymbol{X}^*) \cap \{\boldsymbol{C}_1, \boldsymbol{C}_3, \boldsymbol{C}_4, \boldsymbol{C}_5\} \neq \emptyset$. But, since $\boldsymbol{X}^* \in \mathsf{K}$, we would get $\mathsf{K} \cap \{\boldsymbol{C}_1, \boldsymbol{C}_3, \boldsymbol{C}_4, \boldsymbol{C}_5\} \neq \emptyset$, contradicting the assumption that K omits $\boldsymbol{C}_1, \boldsymbol{C}_3, \boldsymbol{C}_4, \boldsymbol{C}_5$. Thus we conclude that case (i) of Fact 6.6 cannot hold.

Now, suppose that case (ii) of Fact 6.6 holds. Since S is the identity, the poset underlying $\boldsymbol{X}$ is one of the rooted posets depicted below (in which the elements other than $\bot, x, y, z$, and $\top$ are marked with squares):



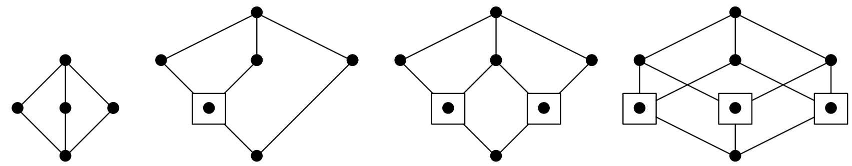
$\boldsymbol{Z}_1$ $\boldsymbol{Z}_2$ $\boldsymbol{Z}_3$ $\boldsymbol{Z}_4$ $\boldsymbol{Z}_5$ $\boldsymbol{Z}_6$

For every $i = 1, \ldots, 6$ there is a bisimulation equivalence T_i on $\boldsymbol{Z}_i$ such that

$$\boldsymbol{Z}_1/T_1 \cong \boldsymbol{Z}_2/T_2 \cong \boldsymbol{Z}_5/T_5 \cong \boldsymbol{Z}_6/T_6 \cong \boldsymbol{C}_{1*} \text{ and } \boldsymbol{Z}_3/T_3 \cong \boldsymbol{C}_{2*} \text{ and } \boldsymbol{Z}_4/T_4 \cong \boldsymbol{C}_{4*}.$$

By Lemma 3.2(iii), this implies that $\mathbb{IS}(\boldsymbol{X}^*) \cap \{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_4\} \neq \emptyset$. But, since $\boldsymbol{X}^* \in \mathsf{K}$, we would get $\mathsf{K} \cap \{\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_4\} \neq \emptyset$, contradicting the assumption that K omits $\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_4$. Thus we conclude that also case (ii) of Fact 6.6 cannot hold.

Thus condition (iii) of Fact 6.6 holds necessarily. Since S is the identity, the poset underlying $\boldsymbol{X}$ is one of the rooted posets depicted below (in which the elements other than $\bot, x, y, z$, and $\top$ are marked with squares):

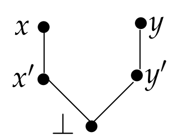


$\boldsymbol{Z}_1$ $\boldsymbol{Z}_2$ $\boldsymbol{Z}_3$ $\boldsymbol{Z}_4$

Observe that $\boldsymbol{Z}_1 \cong \boldsymbol{C}_{4*}$. Moreover, for every $i = 2, 3, 4$ there is a bisimulation equivalence T_i on $\boldsymbol{Z}_i$ such that $\boldsymbol{Z}_2/T_2 \cong \boldsymbol{Z}_3/T_3 \cong \boldsymbol{C}_{2*}$ and $\boldsymbol{Z}_4/T_4 \cong \boldsymbol{C}_{4*}$. As in the previous cases, this implies $\mathsf{K} \cap \{\boldsymbol{C}_2, \boldsymbol{C}_4\} \neq \emptyset$, contradicting the assumption that K omits $\boldsymbol{C}_2$ and $\boldsymbol{C}_4$. Thus we reached the desired contradiction. As a consequence, $\boldsymbol{A}$ has width $\leqslant 2$. This immediately implies that finite members of K validate $\beta(\boldsymbol{P}_1) \approx 1$. ⊠

Lemma 6.8. *Let K be a variety of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$. Every finite member of K satisfies the equation $\beta(\boldsymbol{P}_2) \approx 1$.*

Proof. Suppose, with a view to contradiction, that there is a finite algebra $\boldsymbol{A} \in \mathsf{K}$ in which the equation $\beta(\boldsymbol{P}_2) \approx 1$ fails. By Theorem 6.1 there is a subframe $\boldsymbol{X}$ of $\boldsymbol{A}_*$ and a surjective Esakia morphism from $\boldsymbol{X}$ to the space obtained endowing $\boldsymbol{P}_2$ with the discrete topology. Because of the definition of an Esakia morphism, this implies that there is a subposet of $\boldsymbol{A}_*$ isomorphic to $\boldsymbol{P}_2$. We label the elements of this a copy of $\boldsymbol{P}_2$ inside $\boldsymbol{A}_*$ as follows:



Moreover, by Lemma 3.2(ii) and $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$, we can assume without loss of generality that $\bot$ is the minimum of $\boldsymbol{A}_*$ and the unique common lower bound of x' and y'. In addition, as in the proof of Lemma 6.4, we can assume without loss of generality that A_* contains at most one element $\top$ not in ${\downarrow}\{x, y\}$. By Lemma 6.4 we know that $\boldsymbol{A}_*$ has depth $\leqslant 2$, whence, provided that $\top$ exists, it must be comparable either with x or y. As $\top \nleqslant x, y$, this implies that either $\top$ does not exist or $x < \top$ or $y < \top$. Consequently,

Fact 6.9. One of the following conditions holds:

(i) $A_* = {\downarrow}\{x, y\}$;
(ii) $\boldsymbol{A}_*$ has a maximum $\top$ and $A_* = \{\top\} \cup {\downarrow}\{x, y\}$; or
(iii) $\boldsymbol{A}_*$ has a maximal element $\top$ strictly above exactly one between x and y, and $A_* = \{\top\} \cup {\downarrow}\{x, y\}$.

Observe that in case (iii) if $x < \top$ (resp. $y < \top$), then $y \nleqslant \top$ (resp. $x \nleqslant \top$), and x and y are incomparable.

Our aim is to show that conditions (i), (ii), and (iii) lead to a contradiction. First suppose that condition (i) holds. We shall see that for all $z \in A_*$,

$$\{x, x', y, y'\} \cap {\uparrow}z \in \{\{x\}, \{y\}, \{x, x'\}, \{y, y'\}, \{x, x', y\}, \{y, y', x\}, \{x, x', y, y'\}\}. \quad (9)$$

To prove this, consider $z \in A_*$. Clearly $\{x, x', y, y'\} \cap {\uparrow}z$ is an upset of the copy of $\boldsymbol{P}_2$ in $\boldsymbol{A}_*$ given by $\{\bot, x, x', y, y'\}$. Moreover, this upset must be non-empty by assumption (i), since $z \in {\downarrow}\{x, y\}$. Thus, in order to establish the above display, it suffices to show that $\{x, x', y, y'\} \cap {\uparrow}z \neq \{x, y\}$. Suppose the contrary with a view to contradiction. Then observe that $\bot \leqslant z, x', y'$ and recall that x' and y' are incomparable. Since $\boldsymbol{A}_*$ has width $\leqslant 2$ by Lemma 6.4, this implies that z is comparable either with x' or with y'. We can assume without loss of generality that z is comparable with x'. Since $\{x, x', y, y'\} \cap {\uparrow}z = \{x, y\}$, this implies $x' < z$ and, therefore, $x' \leqslant z \leqslant y'$. But this contradicts the fact that $x' \nleqslant y$, whence establishing (9).

