A simple propositional calculus for compact Hausdorff spaces

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Abstract

In [5] it was shown that subordinations on a Boolean algebra correspond to strict implications. In this paper we study the variety $SIA$ of strict implication algebras, and the corresponding strict implication calculus $SIC$, which is a propositional modal logic with one binary modality of strict implication. We also study the symmetric strict implication calculus $S^2IC$, which is an extension of $SIC$, and prove that $S^2IC$ is strongly sound and complete with respect to de Vries algebras. By de Vries duality, this yields completeness of $S^2IC$ with respect to compact Hausdorff spaces. Since some of the defining axioms of de Vries algebras are $\Pi_2$-sentences, we develop the corresponding theory of non-standard rules, which we term $\Pi_2$-rules. We study the resulting inductive elementary classes of algebras, and give a general criterion of admissibility for $\Pi_2$-rules. We also compare our approach to the existing approaches in the literature that are related to our work[1].

1 Introduction

A subordination on a Boolean algebra $B$ is a binary relation $\prec$ on $B$ satisfying certain conditions (see Definition 2.1). Subordinations were introduced in [5]. They are in one-to-one correspondence with quasi-modal operators of [8] and pre-contact relations of [12]. Subordinations can be modeled dually by closed relations on Stone spaces. This leads to a duality between the category $Sub$ of subordination algebras and the category $StR$ of pairs $(X, R)$ where $X$ is a Stone space and $R$ is a closed relation on $X$ (see [5, Sec. 2.1]). Further conditions on $\prec$ characterize when the relation $R$ is reflexive or symmetric. This yields the subcategories $RSub$ and $Con$ of $Sub$ consisting of reflexive subordination algebras and contact algebras, respectively. As the name suggests, reflexive subordinations dually correspond to reflexive closed relations, while contact relations to reflexive and symmetric closed relations.

Compingent algebras, introduced by de Vries [11], are obtained by adding two additional conditions to the definition of contact algebras. As was shown in [5, Lem. 6.3], they are dually characterized by irreducible equivalence relations (the definition is given in Section 2). Thus, the category $Com$ of compingent algebras is dually equivalent to the subcategory of $StR$ consisting of the pairs $(X, R)$ where $R$ is an irreducible equivalence relation. A de Vries

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1The results presented in this paper were first reported in [21] (see also [4]).
algebra is a complete compingent algebra. Since complete Boolean algebras dually correspond to extremally disconnected Stone spaces, we conclude that the category $\text{DeV}$ of de Vries algebras is dually equivalent to the subcategory of $\text{StR}$ consisting of the pairs $(X, R)$ where the Stone space $X$ is extremally disconnected and $R$ is an irreducible equivalence relation. Such pairs were called Gleason spaces in [5] since they are closely related to Gleason covers of compact Hausdorff spaces. A key result of [5] Thm. 6.13] is that this close correspondence between Gleason spaces and Gleason covers yields that the category $\text{Gle}$ of Gleason spaces is equivalent to the category $\text{KHaus}$ of compact Hausdorff spaces. Since $\text{Gle}$ is dually equivalent to $\text{DeV}$, we arrive at de Vries duality: $\text{KHaus}$ is dually equivalent to $\text{DeV}$.

It was pointed out in [5] Sec. 3] that subordinations on a Boolean algebra $B$ can alternatively be described by binary operations on $B$ called strict implications. In this paper we study the resulting variety of strict implication algebras. The study simplifies considerably if we work with the strict implications that correspond to reflexive subordinations. In Section 3 we prove that the resulting variety $\text{SIA}$ of strict implication algebras is a discriminator variety, and we give its axiomatization. We also prove that $\text{SIA}$ is a locally finite variety. In Section 4 we develop the corresponding strict implication calculus $\text{SIC}$, which is a modal logic with one binary modality that corresponds to the strict implication. Section 5 is devoted to the symmetric strict implication calculus $\text{S}_{2}\text{IC}$ which is an extension of $\text{SIC}$. The corresponding variety $\text{S}_{2}\text{IA}$ is the subvariety of $\text{SIA}$ generated by the strict implication algebras that correspond to contact algebras. One of our main results is that $\text{S}_{2}\text{IA}$ is generated by the strict implication algebras that correspond to de Vries algebras. This yields that $\text{S}_{2}\text{IC}$ is complete with respect to $\text{DeV}$, which coupled with de Vries duality, yields that $\text{S}_{2}\text{IC}$ is the logic of compact Hausdorff spaces.

Our approach is closely related to that of Balbiani et al. [1], which along with [5] inspired the current paper. Balbiani et al. develop two-sorted logical calculi for region-based theories of space. Our calculus is simpler as we work with one-sorted propositional modal logic with one binary modality. In Section 7 we show how to translate the language of [1] into our language.

Two of the defining axioms of de Vries algebras are universal-existential statements or $\Pi_{2}$-statements. They can be expressed in our language by use of non-standard rules, which we call $\Pi_{2}$-rules. That $\text{S}_{2}\text{IC}$ is complete with respect to de Vries algebras shows that these $\Pi_{2}$-rules are admissible in $\text{S}_{2}\text{IC}$. This is again closely related to Balbiani et al. [1] who use similar non-standard rules in the context of region-based theories of space and prove their admissibility. In Section 6 we develop the theory of $\Pi_{2}$-rules, show that they define inductive elementary subclasses of $\text{RSub}$, and that every derivation system axiomatized by $\Pi_{2}$-rules is strongly sound and complete with respect to the subclass of $\text{RSub}$ it defines. We also give a criterion of when a $\Pi_{2}$-rule is admissible. We prove that being a zero-dimensional de Vries algebra is definable by a $\Pi_{2}$-rule, and that this rule is admissible in $\text{S}_{2}\text{IC}$. As a consequence, we obtain that $\text{S}_{2}\text{IC}$ is complete with respect to zero-dimensional de Vries algebra, and hence with respect to zero-dimensional compact Hausdorff spaces, also known as Stone spaces. On the other side of the spectrum from zero-dimensional spaces are connected spaces. We define the connected symmetric strict implication calculus $\text{CS}_{2}\text{IC}$ by adding one axiom to $\text{S}_{2}\text{IC}$, and prove that $\text{CS}_{2}\text{IC}$ is complete with respect to connected de Vries algebras, and consequently with respect to connected compact Hausdorff spaces.
2 Subordinations, contact algebras, and de Vries algebras

In this section we recall the definitions of subordination, contact algebra, compingent algebra, and de Vries algebra, as well as the duality theory for these algebras. We also connect the duality theory for de Vries algebras to de Vries duality for compact Hausdorff spaces via Gleason spaces.

Definition 2.1. ([5])

(1) A subordination on a Boolean algebra \( B \) is a binary relation \( \prec \) satisfying:

(S1) \( 0 \prec 0 \) and \( 1 \prec 1 \);
(S2) \( a \prec b, c \) implies \( a \prec b \land c \);
(S3) \( a, b \prec c \) implies \( a \lor b \prec c \);
(S4) \( a \leq b \prec c \leq d \) implies \( a \prec d \).

(2) We call \( (B, \prec) \) a subordination algebra, and let \( \text{Sub} \) be the class of all subordination algebras.

By Stone duality, Boolean algebras correspond to zero-dimensional compact Hausdorff spaces, known as Stone spaces. Given a Boolean algebra \( B \), its dual Stone space is the space \( X \) of ultrafilters of \( B \), the topology on which is given by the basis \( \{ \beta(a) \mid a \in B \} \), where \( \beta(a) = \{ x \in X \mid a \in x \} \). Then \( \beta \) is an isomorphism from \( B \) to the Boolean algebra \( \text{Clop}(X) \) of clopen subsets of \( X \).

We say that a binary relation \( R \) on a Stone space \( X \) is closed if \( R \) is a closed subset of \( X \times X \) in the product topology. Let \( \text{StR} \) be the class of pairs \( (X, R) \) where \( X \) is a Stone space and \( R \) is a closed relation on \( X \). There is a one-to-one correspondence between \( \text{Sub} \) and \( \text{StR} \), which extends to a categorical duality; see [5, Sec. 2.1]. This one-to-one correspondence can be obtained as follows. As usual, for a binary relation \( R \) on a set \( X \) and \( S \subseteq X \), we write

\[ R[S] := \{ x \in X \mid sRx \text{ for some } s \in S \}. \]

Let \( B \) be a Boolean algebra and \( X \) the Stone space of \( B \). If \( R \) is a closed relation on \( X \), then the binary relation \( \prec \) defined by \( a \prec b \iff R[\beta(a)] \subseteq \beta(b) \) is a subordination on \( B \). Conversely, let \( \prec \) be a subordination on \( B \). For \( S \subseteq B \), write

\[ \overset{\uparrow}{S} := \{ a \in B \mid s \prec a \text{ for some } s \in S \}, \]

and define the binary relation \( R \) on \( X \) by \( xRy \iff \overset{\uparrow}{x} \subseteq y \). Then \( R \) is a closed relation on \( x \), and this correspondence is one-to-one.