Then we shall see that the following relation is a bisimulation equivalence on $\boldsymbol{A}_*$:

$$S \coloneqq \{\langle u, v\rangle \in A_* \times A_* \colon \{x, x', y, y'\} \cap {\uparrow}u = \{x, x', y, y'\} \cap {\uparrow}v\}.$$

As before, it suffices to show that condition (i) in the definition of a bisimulation equivalence holds. To this end, consider $t, u, v \in A_*$ such that $\langle t, u\rangle \in S$, $t \neq u$, $t < v$, and $\langle t, v\rangle \notin S$. We need to find an element $w \geqslant u$ such that $\langle v, w\rangle \in S$. By (9)

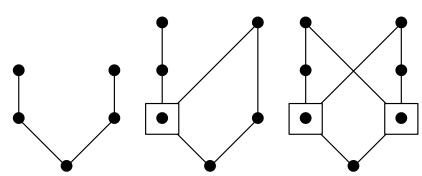
$$\{x, x', y, y'\} \cap {\uparrow}v \in \{\{x\}, \{y\}, \{x, x'\}, \{y, y'\}, \{x, x', y\}, \{y, y', x\}, \{x, x', y, y'\}\}.$$

If $\{x, x', y, y'\} \cap {\uparrow}v = \{x\}$, then $\langle v, x\rangle \in S$. Moreover, from $t \leqslant v \leqslant x$ and $\langle t, u\rangle \in S$ it follows $u \leqslant x$. Thus, setting $w \coloneqq x$, we are done. A similar argument works if $\{x, x', y, y'\} \cap {\uparrow}v$ is $\{y\}$ or $\{x, x'\}$ or $\{y, y'\}$ (take respectively $w \coloneqq y$, $w \coloneqq x'$, and $w \coloneqq y'$). Then it only remains to consider the case where

$$\{x, x', y, y'\} \cap {\uparrow}v \in \{\{x, x', y\}, \{y, y', x\}, \{x, x', y, y'\}\}.$$

But an argument analogous to the one detailed in the last paragraph of the proof of Fact 6.7 shows that this case leads to a contradiction. Hence we conclude the S is a bisimulation equivalence on $\boldsymbol{A}_*$.

In particular, by Lemma 3.2(iii) this implies $(\boldsymbol{A}_*/S)^* \in \mathsf{K}$. Consequently, we can assume without loss of generality that S is the identity relation on $\boldsymbol{A}_*$. Together with (9) and the fact that $\{\bot, x, x', y, y'\}$ forms a subposet of $\boldsymbol{A}_*$ isomorphic to $\boldsymbol{P}_2$, this implies that $\boldsymbol{A}_*$ is isomorphic to one of the following rooted posets (in which the elements other than $\bot, x, x', y, y'$ are marked with squares):



$\boldsymbol{Z}_1$ $\boldsymbol{Z}_2$ $\boldsymbol{Z}_3$

Observe that $\boldsymbol{C}_{1*}$ is isomorphic to an E-subspace of $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$. Moreover, there is a bisimulation equivalence T on $\boldsymbol{Z}_1$ such that $\boldsymbol{Z}_1/T \cong \boldsymbol{C}_{1*}$. By Lemma 3.2(ii, iii) this implies $\boldsymbol{C}_1 \in \mathbb{H}(\boldsymbol{A}) \cup \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$, contradicting the assumption that $\boldsymbol{C}_1 \notin \mathsf{K}$. Thus condition (i) cannot hold.

Next we consider the case where condition (ii) holds. An argument analogous to the one detailed for case (i) shows that for every $z \in A_*$,

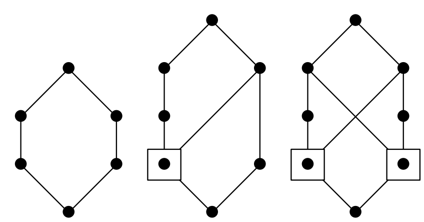
$$\begin{aligned}\{x, x', y, y', \top\} \cap {\uparrow}z \in \{&\{\top\}, \{x, \top\}, \{y, \top\}, \{x, x', \top\}, \{y, y', \top\},\\ &\{x, x', y, \top\}, \{y, y', x, \top\}, \{x, x', y, y', \top\}\}.\end{aligned} \tag{10}$$

We shall see that the following relation is a bisimulation equivalence on $\boldsymbol{A}_*$:

$$S := \{\langle u, v\rangle \in A_* \times A_* \colon \{x, x', y, y', \top\} \cap {\uparrow} u = \{x, x', y, y', \top\} \cap {\uparrow} v\}.$$

To this end, consider $t, u, v \in A_*$ such that $\langle t, u\rangle \in S$, $t \neq u$, $t < v$, and $\langle t, v\rangle \notin S$. We need to find an element $w \geqslant u$ such that $\langle v, w\rangle \in S$. First we consider the case where $v \neq \top$. In this case, $v \in {\downarrow}\{x, y\}$ by assumption (ii). As $t \leqslant v$, we also get $t \in {\downarrow}\{x, y\}$. In turn, this guarantees $u \in {\downarrow}\{x, y\}$, since $\langle t, u\rangle \in S$. Consequently, $t, u, v \in {\downarrow}\{x, y\}$. This allows us to repeat the argument detailed in the case of condition (i), obtaining the desired element w. Then it only remains to consider the case where $v = \top$. But assumption (ii) guarantees $u \leqslant \top = v$. Thus, by setting $w := \top$, we are done. This establishes that S is a bisimulation equivalence on $\boldsymbol{A}_*$.

Consequently, we can assume without loss of generality that S is the identity relation on $\boldsymbol{A}_*$. Together with (10) and the fact that $\{\bot, x, x', y, y', \top\}$ forms a subposet of $\boldsymbol{A}_*$ isomorphic to $\boldsymbol{P}_2$ plus a new top element, this implies that $\boldsymbol{A}_*$ is isomorphic to one of the following rooted posets (in which the elements other than $\bot, x, x', y, y', \top$ are marked with squares):



$\boldsymbol{Z}_1$ $\boldsymbol{Z}_2$ $\boldsymbol{Z}_3$

Observe that $\boldsymbol{C}_{2*}$ is isomorphic to an E-subspace of $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$. Moreover, there is a bisimulation equivalence T on $\boldsymbol{Z}_1$ such that $\boldsymbol{Z}_1 / T \cong \boldsymbol{C}_{2*}$. By Lemma 3.2(ii, iii) this implies $\boldsymbol{C}_2 \in \mathbb{H}(\boldsymbol{A}) \cup \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$, contradicting the assumption that $\boldsymbol{C}_2 \notin \mathsf{K}$. Thus also condition (ii) cannot hold.

Consequently, by Fact 6.9 condition (iii) holds necessarily. We can assume without loss of generality that $\top > x$ and $\top \ngeqslant y$. Observe that for every $z \in A_*$,

$$\{x, x', y'\} \cap {\uparrow} z \in \{\emptyset, \{x\}, \{y'\}, \{x, x'\}, \{x, y'\}, \{x, x', y'\}\}. \tag{11}$$

This is an immediate consequence of the fact that $\{x, x', y'\} \cap {\uparrow} z$ must be an upset of the subposet of $\boldsymbol{A}_*$ with universe $\{x, x', y'\}$.

We shall see that the following relation is a bisimulation equivalence on $\boldsymbol{A}_*$:

$$S := \{\langle u, v\rangle \in A_* \times A_* \colon \{x, x', y'\} \cap {\uparrow} u = \{x, x', y'\} \cap {\uparrow} v\}.$$

To prove this, consider $t, u, v \in A_*$ such that $\langle t, u\rangle \in S$, $t \neq u$, $t < v$, and $\langle t, v\rangle \notin S$. As usual, we need to find an element $w \geqslant u$ such that $\langle v, w\rangle \in S$. First we consider the case where $\{x, x', y'\} \cap {\uparrow} v = \emptyset$. Observe that $\{x, x', y'\} \cap {\uparrow} t \neq \emptyset$, since $\langle t, v\rangle \notin S$. Thus either $t \leqslant x$ or $t \leqslant y'$. As $\langle t, u\rangle \in S$, this implies that either $u \leqslant x$ or $u \leqslant y'$. Consequently, either $u \leqslant \top$ or $u \leqslant y$. Observe that $\{x, x', y'\} \cap {\uparrow}\top = \{x, x', y'\} \cap {\uparrow} y = \emptyset$, whence $\langle \top, v\rangle, \langle y, v\rangle \in S$. Thus there exists some $w \geqslant u$ (namely either $\top$ or y) such that $\langle w, v\rangle \in S$, as desired.

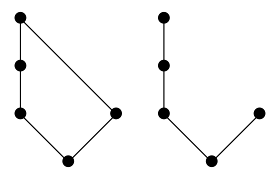
Now we consider the case where $\{x, x', y'\} \cap {\uparrow} v \neq \emptyset$. If $\{x, x', y'\} \cap {\uparrow} v = \{x\}$, then $\langle v, x\rangle \in S$. Moreover, as $\langle t, u\rangle \in S$ and $t \leqslant v \leqslant x$, we have $u \leqslant x$. Thus, by setting

$w := x$, we are done. A similar argument works if $v \cap \uparrow\{x, x', y, y'\}$ is $\{x, x'\}$ or $\{y'\}$ (take respectively $w := x'$ and $w := y'$). By (11) it only remains to consider the case where

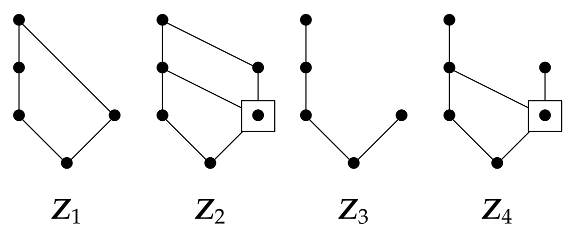
$$\{x, x', y'\} \cap \uparrow v \in \{\{x, y\}, \{x, x', y'\}\}.$$

But an argument analogous to the one detailed in the last paragraph of the proof of Fact 6.7 shows that this case leads to a contradiction. Hence we conclude the S is a bisimulation equivalence on $\boldsymbol{A}_*$.

Consequently, we can assume without loss of generality that S is the identity relation on $\boldsymbol{A}_*$. Observe that the subposet of $\boldsymbol{A}_*$ with universe $\{\bot, x, x', y', \top\}$ is isomorphic to one of the following rooted posets:



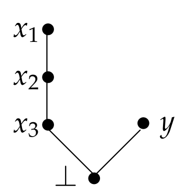
Together with (11) and the fact that S is the identity relation, this implies that $\boldsymbol{A}_*$ is isomorphic to one of the following rooted posets (in which the elements other than $\bot, x, x', y', \top$ are marked with squares):



For every $i = 1, \dots, 4$ there is a bisimulation equivalence T_i on $\boldsymbol{Z}_i$ such that $\boldsymbol{Z}_1/T_1 \cong \boldsymbol{Z}_2/T_2 \cong \boldsymbol{C}_{2*}$ and $\boldsymbol{Z}_3/T_3 \cong \boldsymbol{Z}_4/T_4 \cong \boldsymbol{C}_{1*}$. By Lemma 3.2(iii) this implies $\{\boldsymbol{C}_1, \boldsymbol{C}_2\} \cap \mathbb{IS}(\boldsymbol{A}) \neq \emptyset$, whence either $\boldsymbol{C}_1$ or $\boldsymbol{C}_2$ belongs to K. But this contradicts the fact that K omits $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. Hence we reached the desired contradiction. ⊠

Lemma 6.10. *Let K be a variety of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$. Every finite member of K satisfies the equation $\beta(\boldsymbol{P}_3) \approx 1$.*

Proof. Suppose, with a view to contradiction, that there is a finite algebra $\boldsymbol{A} \in \mathsf{K}$ in which the equation $\beta(\boldsymbol{P}_3) \approx 1$ fails. By Theorem 6.1 there is a subframe $\boldsymbol{X}$ of $\boldsymbol{A}_*$ and a surjective Esakia morphism from $\boldsymbol{X}$ to the space obtained endowing $\boldsymbol{P}_3$ with the discrete topology. Because of the definition of an Esakia morphism, this implies that there is a subposet of $\boldsymbol{A}_*$ isomorphic to $\boldsymbol{P}_3$. We label the elements of this a copy of $\boldsymbol{P}_3$ inside $\boldsymbol{A}_*$ as follows:



Moreover, by Lemma 3.2(ii) and $\mathbb{H}(\mathsf{K}) \subseteq \mathsf{K}$, we can assume without loss of generality that $\bot$ is the minimum of $\boldsymbol{A}_*$ and the unique common lower bound of x_3 and y.

First observe that for every $z \in A_*$,

$$\{x_2, x_3, y\} \cap \uparrow z \in \{\emptyset, \{x_2\}, \{x_2, x_3\}, \{y\}, \{x_2, y\}, \{x_2, x_3, y\}\}. \tag{12}$$

This is an immediate consequence of the fact that $\{x_2, x_3, y\} \cap \uparrow z$ is an upset of the subposet of $\boldsymbol{A}_*$ with universe $\{x_2, x_3, y\}$.

Now, we shall see that the following relation is a bisimulation equivalence on $\boldsymbol{A}_*$:

$$S := \{\langle u, v\rangle \in A_* \times A_* \colon \{x_2, x_3, y\} \cap \uparrow u = \{x_2, x_3, y\} \cap \uparrow v\}.$$

To this end, consider $t, u, v \in A_*$ such that $\langle t, u\rangle \in S$, $t \neq u$, $t < v$, and $\langle t, v\rangle \notin S$. As usual, we need to find an element $w \geqslant u$ such that $\langle v, w\rangle \in S$.

First we consider the case where $\{x_2, x_3, y\} \cap \uparrow v = \emptyset$. If $t \leqslant x_2$, then also $u \leqslant x_2 \leqslant x_1$ (as $\langle t, u\rangle \in S$). Consequently, setting $w := x_1$, we are done. Then suppose that $t \nleqslant x_2$. Since $\langle t, v\rangle \notin S$ and $\{x_2, x_3, y\} \cap \uparrow v = \emptyset$, we get $\{x_2, x_3, y\} \cap \uparrow t \neq \emptyset$. This implies that either $t \leqslant x_2$ or $t \leqslant y$. As $t \nleqslant x_2$, we conclude $t \leqslant y$. First consider the case where $t = y$. Then $y = t < v$. As $\langle t, u\rangle$, we have $u \leqslant y \leqslant v$. Thus, by taking $w := v$, we are done. We shall see that the case where $t < y$ never happens. To this end, suppose the contrary, with a view to contradiction. We shall see that

$$\bot < t < y \text{ and } t \text{ is incomprable with } x_2, x_3. \tag{13}$$

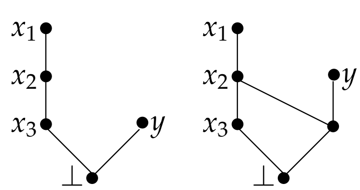
As $t \nleqslant x_2$, clearly $t \neq \bot$, whence $\bot < t$. Consequently, $\bot < t < y$. Now, as $t \nleqslant x_2$, we have $t \nleqslant x_2, x_3$. Moreover, since $t \leqslant y$ and $x_2, x_3 \nleqslant y$, clearly $x_2, x_3 \nleqslant t$. Thus t is incomparable with x_2 and x_3. This establishes (13). Then $\{\bot, x_2, x_3, t, y\}$ is the universe of a subposet of $\boldsymbol{A}_*$ isomorphic to $\boldsymbol{P}_2$. Thus $\boldsymbol{A}$ does not satisfies $\beta(\boldsymbol{P}_2)$. But this contradicts Lemma 6.8. Hence we reached a contradiction, as desired. This concludes the analysis of the case where $\{x_2, x_3, y\} \cap \uparrow v = \emptyset$.