We next consider the following additional properties of \( \prec \):

(S5) \( a \prec b \) implies \( a \leq b \);
(S6) \( a \prec b \) implies \( \neg b \prec \neg a \);
(S7) \( a \prec b \) implies there is \( c \in B \) with \( a \prec c \prec b \);
(S8) \( a \neq 0 \) implies there is \( b \neq 0 \) with \( b \prec a \).

The next lemma gives a dual characterization of (S5)-(S7).
Lemma 2.2 ([12]). Let $B$ be a Boolean algebra, $X$ the Stone space of $B$, $\prec$ a subordination on $B$, and $R$ the corresponding closed relation on $X$.

1. $(B, \prec)$ satisfies (S5) iff $R$ is reflexive.
2. $(B, \prec)$ satisfies (S6) iff $R$ is symmetric.
3. $(B, \prec)$ satisfies (S7) iff $R$ is transitive.

Definition 2.3. Let $(B, \prec)$ be a subordination algebra.

1. We call $(B, \prec)$ reflexive if $(B, \prec)$ satisfies (S5), and let $R_{\text{Sub}}$ be the class of reflexive subordination algebras.
2. We call $(B, \prec)$ a contact algebra if $(B, \prec)$ satisfies (S5) and (S6), and let $\text{Con}$ be the class of contact algebras.

We clearly have that $\text{Con} \subset R_{\text{Sub}} \subset \text{Sub}$, that reflexive subordination algebras dually correspond to the subclass of $\text{StR}$ consisting of reflexive closed relations on Stone spaces, and that contact algebras dually correspond to the subclass of $\text{StR}$ consisting of reflexive and symmetric closed relations on Stone spaces.

Definition 2.4.

1. ([11]) We call a contact algebra $(B, \prec)$ a compingent algebra if it satisfies (S7) and (S8).
2. ([3]) We call a compingent algebra $(B, \prec)$ a de Vries algebra if $B$ is a complete Boolean algebra.
3. Let $\text{Com}$ be the class of compingent algebras and $\text{DeV}$ the class of de Vries algebras.

We clearly have that $\text{DeV} \subset \text{Com} \subset \text{Con}$. Let $(B, \prec)$ be a contact algebra and let $(X, R)$ be its dual. As follows from Lemma 2.2, $R$ is a reflexive and symmetric closed relation. Moreover, $(B, \prec)$ satisfies (S7) iff $R$ is an equivalence relation. By [5, Lem. 6.3], $(B, \prec)$ satisfies (S8) iff $R$ is an irreducible equivalence relation, where we recall (see [5, Def. 6.1 and Rem. 6.2]) that $R$ is irreducible provided $R[U]$ is a proper subset of $X$ for each proper clopen subset $U$ of $X$.

To characterize dually de Vries algebras we recall that a Boolean algebra $B$ is complete iff its Stone space $X$ is extremally disconnected, where a space is extremally disconnected provided the closure of each open set is clopen. Thus, compingent algebras dually correspond to pairs $(X, R)$ where $X$ is a Stone space and $R$ is an irreducible equivalence relation, while de Vries algebras correspond to pairs $(X, R)$ where $X$ is an extremally disconnected Stone space and $R$ is an irreducible equivalence relation. Such pairs were called Gleason spaces in [5, Def. 6.6] because of the close connection to Gleason covers of compact Hausdorff spaces.

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\footnote{We point out that $(B, \prec)$ being reflexive does not mean that $\prec$ is a reflexive relation, rather that the corresponding closed relation on the Stone space is reflexive (see Lemma 2.2(1)).}

\footnote{For details on Gleason covers we refer to [10] and [19, Sec. III.3]. They are not crucial for the content of this paper.}
Let $X$ be a compact Hausdorff space, and let $(Y, \pi)$ be the Gleason cover of $X$. Define $R$ on $Y$ by $xRy$ iff $\pi(x) = \pi(y)$. Then $(Y, R)$ is a Gleason space. Conversely, if $(Y, R)$ is a Gleason space, then the quotient space $X := Y/R$ is compact Hausdorff. This establishes a one-to-one correspondence between Gleason spaces and compact Hausdorff spaces, which extends to a categorical duality (see [5, Sec. 6] for details).

Since $\text{DeV}$ dually corresponds to the class of Gleason spaces, it follows that $\text{DeV}$ dually corresponds to the class $\mathcal{KHaus}$ of compact Hausdorff spaces, which is the object level of the celebrated de Vries duality [11]. The correspondence between $\text{DeV}$ and $\mathcal{KHaus}$ can be obtained directly, as was done by de Vries.

For a compact Hausdorff space $X$, let $\mathcal{RO}(X)$ be the complete Boolean algebra of regular open subsets of $X$. Define $\preceq$ on $\mathcal{RO}(X)$ by $U \preceq V$ iff $\text{Cl}(U) \subseteq V$. Then $(\mathcal{RO}(X), \preceq)$ is a de Vries algebra (that it validates (S7) and (S8) follows from the fact that every compact Hausdorff space is regular and normal; see, e.g., [13, Sec. 3.1]).

Conversely, suppose $(B, \prec)$ is a compingent algebra. A round filter of $(B, \prec)$ is a filter $F$ of $B$ satisfying $\Downarrow F = F$. An end of $(B, \prec)$ is a maximal proper round filter. Let $X$ be the set of ends of $(B, \prec)$. For $a \in B$, let $\beta(a) = \{x \in X \mid a \in x\}$. Then $\{\beta(a) \mid a \in B\}$ generates a compact Hausdorff topology on $X$. Moreover, if $X$ is compact Hausdorff, then it is homeomorphic to the dual of $(\mathcal{RO}(X), \prec)$. If $(B, \prec)$ is a compingent algebra and $X$ is its dual, then $(B, \prec)$ embeds into $(\mathcal{RO}(X), \prec)$, and $(B, \prec)$ is isomorphic to $(\mathcal{RO}(X), \prec)$ iff $(B, \prec)$ is a de Vries algebra. These correspondences extend to contravariant functors, which yield a dual equivalence of the categories $\mathcal{KHaus}$ and $\text{DeV}$. We refer to [11] for missing details and proofs.

3 The variety of strict implication algebras

As was pointed out in [5, Sec. 3], subordinations on $B$ can be described by means of binary operations $\rightsquigarrow: B \times B \to B$ with values in $\{0, 1\}$ satisfying

1. $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$;
2. $(a \lor b) \rightsquigarrow c = (a \rightsquigarrow c) \land (b \rightsquigarrow c)$;
3. $a \rightsquigarrow (b \land c) = (a \rightsquigarrow b) \land (a \rightsquigarrow c)$.

If $\prec$ is a subordination on $B$, then define $\rightsquigarrow: B \times B \to B$ by

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\rightsquigarrow$ has values in $\{0, 1\}$ and satisfies (I1)–(I3). Conversely, given $\rightsquigarrow$, define $\prec$ by setting

$$a \prec b \iff a \rightsquigarrow b = 1.$$  

It is easy to see that $\prec$ is a subordination on $B$, and that this correspondence is one-to-one.

Moreover, the axioms (S5)–(S8) correspond, respectively, to the axioms:

1. $a \rightsquigarrow b \leq a \to b;$
(I5) \( a \leadsto b = \neg b \leadsto \neg a; \)

(I6) \( a \leadsto b = 1 \) implies \( \exists c : a \leadsto c = 1 \) and \( c \leadsto b = 1; \)

(I7) \( a \neq 0 \) implies \( \exists b \neq 0 : b \leadsto a = 1. \)

Note that (I2)-(I3) correspond to (S2)-(S4) which explains why the numbering of the I-axioms is one off the numbering of the S-axioms. As we will see, adding (I4) to (I1)–(I3) is very useful in algebraic as well as logical calculations. Therefore, as our base variety, we will consider the variety generated by the algebras \((B, \leadsto)\), where \(B\) is a Boolean algebra and \(\leadsto\) is a binary operation on \(B\) with values in \(\{0, 1\}\) satisfying (I1)-(I4). From now on, when we write \((B, \leadsto) \in \text{RSub}\), we mean that the corresponding \((B, \prec)\) is reflexive (see Definition 2.3(1)).

**Definition 3.1.** We call \((B, \leadsto)\) a strict implication algebra if \((B, \leadsto)\) belongs to the variety generated by \(\text{RSub}\). Let \(\text{SIA}\) be the variety of strict implication algebras.

**Remark 3.2.** While \(\leadsto\) is not a normal and additive operator on \(B\), it gives rise to the normal and additive operator \(\Delta(a, b) := \neg(a \leadsto \neg b)\). Then \((B, \Delta)\) is a BAO (Boolean algebra with operators), and \(\leadsto\) is definable from \(\Delta\) by \(a \leadsto b = \neg \Delta(a, \neg b)\). We prefer to work with \(\leadsto\) since it arises from subordinations more naturally.

Let \((B, \leadsto) \in \text{SIA}\). For \(a \in B\), define

\[ \square a = 1 \leadsto a. \]

By (I1), \(\square 1 = 1 \leadsto 1 = 1; \) and by (I3), \(\square (a \land b) = 1 \leadsto (a \land b) = (1 \leadsto a) \land (1 \leadsto b) = \square a \land \square b. \) Thus, \((B, \square)\) is a (normal) modal algebra.

Suppose \((B, \leadsto) \in \text{RSub}\). If \(a = 1\), then \(\square a = 1\). If \(a \neq 1\), then by (I4), \(\square a = 1 \leadsto a \leq 1 \rightarrow a = a \neq 1\), so \(\square a \neq 1\). But since \((B, \leadsto) \in \text{RSub}\), we have that \(\leadsto\) only takes values in \(\{0, 1\}\). Therefore, \(\square\) only takes values in \(\{0, 1\}\). Thus, \(\square a = 0\), and so

\[ \square a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases} \]

We let \(\Diamond\) be the dual of \(\square\), i.e., \(\Diamond a = \neg \square \neg a\). Then

\[ \Diamond a = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0, \end{cases} \]

and so \(\Diamond\) is the so-called unary discriminator term [18]. From this, and the fact that the class \(\text{RSub}\) is axiomatized by universal first-order formulas, the following observations are immediate [24, Sec. 8.2].

**Proposition 3.3.**

1. The variety \(\text{SIA}\) is a discriminator variety, and hence a semisimple variety.

2. The simple algebras in \(\text{SIA}\) are exactly the members of \(\text{RSub}\).

We next turn to axiomatization of \(\text{SIA}\). First we observe that \(\square a \leq a\) by (I4). Let \(\forall\) be the variety axiomatized by (I1)–(I4) and the axioms:
We recall that a modal algebra \((B, \square)\) is an S5-algebra if its satisfies \(\square a \leq a, \square a \leq \square \square a,\) and \(\neg \square a \leq \square \neg \square a.\) Thus, if \((B, \leadsto) \in V,\) then \((B, \square)\) is an S5-algebra.

**Theorem 3.4.** \(SIA = V.\)

**Proof.** It is straightforward to see that (I8)–(I11) hold in each member of \(RSub.\) Since \(SIA\) is generated by \(RSub,\) it follows that \(SIA \subseteq V.\) For the reverse inclusion, we utilize [18, Thm. 3], by which a unary term \(\bigcirc\) is a discriminator term in subdirectly irreducible members of a variety \(V\) iff \(V\) satisfies four equations that in our setting amount to:

- \(\square a \leq \square \square a;\)
- \(\neg \square a \leq \square \neg \square a;\)
- \(\square a \leq \neg \square a \leadsto 0;\)
- \(\neg \square a \leq \square a \leadsto 0.\)

Clearly each \((B, \leadsto) \in V\) satisfies the first three. To see that it also satisfies the fourth, since we have \(\neg \square a = \square \neg \square a,\) using (I11), we obtain

\[\neg \square a = \square \neg \square a \leq \neg \square \neg \square a \leadsto 0 = \neg \square a \leadsto 0 = \square a \leadsto 0.\]

Thus, [18, Thm. 3] applies, by which \(\bigcirc\) is a unary discriminator in all subdirectly irreducible members of \(V.\) So if \((B, \leadsto)\) is a subdirectly irreducible member of \(V,\) then \(\square\) only takes the values 0 and 1. Since (I10) holds in \((B, \leadsto),\) we have that \(\leadsto\) also only takes the values 0 and 1. Therefore, as (I1)–(I4) hold in \((B, \leadsto),\) it follows from the definition of \(RSub\) that \((B, \leadsto) \in RSub.\) Thus, each subdirectly irreducible member of \(V\) belongs to \(SIA.\) Since \(V\) is generated by its subdirectly irreducible algebras, we conclude that \(V \subseteq SIA.\)

We next give an alternate axiomatization of \(SIA,\) which will be useful in Section 4. Let \(W\) be the variety axiomatized by (I1)–(I4) and the axioms:

- (I12) \(\square (a \rightarrow b) \land (b \leadsto c) \leq a \leadsto c;\)
- (I13) \((a \leadsto b) \land \square (b \rightarrow c) \leq a \leadsto c;\)
- (I14) \(a \leadsto b \leq c \leadsto (a \leadsto b);\)
- (I15) \(\neg (a \leadsto b) \leq c \leadsto \neg (a \leadsto b).\)

To prove that \(SIA = W,\) we require the following lemma.

**Lemma 3.5.** (I2) and (I3) imply \(a \leq b \Rightarrow (b \leadsto c \leq a \leadsto c \text{ and } c \leadsto a \leq c \leadsto b).\)

**Proof.** By (I2), \(b \leadsto c = (a \lor b) \leadsto c = (a \leadsto c) \land (b \leadsto c).\) Therefore, \(b \leadsto c \leq a \leadsto c.\) Also, by (I3), \(c \leadsto a = c \leadsto (a \land b) = (c \leadsto a) \land (c \leadsto b).\) Thus, \(c \leadsto a \leq c \leadsto b.\)
Theorem 3.6. $\text{SIA} = \mathcal{W}$.

Proof. First we show that $\text{SIA} \subseteq \mathcal{W}$. For this it is sufficient to see that (I12)–(I15) hold in each strict implication algebra $(B, \rightarrow)$. To see that (I12) holds, by (I11),

$$\Box(a \rightarrow b) \land (b \leadsto c) \leq (~\Box(a \rightarrow b) \leadsto 0) \land (b \leadsto c).$$

Since $0 \leq c$, by Lemma 3.5 and (I2),

$$(~\Box(a \rightarrow b) \leadsto 0) \land (b \leadsto c) \leq (~\Box(a \rightarrow b) \lor b) \leadsto c$$

Because $\Box(a \rightarrow b) \leq a \rightarrow b$, we have $(a \rightarrow b) \rightarrow b \leq \Box(a \rightarrow b) \rightarrow b$. Therefore, by Lemma 3.5

$$(\Box(a \rightarrow b) \rightarrow b) \leadsto c \leq ((a \rightarrow b) \rightarrow b) \leadsto c.$$ But $(a \rightarrow b) \rightarrow b = (\neg \neg a \lor b) \lor b = a \lor b$, so applying Lemma 3.5 again yields

$$(a \rightarrow b) \rightarrow b = (a \lor b) \rightarrow c \leq a \rightarrow c.$$ Thus, (I12) holds in $(B, \rightarrow)$.

To see that (I13) holds, by Lemma 3.5 and (I3),

$$(a \leadsto b) \land \Box(b \rightarrow c) = (a \leadsto b) \land (1 \leadsto (b \rightarrow c)) \leq (a \leadsto b) \land (a \leadsto (b \rightarrow c)) = a \leadsto (b \land (b \rightarrow c)).$$

Since $b \land (b \rightarrow c) \leq c$, applying Lemma 3.5 again yields

$$a \leadsto (b \land (b \rightarrow c)) \leq a \leadsto c.$$ Thus, (I13) holds in $(B, \rightarrow)$.

To see that (I14) holds, by (I10) and Lemma 3.5

$$a \leadsto b = \Box(a \leadsto b) = 1 \leadsto (a \leadsto b) \leq c \leadsto (a \leadsto b).$$ Thus, (I14) holds in $(B, \rightarrow)$.

To see that (I15) holds, since $(B, \Box)$ is an $\text{S5}$-algebra, by (I10) and Lemma 3.5

$$\neg(a \leadsto b) = ~\Box(a \leadsto b) = \Box\neg(a \leadsto b) \leq c \leadsto \neg(a \leadsto b).$$ Thus, (I15) holds in $(B, \rightarrow)$. Consequently, $\text{SIA} \subseteq \mathcal{W}$.

It is left to show that $\mathcal{W} \subseteq \text{SIA}$. For this it is sufficient to see that (I8)–(I11) hold in each $(B, \leadsto) \in \mathcal{W}$. It follows from (I14) that $a \leadsto b \leq \Box(a \leadsto b)$, and it follows from (I4) that $\Box(a \leadsto b) \leq a \leadsto b$. Thus, (I10) holds in $(B, \leadsto)$. 8
That (I8) holds in \((B, \leadsto)\) is immediate from (I10):

\[
\Box a = 1 \leadsto a = \Box (1 \leadsto a) = \Box \Box a.
\]

To see that (I11) holds, substituting in (I12) \(-\Box a\) for \(a\) and 0 for both \(b\) and \(c\) yields

\[
\Box (\neg \Box a \rightarrow 0) \land (0 \leadsto 0) \leq \neg \Box a \leadsto 0.
\]

Now, using (I1) and (I8), we have:

\[
\Box (\neg \Box a \rightarrow 0) \land (0 \leadsto 0) = \Box (\neg \Box a) \land 1 = \Box \Box a = \Box a.
\]

Thus, \(\Box a \leq \neg \Box a \leadsto 0\), and so (I11) holds in \((B, \leadsto)\).

Finally, it follows from (I15) that \((a \leadsto b) \leq \Box (a \leadsto b)\), and it follows from (I4) that

\[
\Box (a \leadsto b) \leq (a \leadsto b).
\]

Therefore, \((a \leadsto b) = \Box (a \leadsto b)\). Thus,

\[
\neg \Box a = \neg (1 \leadsto a) = \Box \neg (1 \leadsto a) = \Box \neg \Box a,
\]

and hence (I9) holds in \((B, \leadsto)\). Consequently, \(W \subseteq \text{SIA}\). \(\Box\)

We next show that our base variety \text{SIA} is locally finite, and consider subvarieties and inductive subclasses of \text{SIA}.