If $\{x_2, x_3, y\} \cap \uparrow v$ is equal to $\{x_2\}$, $\{x_2, x_3\}$, or $\{y\}$, then, by taking respectively $w := x_2$, $w := x_3$ and $w := y$, we are done. In the light of (12), the only case that remains to be considered is the one where

$$\{x_2, x_3, y\} \cap \uparrow v \in \{\{x_2, y\}, \{x_2, x_3, y\}\}.$$

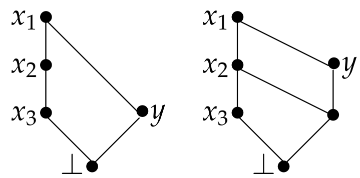
But an argument analogous to the one detailed in the last paragraph of the proof of Fact 6.7 shows that this case leads to a contradiction. Hence we conclude the S is a bisimulation equivalence on $\boldsymbol{A}_*$.

Consequently, we can assume without loss of generality that S is the identity relation on $\boldsymbol{A}_*$. Now, either y is a maximal element $\boldsymbol{A}_*$ or it is not. If y is a maximal element of $\boldsymbol{A}_*$, then the fact that S is the identity relation and condition (12) imply that $\boldsymbol{A}_*$ is isomorphic to one of the following rooted posets:



In both cases, $\boldsymbol{C}_1 \in \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$ by Lemma 3.2(iii). But this contradicts the assumption that K omits $\boldsymbol{C}_1$.

We conclude that y is not a maximal element of $\boldsymbol{A}_*$. Together with the fact that S is the identity relation and condition (12), this implies that $\boldsymbol{A}_*$ is isomorphic to one of the following rooted posets:



In both cases, $\boldsymbol{C}_2 \in \mathbb{IS}(\boldsymbol{A}) \subseteq \mathsf{K}$ by Lemma 3.2(iii). But this contradicts the assumption that K omits $\boldsymbol{C}_2$. Hence we reached a contradiction, as desired. ⊠

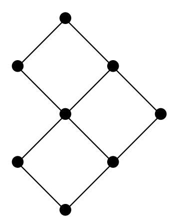
Corollary 6.11. *Let K be a variety of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$. The finite members of K belong to KG.*

Proof. Apply Theorem 6.3 to Lemmas 6.4, 6.8, and 6.10. ⊠

We rely on the following observation, which specializes[4] [9, Cor. 4.3.10]:

Lemma 6.12. *If $\boldsymbol{A} \in \mathsf{KG}$ is a nontrivial finite FSI algebra, then $\boldsymbol{A} = \boldsymbol{B}_1 + \dots + \boldsymbol{B}_n$ for some Heyting algebras $\boldsymbol{B}_1, \dots, \boldsymbol{B}_n \in \mathbb{H}(\boldsymbol{RN})$ such that $\boldsymbol{B}_1$ is the two-element Boolean algebra.*

Let $\boldsymbol{2}$ and $\boldsymbol{4}$ be the, respectively, the two and the four-element Boolean algebras. Moreover, let $\boldsymbol{D}$ be the Heyting algebra depicted below:



The next result describes the structure of varieties of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$:

Theorem 6.13. *Let K be a variety of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$. Then*

(i) *K is a locally finite subvariety of KG.*
(ii) *Every nontrivial finite FSI member of K has the form $\boldsymbol{B}_1 + \dots + \boldsymbol{B}_n$ for some Heyting algebras $\boldsymbol{B}_i$ such that $\boldsymbol{B}_1 \cong \boldsymbol{2}$, and if $n > 1$, then $\boldsymbol{B}_n \in \mathbb{I}\{\boldsymbol{2}, \boldsymbol{4}, \boldsymbol{D}\}$ and $\boldsymbol{B}_j \in \mathbb{I}\{\boldsymbol{2}, \boldsymbol{4}\}$ for all $1 < j < n$.*

Moreover, the above conditions hold for every primitive variety K of Heyting algebras.

Proof. (ii): Let $\boldsymbol{A}$ be a finite nontrivial FSI member of K. By Corollary 6.11 and Lemma 6.12 we obtain that $\boldsymbol{A} = \boldsymbol{B}_1 + \dots + \boldsymbol{B}_n$ for some finite nontrivial $\boldsymbol{B}_1, \dots, \boldsymbol{B}_n \in \mathbb{H}(\boldsymbol{RN})$ such that $\boldsymbol{B}_1 = \boldsymbol{2}$. We may assume without loss of generality that no $\boldsymbol{B}_i$ can be written as a sum with at least two nontrivial components.

Suppose with a view to contradiction that $n > 1$ and $\boldsymbol{B}_n \notin \mathbb{I}\{\boldsymbol{2}, \boldsymbol{4}, \boldsymbol{D}\}$. Then observe

$$\boldsymbol{2} + \boldsymbol{B}_n \in \mathbb{S}(\boldsymbol{2} + \boldsymbol{B}_2 + \dots + \boldsymbol{B}_n) = \mathbb{S}(\boldsymbol{B}_1 + \dots + \boldsymbol{B}_n) = \mathbb{S}(\boldsymbol{A}) \subseteq \mathsf{K}.$$

Now, recall that $\boldsymbol{B}_n$ is a finite nontrivial member of $\mathbb{H}(\boldsymbol{RN})$. To visualize $\boldsymbol{B}_n$, it is convenient to observe that the order-type of finite homomorphic images of $\boldsymbol{RN}$ is that of

[4] We point out that this proof is via Esakia duality

principal nontotal downsets of $\boldsymbol{RN}$. Since $\boldsymbol{B}_n$ cannot be written as a sum with at least two nontrivial components, this implies that the order type of $\boldsymbol{B}_n$ is that of $\downarrow^{\boldsymbol{RN}} a$ for some $a \in \boldsymbol{RN}$ that a is not prime. Together with the fact that $\boldsymbol{B}_n$ is nonisomorphic to $\mathbf{2}, \mathbf{4}, \boldsymbol{D}$ and inspecting the structure of $\boldsymbol{RN}$, this yields

$$\mathbb{S}(\mathbf{2} + \boldsymbol{B}_n) \cap \{\boldsymbol{C}_1, \boldsymbol{C}_2\} \neq \emptyset.$$

Thus either $\boldsymbol{C}_1$ or $\boldsymbol{C}_2$ belongs to K, which contradicts the assumption. We conclude that if $n > 1$, then $\boldsymbol{B}_n \in \mathbb{I}\{\mathbf{2}, \mathbf{4}, \boldsymbol{D}\}$.

Next consider $1 < j < n$ and suppose, with a view to contradiction, that $\boldsymbol{B}_j \notin \mathbb{I}\{\mathbf{2}, \mathbf{4}\}$. As $1 < j < n$, this yields

$$\mathbf{2} + \boldsymbol{B}_j + \mathbf{2} \in \mathbb{S}(\mathbf{2} + \boldsymbol{B}_2 + \cdots + \boldsymbol{B}_n) = \mathbb{S}(\boldsymbol{B}_1 + \cdots + \boldsymbol{B}_n) = \mathbb{S}(\boldsymbol{A}) \subseteq \mathsf{K}.$$

As above, the order type of $\boldsymbol{B}_j$ is that of $\downarrow^{\boldsymbol{RN}} a$ for some $a \in \boldsymbol{RN}$ that a is not prime. Together with the fact that $\boldsymbol{B}_j$ is nonisomorphic to $\mathbf{2}, \mathbf{4}$ and inspecting the structure of $\boldsymbol{RN}$, this yields $\boldsymbol{C}_2 \in \mathbb{S}(\mathbf{2} + \boldsymbol{B}_j + \mathbf{2})$, whence $\boldsymbol{C}_2 \in \mathsf{K}$ that is false. Hence we conclude that if $1 < j < n$, then $\boldsymbol{B}_j \in \mathbb{I}\{\mathbf{2}, \mathbf{4}\}$.