**Proposition 3.7.** The variety \text{SIA} is locally finite.

**Proof.** Let \((B, \leadsto) \in \text{RSub}\) be \(n\)-generated, with generators \(a_1, \ldots, a_n \in B\). For each \(a \in B\), there is a term \(t(x_1, \ldots, x_n)\) such that \(a = t(a_1, \ldots, a_n)\). Since \((B, \leadsto) \in \text{RSub}\), for each \(b, c \in B\), we have \(b \leadsto c \in \{0, 1\}\). Therefore, by replacing each subterm of \(t(x_1, \ldots, x_n)\) of the form \(x \leadsto y\) with either 0 or 1, we obtain a Boolean term \(t'(x_1, \ldots, x_n)\) such that \(a = t'(a_1, \ldots, a_n)\). Thus, \(B\) is \(n\)-generated as a Boolean algebra, and hence has at most \(2^n\) elements. Since \text{RSub} is the class of simple algebras in \text{SIA}, which is a semisimple variety (see Proposition 3.3), there is a uniform bound \(m(n) = 2^n\) on all \(n\)-generated subdirectly irreducible members of \text{SIA}. Consequently, by [2] Thm. 3.7(4), \text{SIA} is locally finite. \(\Box\)

As an immediate consequence we obtain:

**Corollary 3.8.** Every subvariety of \text{SIA} is generated by its finite members.

While \text{SIA} has many subvarieties, we will be interested in the subvariety obtained by postulating the identity (I5). Our interest is motivated by the fact that this variety is exactly the subvariety of \text{SIA} generated by the class \text{Con} of contact algebras. We further restrict \text{Con} to the class \text{Com} of compingent algebras, by postulating (I6) and (I7). But unlike (I5), neither (I6) nor (I7) is an identity. However, both (I6) and (I7) are \(\Pi_2\)-statements (i.e., statements of the form \(\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})\), where \(\bar{x}, \bar{y}\) are tuples of variables and \(\Phi(\bar{x}, \bar{y})\) is a quantifier-free formula). By the Chang-Łoś-Suszko Theorem (see, e.g., [10] Thm. 3.2.3), the elementary classes corresponding to \(\Pi_2\)-statements are inductive classes, where we recall that a class is *inductive* provided it is closed under unions of chains (equivalently, closed under directed limits). While we will be mainly interested in the inductive class \text{Con}, in Section 6 we will show that all inductive subclasses of \text{RSub} can be axiomatized by non-standard rules.

We conclude this section by observing that, unlike subvarieties of \text{SIA}, not every inductive subclass of \text{SIA} is determined by its finite algebras. For example, the inductive elementary
class \( \text{Com} \) is not determined by its finite algebras. To see this, let \( \text{FCom} \) be the class of finite compingent algebras. We show that every algebra in \( \text{FCom} \) validates the equation \( a \rightsquigarrow a = 1 \).

Every finite compingent algebra is a finite de Vries algebra. Therefore, by de Vries duality, every finite compingent algebra \( (B, \prec) \) is isomorphic to the powerset of a finite discrete space \( X \). Since every subset of a discrete space is clopen, we have \( a \prec a \), so \( a \rightsquigarrow a = 1 \) for each \( a \in B \). On the other hand, if we let \( X = [0, 1] \), then the de Vries algebra \( (\mathcal{RO}(X), \prec) \) falsifies \( a \rightsquigarrow a = 1 \). Indeed, if we put \( a = [0, \frac{1}{2}] \), then the closure of \( a \) is \( [0, \frac{1}{2}] \not\subseteq a \). So \( a \not\prec a \), and hence \( a \rightsquigarrow a \neq 1 \). Thus, \( \text{Com} \) is not determined by its finite algebras.

4 The strict implication calculus

We next present a sound and complete deductive system for \( \text{SIA} \). We will work with the language of classical propositional logic, with a countably infinite supply of propositional letters and primitive connectives \( \land, \neg \), which we will enrich with one binary connective \( \rightsquigarrow \) of strict implication. Then \( \top, \bot, \lor, \land, \rightarrow, \leftrightarrow \) are usual abbreviations, and \( \Box \varphi \) abbreviates \( \top \rightsquigarrow \varphi \).

A valuation on \( (B, \rightsquigarrow) \) is an assignment of elements of \( B \) to propositional letters of our language \( \mathcal{L} \), which extends to all formulas of \( \mathcal{L} \) in the usual way. We say that a valuation \( v \) on \( (B, \rightsquigarrow) \) satisfies a formula \( \varphi \) if \( v(\varphi) = 1 \). If all valuations on \( (B, \rightsquigarrow) \) satisfy \( \varphi \), then we say that \( (B, \rightsquigarrow) \) validates \( \varphi \), and write \( (B, \rightsquigarrow) \models \varphi \). For a set of formulas \( \Gamma \), we write \( (B, \rightsquigarrow) \models \Gamma \) if \( (B, \rightsquigarrow) \models \varphi \) for every \( \varphi \in \Gamma \).

Suppose \( \mathcal{U} \subseteq \text{SIA} \), \( \varphi \) is a formula, and \( \Gamma \) is a set of formulas. We say that \( \varphi \) is a semantic consequence of \( \Gamma \) over \( \mathcal{U} \), and write \( \Gamma \models_{\mathcal{U}} \varphi \), provided for each \( (B, \rightsquigarrow) \in \mathcal{U} \) and each valuation \( v \) on \( (B, \rightsquigarrow) \), if \( v(\gamma) = 1 \) for each \( \gamma \in \Gamma \), then \( v(\varphi) = 1 \).

Consider the following axiom schemes:

\[
\begin{align*}
\text{(A1)} & \quad (\bot \rightsquigarrow \varphi) \land (\varphi \rightsquigarrow \top), \\
\text{(A2)} & \quad [(\varphi \lor \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \land (\psi \rightsquigarrow \chi)], \\
\text{(A3)} & \quad (\varphi \rightsquigarrow (\psi \land \chi)) \leftrightarrow [(\varphi \rightsquigarrow \psi) \land (\varphi \rightsquigarrow \chi)], \\
\text{(A4)} & \quad (\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi), \\
\text{(A5)} & \quad (\varphi \rightsquigarrow \psi) \leftrightarrow (\neg \psi \rightsquigarrow \neg \varphi), \\
\text{(A8)} & \quad \Box \varphi \rightarrow \Box \Box \varphi, \\
\text{(A9)} & \quad \neg \Box \varphi \rightarrow \neg \Box \neg \varphi, \\
\text{(A10)} & \quad (\varphi \rightsquigarrow \psi) \leftrightarrow \Box (\varphi \rightsquigarrow \psi), \\
\text{(A11)} & \quad \Box \varphi \rightarrow (\neg \Box \varphi \rightsquigarrow \bot), \\
\text{(A12)} & \quad [(\Box (\varphi \rightarrow \psi) \land (\psi \rightsquigarrow \chi))] \rightarrow (\varphi \rightsquigarrow \chi), \\
\text{(A13)} & \quad [(\varphi \rightarrow \psi) \land \Box (\psi \rightarrow \chi)] \rightarrow (\varphi \rightsquigarrow \chi), \\
\text{(A14)} & \quad (\varphi \rightsquigarrow \psi) \rightarrow [\chi \rightsquigarrow (\varphi \rightsquigarrow \psi)], \\
\text{(A15)} & \quad (\neg (\varphi \rightsquigarrow \psi) \rightarrow [\chi \rightsquigarrow (\varphi \rightsquigarrow \psi)]).
\end{align*}
\]

Clearly \( \text{(A1)}–\text{(A5)} \) correspond to \( \text{(I1)}–\text{(I5)} \) and \( \text{(A8)}–\text{(A15)} \) to \( \text{(I8)}–\text{(I15)} \).

**Definition 4.1.** The strict implication calculus \( \text{SIC} \) is the derivation system containing:

- all the theorems of the classical propositional calculus \( \text{CPC} \),
- the axiom schemes \( \text{(A1)}–\text{(A5)} \) and \( \text{(A8)}–\text{(A11)} \),
and closed under the inference rules:

\[
\begin{align*}
\text{(MP)} & \quad \varphi, \varphi \rightarrow \psi \\
\text{(N)} & \quad \varphi \quad \square \varphi
\end{align*}
\]

The definition of derivability in SIC is standard:

**Definition 4.2.**

1. A proof of a formula \( \varphi \) from a set of formulas \( \Gamma \) is a finite sequence \( \psi_1, \ldots, \psi_n \) such that \( \psi_n = \varphi \) and each \( \psi_i \) is in \( \Gamma \) or is an instance of an axiom of SIC or is obtained from \( \psi_j, \psi_k \) for some \( j, k < i \) by applying (MP), or is obtained from \( \psi_j \) for some \( j < i \) by applying (N). Elements of \( \Gamma \) are referred to as assumptions.