(i): Using the layer-structure given by condition (ii) and the fact that an n-generated Heyting algebra cannot be a sum of more than $2n + 1$ nontrivial algebras, it is not hard to see that for every $n \in \omega$ there are, up to isomorphism, only finitely many n-generated finite FSI algebras in K. By Theorem 4.3 we conclude that K is locally finite.

Now, Corollary 6.11 guarantees that the finite members of K belong to KG. As K is locally finite and, therefore, generated by its finite members, this implies that $\mathsf{K} \subseteq \mathsf{KG}$.

Finally, the fact that conditions (i) and (ii) hold for all primitive varieties of Heyting algebras is a consequence of Lemma 5.1. ⊠

7. Primitive varieties of Heyting algebras

For the present purpose, it is convenient to describe Esakia spaces dual to sums of Heyting algebras. Let $\mathbb{X} = \langle X; \leqslant^{\mathbb{X}} \rangle$ and $\mathbb{Y} = \langle Y; \leqslant^{\mathbb{Y}} \rangle$ be two posets (with disjoint universes). Their *sum* $\mathbb{X} + \mathbb{Y}$ is the poset with universe $X \cup Y$ and whose order relation $\leqslant$ is defined as follows for every $x, y \in X \cup Y$:

$$\begin{aligned} x \leqslant y \iff & \text{either } (x, y \in X \text{ and } x \leqslant^{\mathbb{X}} y) \text{ or } (x, y \in Y \text{ and } x \leqslant^{\mathbb{Y}} y) \\ & \text{or } (x \in X \text{ and } y \in Y). \end{aligned}$$

So, $\mathbb{X} + \mathbb{Y}$ is the poset obtained by placing $\mathbb{Y}$ *above* $\mathbb{X}$.

Now, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two Esakia spaces (with disjoint universes). The *sum* $\boldsymbol{X} + \boldsymbol{Y}$ is the Esakia space, whose underlying poset is $\langle X; \leqslant^{\boldsymbol{X}} \rangle + \langle Y; \leqslant^{\boldsymbol{Y}} \rangle$, endowed with the topology consisting of the sets $U \subseteq X \cup Y$ such that $U \cap X$ and $U \cap Y$ are open respectively in $\boldsymbol{X}$ and $\boldsymbol{Y}$. The following result is [9, Thm. 4.1.16] and [48, Lem. 5.1].

Lemma 7.1. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are Heyting algebras, then the Esakia spaces $(\boldsymbol{A} + \boldsymbol{B})_*$ and $\boldsymbol{A}_* + \boldsymbol{B}_*$ are isomorphic.*

Moreover, we shall recall a basic concept from universal algebra. Let K be a variety. A nontrivial algebra $\boldsymbol{A} \in \mathsf{K}$ is said to be a *splitting algebra* in K [45, 59] if there exists the largest subvariety V of K omitting $\boldsymbol{A}$. In this case, V is always axiomatized relative to K by a single equation, sometimes called the *splitting equation*. In the realm of Heyting algebras, this phenomenon was first discovered by Jankov [38][5], who associated a special formula $\chi(\boldsymbol{A})$—now known as *the Jankov formula*—with every finite nontrivial FSI (equiv. finite subdirectly irreducible) Heyting algebra $\boldsymbol{A}$, validating the following result:

Theorem 7.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras such that $\boldsymbol{A}$ is finite, nontrivial, and FSI.*

$$\boldsymbol{B} \vDash \chi(\boldsymbol{A}) \approx 1 \Longleftrightarrow \boldsymbol{A} \notin \mathbb{SH}(\boldsymbol{B}).$$

Moreover, $\chi(\boldsymbol{A}) \approx 1$ axiomatizes the largest variety of Heyting algebras omitting $\boldsymbol{A}$.

Bearing this in mind, let Citk be the largest variety of Heyting algebras omitting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$, i.e., the variety of Heyting algebras axiomatized by the equations

$$\chi(\boldsymbol{C}_i) \approx 1, \text{ for all } i = 1, \dots, 5.$$

Citikin's Theorem [16] can be phrased in purely algebraic terms as follows:

Theorem 7.3. *The following conditions are equivalent for a variety K of Heyting algebras:*

(i) K *is primitive;*
(ii) K *is a subvariety of* Citk*;*
(iii) K *omits the algebras $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$;*
(iv) *Every nontrivial finite FSI member of K has the form $\boldsymbol{B}_1 + \dots + \boldsymbol{B}_n$ for some Heyting algebras $\boldsymbol{B}_i$ such that $\boldsymbol{B}_1 \cong \mathbf{2}$, and $\boldsymbol{B}_j \in \mathbb{I}\{\mathbf{2}, \mathbf{4}\}$ for all $1 < j < n$, and, if $n > 1$, then $\boldsymbol{B}_n \in \mathbb{I}\{\mathbf{2}, \mathbf{4}, \boldsymbol{D}\}$.*

Consequently, Citk *is the largest primitive variety of Heyting algebras.*

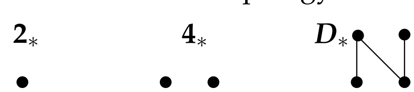
Proof. Parts (i)⇒(iii) and (iii)⇒(iv) follow, respectively, from Lemma 5.1 and Theorem 6.13. Moreover, conditions (ii) and (iii) are equivalent by definition of Citk.

(iv)⇒(i): The proof of Theorem 6.13(ii) shows that K is locally finite. Hence to establish that K is primitive it suffices, by Theorem 2.2, to show that the finite nontrivial FSI members of K are weakly projective in K.

Consider a finite nontrivial FSI algebra $\boldsymbol{A} \in \mathsf{K}$. Let also $\boldsymbol{E} \in \mathsf{K}$ be such that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{E})$. Clearly, there is a surjective homomorphism $f \colon \boldsymbol{E} \to \boldsymbol{A}$. Since $\boldsymbol{A}$ is finite, there is a finitely generated subalgebra $\boldsymbol{B} \leqslant \boldsymbol{E}$ such that $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$. Therefore, to conclude that $\boldsymbol{A}$ is weakly projective in K, it will be enough to show that $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$ and, therefore, $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{E})$. Instead of proving directly that $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$, we will establish the existence of a surjective Esakia morphism $g \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ (see Lemma 3.2(iii)).

To this end, observe that $\boldsymbol{B}$ is finite, since K is locally finite and $\boldsymbol{B}$ finitely generated. Thus the Esakia space $\boldsymbol{B}_*$ is finite and, therefore, endowed with the discrete topology. Moreover, in the light of Lemma 3.2(ii) and $\boldsymbol{A} \in \mathbb{H}(\boldsymbol{B})$, we can identify $\boldsymbol{A}_*$ with an E-subspace of $\boldsymbol{B}_*$. From now on we work under this identification.

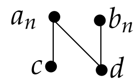
Observe that the Esakia spaces dual to the algebras $\mathbf{2}, \mathbf{4}$, and $\boldsymbol{D}$ are respectively the following posets endowed with the discrete topology:



Thus, by the assumption (i.e., condition (iv) of Theorem 7.3) and Lemma 7.1, the Esakia space $\boldsymbol{A}_*$ is isomorphic to $\boldsymbol{X}_1 + \dots + \boldsymbol{X}_n$ for some Esakia spaces $\boldsymbol{X}_i$ such that $\boldsymbol{X}_1 = \mathbf{2}_*$, and $\boldsymbol{X}_j \in \{\mathbf{2}_*, \mathbf{4}_*\}$ for all $1 < j < n$, and, if $n > 1$, then $\boldsymbol{X}_n \in \{\mathbf{2}_*, \mathbf{4}_*, \boldsymbol{D}_*\}$.

[5] His approach was subsequently generalized to arbitrary varieties with EDPC in [12, Cor. 3.2] (see also [22, Cor. 3.8] for a similar result).

Let us label the elements of $\boldsymbol{A}_*$ (equiv. of $\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_n$). Consider one of the various $\boldsymbol{X}_i$. If $\boldsymbol{X}_i = \mathbf{2}_*$ (resp. $\boldsymbol{X}_i = \mathbf{4}_*$), then we denote the unique element (resp. the two elements) of $\boldsymbol{X}_i$ by a_i (resp. a_i and b_i). Then suppose that $\boldsymbol{X}_i = \boldsymbol{D}_*$. In this case, we have necessarily $i = n$ and denote the elements of $\boldsymbol{X}_i$ as follows:



So far we labeled all elements of $\boldsymbol{A}_*$. This will allow us to define a surjective Esakia morphism from $\boldsymbol{B}_*$ to $\boldsymbol{A}_*$. To this end, recall that $\boldsymbol{A}_*$ is an E-subspace of $\boldsymbol{B}_*$. For every element $x \in \boldsymbol{B}_*$, let $\boldsymbol{m}(x)$ be the set of minimal elements in the upset $\boldsymbol{A}_* \cap {\uparrow}^{\boldsymbol{B}_*} x$.

Fact 7.4. For every $x \in \boldsymbol{B}_*$ either $\boldsymbol{m}(x)$ is empty, or a singleton, or one of the following sets: $\{c, d\}$, $\{b_n, c\}$, or $\{a_i, b_i\}$ for some $i \in \omega$.

Proof. To prove this, consider $x \in \boldsymbol{B}_*$ such that $\boldsymbol{m}(x)$ has at least two distinct elements y and z. Suppose, with a view to contradiction, that $\boldsymbol{m}(x) \neq \{y, z\}$. Then there is an element $v \in \boldsymbol{m}(x) \smallsetminus \{y, z\}$. As v, y, and z are minimal in $\boldsymbol{A}_* \cap {\uparrow}^{\boldsymbol{B}_*} x$, they must be incomparable in $\boldsymbol{A}_*$. But this contradicts the fact that $\boldsymbol{A}_*$ is a rooted poset of width $\leqslant 2$. Hence we conclude that $\boldsymbol{m}(x) = \{y, z\}$. Bearing in mind that $\boldsymbol{A}_* = \boldsymbol{X}_1 + \cdots + \boldsymbol{X}_n$ and that y and x are incomparable, this easily implies that $\boldsymbol{m}(x)$ is one of the sets $\{c, d\}, \{b_n, c\}$, and $\{a_i, b_i\}$ for some $i \in \omega$. $\boxtimes$

Recall that $\boldsymbol{A}_*$ is an E-subspace of $\boldsymbol{B}_*$. Then consider the map $g \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ defined as follows for every $x \in \boldsymbol{B}_*$:

$$g(x) := \begin{cases} a_n & \text{if } \boldsymbol{m}(x) = \emptyset \\ y & \text{if } \boldsymbol{m}(x) = \{y\} \text{ for some } y \in \boldsymbol{A}_* \\ a_{n-1} & \text{if } \boldsymbol{m}(x) \in \{\{c, d\}, \{b_n, c\}\} \\ a_{i-1} & \text{if } \boldsymbol{m}(x) = \{a_i, b_i\} \text{ for some } i \in \omega \text{ such that } \boldsymbol{X}_i \neq \boldsymbol{D}_* \\ d & \text{if } \boldsymbol{m}(x) = \{a_n, b_n\} \text{ and } \boldsymbol{X}_n = \boldsymbol{D}_*. \end{cases}$$

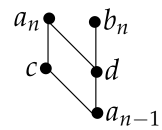
From Fact 7.4 it follows that g is well defined. Hence, to conclude the proof, it only remains to show that $g \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ is a surjective Esakia morphism.

First observe that g is surjective, since $g(x) = x$ for every $x \in \boldsymbol{A}_*$, and that it is order-preserving. Moreover, g is trivially continuous, as the spaces $\boldsymbol{B}_*$ and $\boldsymbol{A}_*$ are discrete. Consequently, it only remains to show that g satisfies condition (1) of the definition of an Esakia morphism. To this end, consider $x \in \boldsymbol{B}_*$ and $y \in \boldsymbol{A}_*$ such that $g(x) \leqslant^{\boldsymbol{A}_*} y$. We need to find $z \in \boldsymbol{B}_*$ such that $x \leqslant^{\boldsymbol{B}_*} z$ and $g(z) = y$. We have two cases:

(i) ${\uparrow}^{\boldsymbol{A}_*} g(x) \subseteq \{g(x)\} \cup {\uparrow}^{\boldsymbol{A}_*} \boldsymbol{m}(x)$; or
(ii) ${\uparrow}^{\boldsymbol{A}_*} g(x) \nsubseteq \{g(x)\} \cup {\uparrow}^{\boldsymbol{A}_*} \boldsymbol{m}(x)$.

(i): In this case, $y \in \{g(x)\} \cup {\uparrow}^{\boldsymbol{A}_*} \boldsymbol{m}(x)$, since $g(x) \leqslant^{\boldsymbol{A}_*} y$. If $y = g(x)$, then, by taking $z := x$, we are done. Now suppose that $y \in {\uparrow}^{\boldsymbol{A}_*} \boldsymbol{m}(x)$, i.e., that there is $v \in \boldsymbol{m}(x)$ such that $v \leqslant^{\boldsymbol{A}_*} y$. In particular, this means that $x \leqslant^{\boldsymbol{B}_*} v \leqslant^{\boldsymbol{B}_*} y$, whence $x \leqslant^{\boldsymbol{B}_*} y$. Since $g(y) = y$, by taking $z := y$, we are done.

(ii): By interrogating the definition of g, it is not hard to see that ${\uparrow}^{\boldsymbol{A}_*} g(x) \nsubseteq \{g(x)\} \cup {\uparrow}^{\boldsymbol{A}_*} \boldsymbol{m}(x)$ implies $\boldsymbol{m}(x) = \{b_n, c\}$. As a consequence, $\boldsymbol{X}_n = \boldsymbol{D}_*$ and $g(x) = a_{n-1}$. In particular, the upset $\uparrow^{A_*} g(x)$ is depicted below:



Now, if $y \in \{g(x)\} \cup \uparrow^{A_*} \boldsymbol{m}(x)$, we repeat the the argument described for case (i), and we are done. Then suppose that $y \notin \{g(x)\} \cup \uparrow^{A_*} \boldsymbol{m}(x) = \{a_{n-1}, a_n, b_n, c\}$. Bearing in mind that $g(x) \leqslant^{A_*} y$, and that the upset $\uparrow^{A_*} g(x)$ is depicted above, this implies $g(x) = d$. Thus, to conclude the proof, it suffices to find an element $z \in \boldsymbol{B}_*$ such that $x \leqslant^{\boldsymbol{B}_*} z$ and $g(z) = d$.