2. If there is a proof of \( \varphi \) from \( \Gamma \), then we say that \( \varphi \) is derivable in SIC from \( \Gamma \) and write \( \Gamma \vdash_{\text{SIC}} \varphi \).

3. If \( \Gamma = \emptyset \), then we say that \( \varphi \) is derivable in SIC and write \( \vdash_{\text{SIC}} \varphi \).

**Remark 4.3.** Since \( \Box (\varphi \land \psi) \leftrightarrow (\Box \varphi \land \Box \psi) \) is an instance of (A3) and \( \Box \varphi \rightarrow \varphi \) is an instance of (A4), we see that all the theorems of the modal system S5 are derivable in SIC.

The deduction theorem for SIC is proved as for S5:

**Theorem 4.4.** For any set of formulas \( \Gamma \) and for any formulas \( \varphi, \psi \), we have:

\[
\Gamma \cup \{ \varphi \} \vdash_{\text{SIC}} \psi \iff \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi.
\]

**Proof.** (\( \Leftarrow \)) This is the easy direction since a proof \( \psi_1, \ldots, \psi_n \) of \( \Box \varphi \rightarrow \psi \) from \( \Gamma \) can easily be extended to a proof of \( \psi \) from \( \Gamma \cup \{ \varphi \} \) as follows:

\[
\begin{align*}
n & \quad \Box \varphi \rightarrow \psi \\
n + 1 & \quad \varphi \text{ (assumption)} \\
n + 2 & \quad \Box \varphi \text{ (by (N) from } n + 1) \\
n + 3 & \quad \psi \text{ (by (MP) from } n \text{ and } n + 2).
\end{align*}
\]

(\( \Rightarrow \)) Suppose there is a proof \( \psi_1, \ldots, \psi_n \) of \( \psi \) from \( \Gamma \cup \{ \varphi \} \). We show by induction on \( i = 1, \ldots, n \) that we can obtain a proof of \( \Box \varphi \rightarrow \psi_i \) from \( \Gamma \). If \( \psi_i = \varphi \), then \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_i \) since \( \vdash_{\text{SIC}} \Box \varphi \rightarrow \varphi \). If \( \psi_i \in \Gamma \), there is nothing to prove. If \( \psi_i \) is an instance of an axiom of SIC, then since \( \vdash_{\text{SIC}} \psi_1 \rightarrow (\Box \varphi \rightarrow \psi_1) \), by applying (MP) we obtain \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_i \). If \( \psi_i \) is obtained by applying (MP) to \( \psi_j \) and \( \psi_k = \psi_j \rightarrow \psi_i \) with \( j, k < i \), then by the inductive hypothesis, \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_j, \Box \varphi \rightarrow (\psi_j \rightarrow \psi_i) \). But then \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_j, \psi_j \rightarrow (\Box \varphi \rightarrow \psi_i) \), which yields \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow (\Box \varphi \rightarrow \psi_i) \), so \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_i \). Finally, if \( \psi_i \) is obtained by applying (N) to \( \psi_j \) with \( j < i \), then by the inductive hypothesis, \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_j \). Applying (N) yields \( \Gamma \vdash_{\text{SIC}} \Box (\Box \varphi \rightarrow \psi_i) \). Therefore, by applying the (K) axiom for \( \Box \) and (MP), we obtain \( \Gamma \vdash_{\text{SIC}} \Box \Box \varphi \rightarrow \Box \psi_j \). Thus, since \( \vdash_{\text{SIC}} \Box \varphi \rightarrow \Box \Box \varphi \), we have \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \Box \psi_j \), and so \( \Gamma \vdash_{\text{SIC}} \Box \varphi \rightarrow \psi_i \), concluding the proof.

\( \square \)
Proposition 4.7. For
\[\text{as follows from Theorem 3.6, Remark 4.6.}\]
\[\text{congruences of Boolean algebras correspond to filters, and this correspondence is obtained as}\]
\[\Gamma^{(1)}\text{ for any } a \text{.}\]
\[\text{Therefore, } (c \rightarrow d) \rightarrow (c \rightarrow a) \rightarrow (c \rightarrow b) \in F;\]
\[\text{filters of } B \text{ such that } a \rightarrow b \in F \text{ implies } (b \rightarrow c) \rightarrow (a \rightarrow c), (c \rightarrow a) \rightarrow (c \rightarrow b) \in F;\]
\[\text{filters of } B \text{ such that } a \rightarrow b, b \rightarrow c, c \rightarrow d \in F \text{ imply } a \rightarrow d \in F.\]

**Proposition 4.7.** For \((B, \sim) \in \text{SIA}\), there is a one-to-one correspondence between

1. congruences of \((B, \sim)\);
2. congruences \(\theta\) of \(B\) such that \(a \theta b\) implies \((a \sim c) \theta (b \sim c)\) and \((c \sim a) \theta (c \sim b)\);
3. filters \(F\) of \(B\) such that \(a \in F\) implies \(\square a \in F\);
4. filters \(F\) of \(B\) such that \(a \rightarrow b \in F\) implies \((b \sim c) \rightarrow (a \sim c), (c \sim a) \rightarrow (c \sim b) \in F;\)
5. filters \(F\) of \(B\) such that \(a \rightarrow b, b \rightarrow c, c \rightarrow d \in F\) imply \(a \rightarrow d \in F.\)

**Proof.** (1)\(\Rightarrow\)(2): This is obvious.

(2)\(\Rightarrow\)(1): Suppose \(a \theta b\) and \(c \theta d\). By (2), \((a \sim c) \theta (b \sim c)\) and \((b \sim c) \theta (b \sim d)\). Therefore, \((a \sim c) \theta (b \sim d)\). Thus, \(\theta\) is a congruence of \((B, \sim)\).

(3)\(\Rightarrow\)(4): Suppose \(F\) satisfies (3) and \(a \rightarrow b \in F\). Then \(\Box (a \rightarrow b) \in F\). By (I12) and (I13), for any \(c \in B\), we have \(\Box (a \rightarrow b) \leq (b \sim c) \rightarrow (a \sim c)\) and \(\Box (a \rightarrow b) \leq (c \sim a) \rightarrow (c \sim b)\). Therefore, \((b \sim c) \rightarrow (a \sim c), (c \sim a) \rightarrow (c \sim b) \in F\), and so \(F\) satisfies (4).

(4)\(\Rightarrow\)(5): Suppose \(F\) satisfies (4) and \(a \rightarrow b, b \sim c, c \rightarrow d \in F\). From \(a \rightarrow b \in F\) it follows that \((b \sim c) \rightarrow (a \sim c) \in F\). Therefore, since \(b \sim c \in F\), we have \(a \sim c \in F\). Also, from \(c \rightarrow d \in F\) it follows that \((a \sim c) \rightarrow (a \sim d) \in F\). This together with \(a \sim c \in F\) yields \(a \sim d \in F\). Thus, \(F\) satisfies (5).

(5)\(\Rightarrow\)(3): Suppose \(F\) satisfies (5) and \(a \in F\). Since \(1 \rightarrow 1 = 1 \sim 1 = 1\) and \(1 \rightarrow a = a\), we have \(1 \rightarrow 1, 1 \sim 1, 1 \in a \in F\). Therefore, by (4), \(\Box a = 1 \sim a \in F\). Thus, \(F\) satisfies (3).

(2)\(\Rightarrow\)(3): Suppose \(\theta\) is a congruence of \(B\) and \(a \in F_{\theta}\). Then \(a \theta 1\). Therefore, \((1 \sim a) \theta (1 \sim 1)\). Thus, \(\Box a \theta 1\), and so \(\Box a \in F_{\theta}\).

(4)\(\Rightarrow\)(2): Suppose \(F\) satisfies (4), \(a \theta_F b\), and \(c \in B\). Then \(a \rightarrow b \in F\) and \(b \rightarrow a \in F\). Therefore, by (4), \((b \sim c) \rightarrow (a \sim c), (c \sim a) \rightarrow (c \sim b) \in F\) and \((a \sim c) \rightarrow (b \sim c), (c \sim b) \rightarrow (a \sim a) \in F\). Thus, \((a \sim c) \leftrightarrow (b \sim c), (c \sim a) \leftrightarrow (c \sim b) \in F\). Consequently, \((a \sim c) \theta_F (b \sim c)\) and \((c \sim a) \theta_F (c \sim b)\), and hence \(\theta_F\) satisfies (2).

**Definition 4.8.** Let \((B, \sim)\) be a strict implication algebra. We call a filter \(F\) of \(B\) a \(\Box\)-filter provided \(F\) satisfies Proposition 4.7(3); that is, \(a \in F\) implies \(\Box a \in F\).
By Proposition 4.7, congruences of strict implication algebras correspond to their \(\square\)-filters. This is a generalization of a similar characterization of congruences of modal algebras (see, e.g., [8, Sec. 7.7]). For a strict implication algebra \((B, \leadsto)\) and \(a \in B\), we use the usual abbreviation
\[
\uparrow a := \{b \in B \mid a \leq b\}.
\]

**Lemma 4.9.** Let \((B, \leadsto)\) be a strict implication algebra, \(a \in B\), and \(F\) a \(\square\)-filter. Then the filter generated by \(F \cup \{\square a\}\) is a \(\square\)-filter. In particular, we have that \(\uparrow \square a\) and \(\downarrow \neg \square a\) are \(\square\)-filters.

**Proof.** Let \(F'\) be the filter generated by \(F \cup \{\square a\}\), and let \(b \in F'\). Then there is \(c \in F\) such that \(c \land \square a \leq b\). As \(\square\) is an \(S5\)-operator, we have \(\square (c \land \square a) \leq \square b\). Since \(F\) is a \(\square\)-filter, \(\square c \in F\). Therefore, again using the fact that \(\square\) is an \(S5\)-operator, we obtain \(\square (c \land \square a) = \square c \land \square \square a = \square c \land \square a \in F'\). Thus, \(\square b \in F'\), which shows that \(F'\) is a \(\square\)-filter.

In particular, as \(\{1\}\) is a \(\square\)-filter, it follows that \(\uparrow \square a\) is a \(\square\)-filter, and by (I9) the same holds for \(\downarrow \neg \square a\). \(\Box\)

For a strict implication algebra \((B, \leadsto)\) and a \(\square\)-filter \(F\), let \((B/F, \leadsto_F)\) be the quotient algebra. For \(a \in B\), let \([a]\) be the corresponding element of \(B/F\).

**Lemma 4.10.** Let \((B, \leadsto) \in \text{SIA}\).

1. For a proper \(\square\)-filter \(F\) in \((B, \leadsto)\), the following are equivalent:
   a. \(F\) is a maximal \(\square\)-filter.
   b. For each \(a \in B\), we have \(\square a \in F\) or \(\neg \square a \in F\).
   c. \((B/F, \leadsto_F) \in \text{RSub}\).

2. If \(F\) is a \(\square\)-filter and \(a \notin F\), then there is a maximal \(\square\)-filter \(M\) such that \(F \subseteq M\) and \(a \notin M\).

**Proof.** (1) (a)\(\Rightarrow\)(b): Suppose \(\square a \notin F\). Let \(G\) be the filter generated by \(F\) and \(\square a\). By Lemma 4.9, \(G\) is a \(\square\)-filter. Since \(F\) is a maximal \(\square\)-filter, \(G\) is improper. Therefore, \(0 = \square a \land b\) for some \(b \in F\). Thus, \(b \leq \neg \square a\), and so \(\neg \square a \in F\).

(b)\(\Rightarrow\)(c): Let \(a \in B\). Then \(\square a \in F\) or \(\neg \square a \in F\). If \(\square a \in F\), then \(\square_F[a] = [\square a] = 1_F\), where \(\square_F[a] = 1_F \leadsto_F [a]\). On the other hand, if \(\square a \notin F\), then \(\neg \square a \in F\), so \(\neg \square_F[a] = [\neg \square a] = 1_F\), and hence \(\square_F[a] = 0_F\). This implies that \(\{1_F\}\) and \(B/F\) are the only two \(\square_F\)-filters in \((B/F, \leadsto_F)\). Thus, \((B/F, \leadsto_F)\) is a simple algebra, and hence \((B/F, \leadsto_F) \in \text{RSub}\) by Proposition 3.3(2).

(c)\(\Rightarrow\)(a): Suppose \(G\) is a \(\square\)-filter properly containing \(F\). Then there is \(a \in G \setminus F\). Since \(G\) is a \(\square\)-filter and \(\square a \leq a\), we see that \(\square a \in G \setminus F\). Therefore, \([\square a] \neq 1_F\). Since \((B/F, \leadsto_F) \in \text{RSub}\), we conclude that \([\square a] = 0_F\). Thus, \([\neg \square a] = 1_F\), yielding that \(\neg \square a \in F \subseteq G\). Consequently, \(G\) is an improper \(\square\)-filter, and hence \(F\) is a maximal \(\square\)-filter.

(2) Since \(a \notin F\), by Zorn’s lemma there is a \(\square\)-filter \(M\) such that \(F \subseteq M\), \(a \notin M\), and \(M\) is maximal with this property. If \(M\) is not a maximal \(\square\)-filter, then by (1), there is \(b \in B\) such that \(\square b, \neg \square b \notin M\). Let \(G\) be the filter generated by \(M\) and \(\square b\) and \(H\) the filter generated by \(M\) and \(\neg \square b\). By Lemma 4.9, both \(G\) and \(H\) are \(\square\)-filters that properly extend \(F\). Therefore, \(a \in G, H\), so there exist \(c, d \in M\) such that \(a \geq \square b \land c\) and \(a \geq \neg \square b \land d\). Thus,
\[ a \geq (\Box b \land c) \lor (\neg \Box b \land d) = (\Box b \lor \neg \Box b) \land (\Box b \lor d) \land (c \lor \neg \Box b) \land (c \lor d) \in M. \] The obtained contradiction proves that \( M \) is a maximal \( \Box \)-filter. \( \square \)

**Theorem 4.11.** For a set of formulas \( \Gamma \) and a formula \( \varphi \), we have:

\[
\Gamma \vdash_{\text{SIC}} \varphi \iff \Gamma \models_{\text{SIA}} \varphi \iff \Gamma \models_{\text{RSub}} \varphi.
\]

**Proof.** We already observed in Proposition 4.5 that \( \Gamma \vdash_{\text{SIC}} \varphi \iff \Gamma \models_{\text{SIA}} \varphi \). This together with \( \text{RSub} \subseteq \text{SIA} \) yields that \( \Gamma \vdash_{\text{SIC}} \varphi \) implies \( \Gamma \models_{\text{RSub}} \varphi \). Conversely, if \( \Gamma \not\vdash_{\text{SIC}} \varphi \), then in the Lindenbaum algebra \((B, \sim)\) of SIC, the \( \Box \)-filter generated by \( \{ \lbrack \psi \rbrack \mid \psi \in \Gamma \} \) does not contain \( \lbrack \varphi \rbrack \). By Lemma 4.10(2), there is a maximal \( \Box \)-filter \( F \) such that \( \{ \lbrack \psi \rbrack \mid \psi \in \Gamma \} \subseteq F \) and \( \lbrack \varphi \rbrack \notin F \). But then \((B/F, \sim_F)\) satisfies \( \Gamma \) and refutes \( \varphi \). By Lemma 4.10(1), \((B/F, \sim_F) \in \text{RSub}\). Thus, \( \Gamma \not\models_{\text{RSub}} \varphi \). \( \square \)

## 5 The symmetric strict implication calculus and its topological completeness

In this section we define the symmetric strict implication calculus \( S^2\text{IC} \) obtained by adding \( (A5) \) to \( \text{SIC} \), and the corresponding variety \( S^2\text{IA} \) of symmetric strict implication algebras. We prove that \( S^2\text{IC} \) is strongly sound and complete with respect to \( \text{Con} \), as well as with respect to \( \text{Com} \) and \( \text{DeV} \). The last completeness together with de Vries duality allows us to introduce topological models for \( S^2\text{IC} \) based on compact Hausdorff spaces, and prove that \( S^2\text{IC} \) is strongly sound and complete with respect to the class of compact Hausdorff spaces.

**Definition 5.1.** 

1. We call a strict implication algebra \((B, \sim)\) symmetric if it satisfies \( (I5) \). Let \( S^2\text{IA} \) be the variety of symmetric strict implication algebras.

2. The **symmetric strict implication calculus** \( S^2\text{IC} \) is obtained from the strict implication calculus \( \text{SIC} \) by postulating \( (A5) \).

Since \( (A5) \) corresponds to \( (I5) \), it follows from Proposition 4.5 that \( S^2\text{IC} \) is strongly sound and complete with respect to \( S^2\text{IA} \). Moreover, since for \((B, \sim) \in \text{RSub}\) we have \((B, \sim) \in \text{Con}\) iff \((B, \sim) \) satisfies \( (I5) \), the following is an immediate consequence of Theorem 4.11.

**Theorem 5.2.** For a set of formulas \( \Gamma \) and a formula \( \varphi \), we have:

\[
\Gamma \vdash_{S^2\text{IC}} \varphi \iff \Gamma \models_{S^2\text{IA}} \varphi \iff \Gamma \models_{\text{Con}} \varphi.
\]

We next show that each contact algebra can be embedded into a compingent algebra. For this we utilize the representation of subordination algebras discussed in Section 2, as well as the following result (cf. Lemma 2.2).

**Lemma 5.3** ([12]). Let \( R \) be a binary relation on a set \( X \). Define \( \prec_R \) on \( \mathcal{P}(X) \) by \( U \prec_R V \) iff \( R[U] \subseteq V \).

1. \( \prec_R \) is a subordination on \( \mathcal{P}(X) \).
2. \( R \) is reflexive iff \( (\mathcal{P}(X), \prec_R) \) satisfies \( (S5) \).
3) $R$ is symmetric iff $(P(X), \prec_R)$ satisfies (S6).

4) $R$ is transitive iff $(P(X), \prec_R)$ satisfies (S7).