To this end, recall that $\uparrow^{B_*} x$ is the dual of a finite nontrivial FSI member of K by Lemma 3.2(i,ii). By the assumption (i.e., condition (iv) of Theorem 7.3) and Lemma 7.1, the Esakia space $\uparrow^{B_*} x$ is isomorphic to $\boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_m$ for some Esakia spaces $\boldsymbol{Y}_i$ such that $\boldsymbol{Y}_1 = \mathbf{2}_*$, and $\boldsymbol{Y}_j \in \{\mathbf{2}_*, \mathbf{4}_*\}$ for all $1 < j < m$, and, if $m > 1$, then $\boldsymbol{Y}_m \in \{\mathbf{2}_*, \mathbf{4}_*, \boldsymbol{D}_*\}$. Now, as $b_n, c \in \boldsymbol{m}(x)$, we get $x \leqslant^{B_*} a_n, b_n, c$. Consequently, $\boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_m$ contains an element b_n incomprable with two elements $c < a_n$. This implies that $\boldsymbol{Y}_m$ is isomorphic to $\boldsymbol{D}$ under a map that is the identity on a_n, b_n, c. Let $z \in \boldsymbol{Y}_m$ be the element playing the role of d. We have $\uparrow^{B_*} z = \{z, a_n, b_n\}$, where $z < a_n, b_n$ and $\boldsymbol{X}_n = \boldsymbol{D}$. Bearing this in mind, it is easy to see that $g(z) = d$. Thus $x \leqslant^{B_*} z$ and $g(z) = d$, as desired. ⊠

Letting **Citk** be the intermediate logic axiomatized by the formulas $\chi(\boldsymbol{C}_1), \dots, \chi(\boldsymbol{C}_5)$, we obtain the classical formulation of Citkin's Theorem:

Corollary 7.5. *An intermediate logic is hereditarily structurally complete if and only if it extends* **Citk**. *Consequently,* **Citk** *is the least hereditarily structurally complete intermediate logic.*

Proof. As explained in Section 2, an intermediate logic is hereditarily structurally complete if and only if the variety of Heyting algebras naturally associated with it is primitive. Thus the result follows from Theorem 7.3. ⊠

8. Primitive varieties of Brouwerian algebras

It is well known that the $\langle \wedge, \vee, \to \rangle$-fragment of **IPC**, here denoted by **IPC**$^+$, is algebraized by the variety of *Brouwerian algebras*, i.e., $\langle \wedge, \vee, \to \rangle$-subreducts of Heyting algebras. As a consequence of the algebraization phenomenon, the lattice of varieties of Brouwerian algebras is dually isomorphic to that of *positive logics*, i.e., axiomatic extensions of **IPC**$^+$. Moreover, a positive logic is hereditarily structurally complete if and only if the variety of Brouwerian algebras associated with it is primitive.

As structural completeness and its variants are very sensitive to (even small) changes of signature, it was natural to wonder whether Citkin's description of hereditarily structurally complete intermediate logics could be extended to positive logics. Recently, a positive solution to this question was supplied by Citkin himself in [19]. However, as we will show below, the results and techniques of the previous sections of this paper yield a very short alternative proof of this result.

Given a Brouwerian algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}_\bot$ the unique Heyting algebra obtained by adding a new bottom element $\bot$ to $\boldsymbol{A}$. As the characterization of FSI algebras given in Lemma 3.2(i) holds for Brouwerian algebras as well, $\boldsymbol{A}$ is FSI if and only if so is $\boldsymbol{A}_\bot$. Given a class of Brouwerian algebras K, define

$$\mathsf{K}_\bot \coloneqq \{\boldsymbol{A}_\bot \colon \boldsymbol{A} \in \mathsf{K}\}.$$

Observe that for every class K of Brouwerian algebras,

$$\mathbb{H}(\mathsf{K}_\bot) = \mathbb{H}(\mathsf{K})_\bot \text{ and } \mathbb{S}(\mathsf{K}_\bot) = \mathbb{S}(\mathsf{K})_\bot \tag{14}$$

where the class operators $\mathbb{H}$ and $\mathbb{S}$ are computed in the language of Heyting algebras for $\mathbb{H}(\mathsf{K}_\bot)$ and $\mathbb{S}(\mathsf{K}_\bot)$, and in the language of Brouwerian algebras for $\mathbb{H}(\mathsf{K})$ and $\mathbb{S}(\mathsf{K})$. Finally, given a Heyting algebra $\boldsymbol{A}$, we denote by $\boldsymbol{A}^+$ its $\langle \wedge, \vee, \to \rangle$-reduct.

Lemma 8.1. *Let K be a variety of Brouwerian algebras. If K omits $\boldsymbol{C}_1^+$ and $\boldsymbol{C}_3^+$, then $\mathbb{V}(\mathsf{K}_\bot)$ omits $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$.*

Proof. Suppose the contrary, with a view to contradiction. Then there is $i = 1, \dots, 5$ such that $\boldsymbol{C}_i \in \mathbb{V}(\mathsf{K}_\bot)$. By Theorem 7.2 the variety $\mathbb{V}(\mathsf{K}_\bot)$ does not validate $\chi(\boldsymbol{C}_i) \approx 1$. Thus there is $\boldsymbol{A} \in \mathsf{K}$ such that $\boldsymbol{A}_\bot$ rejects $\chi(\boldsymbol{C}_i) \approx 1$. By Theorem 7.2 and (14) this implies

$$\boldsymbol{C}_i \in \mathbb{SH}(\boldsymbol{A}_\bot) = (\mathbb{SH}(\boldsymbol{A}))_\bot.$$

As a consequence, $\boldsymbol{C}_i$ has the form $\boldsymbol{B}_\bot$ for some Brouwerian algebra $\boldsymbol{B}$ such that

$$\boldsymbol{B} \in \mathbb{SH}(\boldsymbol{A}). \tag{15}$$

As $\boldsymbol{C}_i = \boldsymbol{B}_\bot$, the bottom element of $\boldsymbol{C}_i$ is meet-irreducible. By inspecting $\boldsymbol{C}_1, \dots, \boldsymbol{C}_5$, this guarantees that $\boldsymbol{C}_i \in \{\boldsymbol{C}_2, \boldsymbol{C}_4\}$. Together with $\boldsymbol{B}_\bot = \boldsymbol{C}_i$, this implies $\boldsymbol{B} \in \{\boldsymbol{C}_1^+, \boldsymbol{C}_3^+\}$. By (15) we conclude that

$$\text{either } \boldsymbol{C}_1^+ \in \mathbb{SH}(\boldsymbol{A}) \subseteq \mathsf{K} \text{ or } \boldsymbol{C}_3^+ \in \mathbb{SH}(\boldsymbol{A}) \subseteq \mathsf{K}.$$

But this contradicts the fact that K omits $\boldsymbol{C}_1^+$ and $\boldsymbol{C}_3^+$. ⊠

As shown by Jankov [38], Theorem 7.2 generalizes to the case of Brouwerian algebras. More precisely, every finite nontrivial FSI (equiv. finite subdirectly irreducible) Brouwerian algebra $\boldsymbol{A}$ can be associated with a formula $\chi(\boldsymbol{A})^+$ such that the largest variety of Brouwerian algebras omitting $\boldsymbol{A}$ exists and is axiomatized by $\chi(\boldsymbol{A})^+ \approx 1$. Bearing this in mind, let Citk^+ be the largest variety of Brouwerian algebras omitting $\boldsymbol{C}_1^+$ and $\boldsymbol{C}_3^+$, i.e., the variety of Brouwerian algebras axiomatized by the equations

$$\chi(\boldsymbol{C}_1^+)^+ \approx 1 \text{ and } \chi(\boldsymbol{C}_3^+)^+ \approx 1.$$

Citkin's description of hereditarily structurally complete positive logics can be phrased algebraically as follows:

Theorem 8.2. *The following conditions are equivalent for a variety K of Brouwerian algebras:*

(i) *K is primitive;*
(ii) *K is a subvariety of Citk^+;*
(iii) *K omits the algebras $\boldsymbol{C}_1^+$ and $\boldsymbol{C}_3^+$;*
(iv) *Every nontrivial finite FSI member of K has the form $\boldsymbol{B}_1 + \dots + \boldsymbol{B}_n$ for some Brouwerian algebras $\boldsymbol{B}_i$ such that $\boldsymbol{B}_1 \cong \boldsymbol{2}^+$, and $\boldsymbol{B}_j \in \mathbb{I}\{\boldsymbol{2}^+, \boldsymbol{4}^+\}$ for all $j > 1$.*

Consequently, Citk^+ is the largest primitive variety of Brouwerian algebras.