We use Lemma 5.3 to show an analogue of [Lem. 2.5] in our setting. Let $(B, \prec)$ and $(C, \prec)$ be in RSub. We say that $(B, \prec)$ is embedded into $(C, \prec)$ if there is a Boolean embedding $h : B \rightarrow C$ such that $a \prec b$ if $h(a) \prec h(b)$ for each $a, b \in B$.

**Lemma 5.4.** Every $(B, \prec) \in \text{Con}$ can be embedded into $(C, \prec) \in \text{Con}$ satisfying (S7).

**Proof.** Suppose $(X, R)$ is the dual of $(B, \prec)$. By Lemma 2.2 $R$ is reflexive and symmetric. Let $Y = \{\{x, y\} \subseteq X \mid xRy\}$ and let

$$X' = \{(x, \alpha) \in X \times Y \mid x \in \alpha\}.$$ 

Define $R'$ on $X'$ by

$$(x, \alpha)R'(y, \beta) \iff \alpha = \beta.$$ 

Clearly $R'$ is an equivalence relation on $X'$ and $f : X' \rightarrow X$ given by $f(x, \alpha) = x$ is onto. Therefore, $f^{-1} : \text{Clop}(X) \rightarrow P(X')$ is a Boolean embedding. Since $R$ is reflexive and symmetric, it follows from the definition of $R'$ that $(x, \alpha)R'(y, \beta)$ implies $xRy$.

**Claim.** For $U, V \in \text{Clop}(X)$, we have $U \prec_R V$ iff $f^{-1}(U) \prec_{R'} f^{-1}(V)$.

**Proof of claim.** Since $(x, \alpha)R'(y, \beta)$ implies $f(x, \alpha)Rf(y, \beta)$, we see that $U \prec_R V$ implies $f^{-1}(U) \prec_{R'} f^{-1}(V)$. For the converse, suppose $U \not\prec_R V$. Then $R[U] \not\subseteq V$. Therefore, there are $x \in U$ and $y \notin V$ such that $xRy$. Let $\alpha = \{x, y\}$. Then $(x, \alpha)R'(y, \alpha)$, $(x, \alpha) \in f^{-1}(U)$, and $(y, \alpha) \notin f^{-1}(V)$. Thus, $R'[f^{-1}(U)] \not\subseteq f^{-1}(V)$, and hence $f^{-1}(U) \not\prec_{R'} f^{-1}(V)$. □

Let $(C, \prec) = (P(X'), \prec_{R'})$. By Lemma 5.3 $(C, \prec)$ satisfies (S1)–(S7), and by the Claim, $f^{-1}$ is an embedding of $(B, \prec)$ into $(C, \prec)$.

**Lemma 5.5.** Suppose $(B, \prec) \in \text{RSub}$. Let $B' = B \times B$ and define $\prec'$ on $B'$ by

$$(a, b) \prec' (c, d) \iff a \prec c \text{ and } b \leq d.$$ 

Then $(B', \prec') \in \text{RSub}$. Moreover, if $(B, \prec) \in \text{Con}$, then $(B', \prec') \in \text{Con}$.

**Proof.** Since $(B, \prec) \in \text{RSub}$, it satisfies (S1)–(S5). We show that $(B', \prec')$ also satisfies (S1)–(S5).

(S1) Since $0 < 0$ and $1 < 1$, it is obvious that $(0, 0) \prec' (0, 0)$ and $(1, 1) \prec' (1, 1)$.

(S2) Suppose $(a, b) < (c, d), (c', d')$. Then $a < c, c' \text{ and } b \leq d, d'$. Therefore, $a < c \land c'$ and $b \leq d \land d'$. Thus, $(a, b) <' (c, c') \land (d, d')$.

(S3) Suppose $(a, b), (a', b') <' (c, d)$. Then $a, a' < c \text{ and } b, b' \leq d$. Therefore, $a \lor a' < c \text{ and } b \lor b' \leq d$. Thus, $(a \lor a', b \lor b') <' (c, d)$.

(S4) Suppose $(a, b) \leq (a', b') < (c', d') \leq (c, d)$. Then $a \leq a' < c \text{ and } b \leq b' \leq d \leq d'$. Thus, $a < c \text{ and } b \leq d, \text{ and so } (a, b) <' (c, d)$.

(S5) Suppose $(a, b) <' (c, d)$. Then $a < c \text{ and } b \leq d$. Therefore, it follows that $a \leq c$. Thus, $(a, b) \leq (c, d)$.
Now suppose that in addition \((B, \prec) \in \text{Con}\). Then \((B, \prec)\) satisfies (S6). We show that \((B', \prec')\) also satisfies (S6).

(S6) Suppose \((a, b) \prec (c, d)\). Then \(a \prec c\) and \(b \leq d\). Therefore, \(\neg c \prec \neg a\) and \(\neg d \leq \neg b\). Thus, \(\neg (c, d) \prec \neg (a, b)\).

Consequently, if \((B, \prec) \in \text{Con}\), then \((B', \prec') \in \text{Con}\).

\[\text{Lemma 5.6.} \] Every \((B, \prec) \in \text{RSub}\) can be embedded into \((C, \prec) \in \text{RSub}\) satisfying (S8). In addition, if \((B, \prec)\) satisfies (S7), then \((C, \prec)\) satisfies (S7), and if \((B, \prec) \in \text{Con}\), then \((C, \prec) \in \text{Con}\).

\[\text{Proof.}\] Starting from \((B, \prec)\), we inductively build a chain

\[(B, \prec) \hookrightarrow (B_1, \prec) \hookrightarrow (B_2, \prec) \hookrightarrow (B_3, \prec) \hookrightarrow \cdots\]

in \text{RSub} such that the union \((C, \prec) := \bigcup_{n \in \omega} (B_n, \prec)\) satisfies (S8).

If \((B_n, \prec)\) is already defined, define \((B_{n+1}, \prec) := (B_n, \prec) \times (B_n, \leq)\). By Lemma 5.5, \((B_{n+1}, \prec) \in \text{RSub}\). Moreover, \(a \mapsto (a, a)\) is an embedding of \((B_n, \prec)\) into \((B_{n+1}, \prec)\). We prove that \((C, \prec)\) satisfies (S8).

Let \(0 \neq a \in C\). Then there is \(n\) such that \(a \in B_n\). Therefore, \((a, a) \in B_{n+1}\). Let \(b := (0, a) \in B_{n+1}\). We have \(b \neq 0\) and \(b \prec (a, a)\). Thus, \((C, \prec)\) satisfies (S8).

Suppose \((B, \prec)\) in addition satisfies (S7). We show that if \((B_n, \prec)\) satisfies (S7), then so does \((B_{n+1}, \prec)\). Let \((a_1, a_2) \prec (b_1, b_2)\) in \((B_{n+1}, \prec)\). Then \(a_1 \prec b_1\) and \(a_2 \leq b_2\) in \(B_n\). By (S7), there exists \(c \in B_n\) such that \(a_1 \prec c\) and \(c \prec b_1\). So, for \((c, a_2) \in B_{n+1}\), we have \((a_1, a_2) \prec (c, a_2) \prec (b_1, b_2)\). Therefore, \((B_{n+1}, \prec)\) satisfies (S7). Thus, by induction, each \((B_n, \prec)\) satisfies (S7). By the Chang-Łoś-Suszko theorem (see, e.g., [10] Thm. 3.2.3), \(\Pi_2\)-sentences are preserved by direct limits. Since (S7) is a \(\Pi_2\)-sentence, the direct limit \((C, \prec)\) of the chain also satisfies (S7).

Finally, if \((B, \prec) \in \text{Con}\), then each \((B_n, \prec) \in \text{Con}\) by Lemma 5.5. Therefore, \((C, \prec) \in \text{Con}\). Since \((C, \prec)\) satisfies (S7) and (S8), we conclude that \((C, \prec) \in \text{Com}\).