Proof. Observe that conditions (ii) and (iii) are equivalent by definition of Citk^+. Moreover, the proof of (i)⇒(iii) is analogous to that of Lemma 5.1.

(iii)⇒(iv): Let $\boldsymbol{A}$ be a nontrivial FSI member of K. Then $\boldsymbol{A}_\bot$ is a finite nontrivial FSI member of $\mathbb{V}(\mathsf{K}_\bot)$. From Lemma 8.1 and Theorem 7.3 it follows that $\boldsymbol{A}_\bot = \boldsymbol{B}_1 + \cdots + \boldsymbol{B}_n$ for some Heyting algebras $\boldsymbol{B}_i$ such that $\boldsymbol{B}_1 \cong \boldsymbol{2}$, and $\boldsymbol{B}_j \in \mathbb{I}\{\boldsymbol{2}, \boldsymbol{4}\}$ for all $1 < j < n$, and, if $n > 1$, then $\boldsymbol{B}_n \in \mathbb{I}\{\boldsymbol{2}, \boldsymbol{4}, \boldsymbol{D}\}$. By construction of $\boldsymbol{A}_\bot$, its bottom element is meet-irreducible. Consequently, necessarily $\boldsymbol{B}_n \cong \boldsymbol{2}$. Also, as $\boldsymbol{A}$ is nontrivial, $\boldsymbol{A}_\bot$ has at least three elements, whence $n > 1$. Thus

$$\boldsymbol{A}_\bot \cong \boldsymbol{2} + \boldsymbol{B}_2 + \cdots + \boldsymbol{B}_{n-1} + \boldsymbol{2}.$$

As a consequence,

$$\boldsymbol{A}_\bot \cong \boldsymbol{2}^+ + \boldsymbol{B}_2^+ + \cdots + \boldsymbol{B}_{n-1}^+$$

where each $\boldsymbol{B}_i^+$ is isomorphic either to $\boldsymbol{2}^+$ or to $\boldsymbol{4}^+$.

(iv)⇒(i): First observe that K omits $\boldsymbol{C}_1^+$ and $\boldsymbol{C}_3^+$. Therefore, Lemma 8.1 and Theorem 6.13(i) imply that $\mathbb{V}(\mathsf{K}_\bot)$ is locally finite. This, in turn, guarantees that K is also locally finite. By Theorem 2.2 we conclude that, in order to prove that K is primitive, it suffices to show that its finite nontrivial FSI members are weakly projective in K.

Accordingly, consider a finite nontrivial FSI member $\boldsymbol{A}$ of K. Then let $\boldsymbol{B} \in \mathsf{K}$ and $f\colon \boldsymbol{B} \to \boldsymbol{A}$ be a surjective homomorphism. Observe that the unique map $f_\bot\colon \boldsymbol{B}_\bot \to \boldsymbol{A}_\bot$ which extends f by $f_\bot(\bot) := \bot$ is a homomorphism of Heyting algebras. By assumption, $\boldsymbol{A}$ is a finite linear sum of copies of $\boldsymbol{2}$ and $\boldsymbol{4}$, whence the same holds for $\boldsymbol{A}_\bot$. By [1, Thm. 4.10] this implies that $\boldsymbol{A}$ is projective in the standard sense. Therefore, as $f_\bot$ is surjective, there is an embedding $g\colon \boldsymbol{A}_\bot \to \boldsymbol{B}_\bot$. Observe that g restricts to an embedding $g\colon \boldsymbol{A} \to \boldsymbol{B}$ of Brouwerian algebras. Consequently, $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B})$. Hence we conclude that $\boldsymbol{A}$ is weakly projective in K. ⊠

Letting $\mathbf{Citk}^+$ be the positive logic axiomatized by $\chi(\boldsymbol{C}_1^+)$ and $\chi(\boldsymbol{C}_3^+)$, we get:

Corollary 8.3. *A positive logic is hereditarily structurally complete if and only if it extends* $\mathbf{Citk}^+$*. Consequently,* $\mathbf{Citk}^+$ *is the least hereditarily structurally complete positive logic.*

9. Properties of primitive varieties

Primitive varieties of Heyting and Brouwerian algebras have a number of interesting properties, as we proceed to explain. Recall that a variety is said to be *finitely based* if it can be axiomatized by finitely many equations.

Theorem 9.1. *The following conditions hold:*

(i) *Primitive varieties of Heyting (resp. Brouwerian) algebras are locally finite.*
(ii) *Primitive varieties of Heyting (resp. Brouwerian) algebras are finitely based.*
(iii) *There are only countably many primitive varieties of Heyting (resp. Brouwerian) algebras.*

We conclude the paper by sketching a proof of the above result.[6]

[6] The reason why in this case we opted for providing a proof sketch only is that, for the case of Brouwerian algebras, a detailed proof of Theorem 9.1 is given in [19] and there is no reason for repeating it here. Moreover, the version of Theorem 9.1 for Heyting algebras is proved by a simple modification of the Brouwerian case.

Proof sketch. We consider the case of Heyting algebras only, as that of Brouwerian algebras is analogous. First observe that primitive varieties of Heyting algebras are locally finite by Theorem 7.3 and the last part of Theorem 6.13. This establishes condition (i). Moreover, condition (iii) is an immediate consequence of (ii). Thus, to conclude the proof, it suffices to establish (ii).

We shall provide a proof sketch only. To this end, recall from Theorem 7.3 that Citk is the largest primitive variety of Heyting algebras. Therefore, to conclude the proof, it only remains to show that all subvarieties of Citk are finitely based. Observe that Citk is finitely based by definition. Moreover, it is locally finite by condition (i). Thus, by general arguments related to Jankov formulas, e.g., [9, Thm. 3.4.14] and [15, Ch. 9], one can reduce the problem of proving that all subvarieties of Citk are finitely based to that of showing that the poset $\mathbb{O}\mathrm{rd}(\mathsf{Citk})$ of finite nontrivial FSI members of Citk ordered under the the relation

$$\boldsymbol{A} \preccurlyeq \boldsymbol{B} \iff \boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B})$$

has no infinite antichain. Recall from Theorem 2.2 that all nontrivial FSI members of Citk are weakly projective in Citk. As a consequence for every $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{O}\mathrm{rd}(\mathsf{Citk})$,

$$\boldsymbol{A} \preccurlyeq \boldsymbol{B} \iff \boldsymbol{A} \in \mathbb{HS}(\boldsymbol{B}) \iff \boldsymbol{A} \in \mathbb{S}(\boldsymbol{B}). \tag{16}$$

Thus, to conclude the proof, it suffices to show that there is no infinite antichain in the poset of finite nontrivial FSI members of Citk ordered under the relation

$$\boldsymbol{A} \preccurlyeq \boldsymbol{B} \iff \boldsymbol{A} \in \mathbb{S}(\boldsymbol{B}).$$

This can be shown by a combinatorial argument similar to the one detailed in [19, Sec. 7] for the case of Brouwerian algebras, using the description of finite nontrivial FSI members of Citk given in Theorem 7.3. ⊠

Thus we arrive at the following corollary.

Corollary 9.2. *Hereditarily structurally complete intermediate logics (resp. positive logics) are locally tabular and finitely axiomatizable. Moreover, there are only countably many such logics.*

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