Let \((B, \dashv \vdash)\) and \((C, \vdash \vdash)\) be the strict implication algebras corresponding to \((B, \prec)\) and \((C, \prec)\), respectively. It is straightforward to check that \(h : B \rightarrow C\) is an embedding of \((B, \prec)\) into \((C, \prec)\) if \(h\) is an isomorphism from \((B, \dashv \vdash)\) to a subalgebra of \((C, \vdash \vdash)\). For a class \(\mathcal{K}\) of strict implication algebras, let \(\text{IS}(\mathcal{K})\) be the class of isomorphic copies of subalgebras of algebras in \(\mathcal{K}\).

\[\text{Theorem 5.7.} \] \(\text{IS(Com)} = \text{Con}\).

\[\text{Proof.}\] Obviously, \(\text{Com} \subseteq \text{Con}\) and as \(\text{Con}\) is a universal class, we have that \(\text{IS(Com)} \subseteq \text{Con}\). Conversely, suppose \((B, \dashv \vdash) \in \text{Con}\). Then by Lemma 5.6 it is isomorphic to a subalgebra of \((C, \vdash \vdash) \in \text{Com}\). Therefore, \(\text{Con} \subseteq \text{IS(Com)}\).

\[\text{Theorem 5.8.} \] \(\text{SIC}^2\) is strongly sound and complete with respect to \(\text{Com}\) i.e., for a set of formulas \(\Gamma\) and a formula \(\varphi\), we have:

\[\Gamma \vdash_{\text{SIC}^2} \varphi \iff \Gamma \models_{\text{Com}} \varphi.\]
Proof. The left to right direction follows from Theorem 5.2 and the fact that \( \text{Com} \subseteq \text{Con} \). Now suppose \( \Gamma \not\vdash_{\text{S2IC}} \phi \). Applying Theorem 5.2 again yields a contact algebra \((B, \sim)\) and a valuation \( v \) on \( B \) such that \( v(\gamma) = 1_B \) for each \( \gamma \in \Gamma \) and \( v(\varphi) \neq 1_B \). By Theorem 5.7, there is \( (C, \sim) \in \text{Com} \) such that \((B, \sim)\) is isomorphic to a subalgebra of \((C, \sim)\). We may view \( v \) as a valuation on \( C \), so \( v(\gamma) = 1_C \) for each \( \gamma \in \Gamma \) and \( v(\varphi) \neq 1_C \). Thus, \( \Gamma \not\vdash_{\text{Com}} \varphi \). \( \qed \)

We recall that the de Vries algebra of a compact Hausdorff space \( X \) is the pair \((\mathcal{RO}(X), \prec)\), where \( \mathcal{RO}(X) \) is the complete Boolean algebra of regular open subsets of \( X \) and \( U \prec V \) iff \( \text{Cl}(U) \subseteq V \). By de Vries duality \([11]\), every de Vries algebra is isomorphic to the de Vries algebra of some compact Hausdorff space. This allows us to define topological semantics for our language.

**Definition 5.9.** A *compact Hausdorff model* is a pair \((X, v)\), where \( X \) is a compact Hausdorff space and \( v \) is a valuation assigning a regular open set to each propositional letter.

If \( \prec \) is the strict implication corresponding to \( \prec \), then the formulas of our language are interpreted in \((\mathcal{RO}(X), \prec)\) \( \in \text{Dev} \).

**Theorem 5.10.**

1. The system \( S^2IC \) is strongly sound and complete with respect to \( \text{Dev} \).
2. The system \( S^2IC \) is strongly sound and complete with respect to compact Hausdorff models.

**Proof.** (1) Since \( \text{Dev} \subseteq \text{Com} \), by Theorem 5.8, we have \( \Gamma \vdash_{S^2IC} \phi \) implies \( \Gamma \models_{\text{Dev}} \phi \). Conversely, suppose \( \Gamma \not\vdash_{S^2IC} \phi \). Applying Theorem 5.8 again yields a compingent algebra \((B, \sim)\) and a valuation \( v \) on \( B \) such that \( v(\gamma) = 1_B \) for each \( \gamma \in \Gamma \) and \( v(\varphi) \neq 1_B \). By de Vries duality, there is a compact Hausdorff space \( X \) such that \((B, \sim)\) embeds into \((\mathcal{RO}(X), \prec)\). We may view \( v \) as a valuation on \( \mathcal{RO}(X) \), so \( v(\gamma) = 1_X \) for each \( \gamma \in \Gamma \) and \( v(\varphi) \neq 1_X \). Since \((\mathcal{RO}(X), \prec) \in \text{Dev} \), we conclude that \( \Gamma \not\vdash_{\text{Dev}} \phi \).

(2) This follows from (1) and de Vries duality. \( \qed \)

**Remark 5.11.** Let \((B, \prec) \in \text{Com} \) and let \( \beta : B \to \mathcal{RO}(X) \) be the embedding. By \([11]\) Thm. I.3.9], \( \beta[B] \) is dense in \( \mathcal{RO}(X) \) (for every \( a, b \in \mathcal{RO}(X) \) with \( a \leq b \), there is \( c \in \beta[B] \) with \( a \leq c \leq b \)). Therefore, \( \mathcal{RO}(X) \) is isomorphic to the MacNeille completion of \( B \). Moreover, for \( U, V \in \mathcal{RO}(X) \), there are \( a, b \in B \) with \( a \prec b \), \( U \subseteq \beta(a) \), and \( \beta(b) \subseteq V \). Thus, it is possible to prove Theorem 5.10(1) without using the de Vries representation of compingent algebras. Namely, for \((B, \prec) \in \text{Com} \) let \( \overline{B} \) be the MacNeille completion of \( B \). By identifying \( B \) with its image, we may view \( B \) as a subalgebra of \( \overline{B} \), and define \( \prec \) on \( \overline{B} \) by setting

\[
x \prec y \text{ iff there exist } a, b \in B \text{ such that } x \leq a \prec b \leq y.
\]

A direct verification shows that \((\overline{B}, \prec) \in \text{Dev} \), which yields a point-free proof of Theorem 5.10(1); see \([4]\) Lem. 6.5] for details.
6 \(\Pi_2\)-rules, admissibility, and further completeness results

6.1 \(\Pi_2\)-rules

As we saw in the previous section, \(S^2IC\) is strongly sound and complete with respect to \(\text{DeV}\), and hence is the strict implication logic of compact Hausdorff models. Note that neither (I6) nor (I7) is expressible in our language. In this section we show that we can express (I6) and (I7) by means of \(\Pi_2\)-rules.

We first rewrite (I6) and (I7) as \(\forall\exists\)-statements:

\[
\begin{align*}
\text{(I6)} & \forall x_1, x_2, y \left( x_1 \rightsquigarrow x_2 \not\approx y \rightarrow \exists z : (x_1 \rightsquigarrow z) \land (z \rightsquigarrow x_2) \not\approx y \right); \\
\text{(I7)} & \forall x, y \left( x \not\approx y \rightarrow \exists z : z \land (z \rightsquigarrow x) \not\approx y \right)
\end{align*}
\]

**Lemma 6.1.** Let \((B, \rightsquigarrow) \in \text{RSub}\).

1. \((B, \rightsquigarrow) \models (I6)\) if and only if \((B, \rightsquigarrow) \models (\Pi 6)\).
2. \((B, \rightsquigarrow) \models (I7)\) if and only if \((B, \rightsquigarrow) \models (\Pi 7)\).

**Proof.** (1) \(\Rightarrow\) Suppose \((B, \rightsquigarrow) \models (I6)\). Let \(a, b, d \in B\) be such that \(a \rightsquigarrow b \not\approx d\). Then \(d \neq 1\) and \(a \rightsquigarrow b \neq 0\). Therefore, there is \(c \in B\) such that \(a \rightsquigarrow c \land (c \rightsquigarrow b) \not\approx d\). Thus, \((B, \rightsquigarrow) \models (\Pi 6)\).

(\(\Leftarrow\)) Suppose \((B, \rightsquigarrow) \models (\Pi 6)\). Let \(a, b \in B\) be such that \(a \rightsquigarrow b = 1\). Then \(a \rightsquigarrow b \not\approx 0\). By (I6), there is \(c \in B\) such that \((a \rightsquigarrow c) \land (c \rightsquigarrow b) \not\approx 0\). Therefore, \((B, \rightsquigarrow) \in \text{RSub}\), we have \(a \rightsquigarrow c = c \rightsquigarrow b = 1\). Thus, \((B, \rightsquigarrow) \models (I6)\).

(2) \(\Rightarrow\) Suppose \((B, \rightsquigarrow) \models (I7)\). Let \(a, c \in B\) be such that \(a \not\approx c\). Then \(a \land \neg c \neq 0\). By (I7), there is \(b \neq 0\) such that \(b \rightsquigarrow (a \land \neg c) = 1\). By (I3), \(b \rightsquigarrow (a \land \neg c) = (b \rightsquigarrow a) \land (b \rightsquigarrow \neg c)\). Therefore, \(b \rightsquigarrow a = 1\) and \(b \rightsquigarrow \neg c = 1\). The latter equality, by (I4), yields \(b \leq \neg c\). Since \(b \neq 0\), we must have \(b \not\approx c\). Thus, we have found \(b \in B\) such that \(b \land (b \rightsquigarrow a) \not\approx c\). This shows that \((B, \rightsquigarrow) \models (\Pi 7)\).

(\(\Leftarrow\)) Suppose \((B, \rightsquigarrow) \models (\Pi 7)\). Let \(a \neq 0\) be an element of \(B\). By (I7), there is \(b \in B\) such that \(b \land (b \rightsquigarrow a) \not\approx 0\). Therefore, \(b \neq 0\) and \(b \rightsquigarrow a = 1\). Thus, \((B, \rightsquigarrow) \models (I7)\). \(\square\)

We next show that \(\forall\exists\)-statements can be expressed by means of non-standard rules, which we call \(\Pi_2\)-rules. The use of non-standard rules in modal logic is not new. One of the pioneers of this approach was Gabbay [14], who introduced a non-standard rule for irreflexivity. A precursor to this work was Burgess [7] who used such rules in the context of branching time logic. We also refer to [15] for the application of non-standard rules to axiomatize the logic of the real line in the language with the Since and Until modalities, and to [23] for a general completeness result for modal languages that are sufficiently expressive to define the so-called difference modality. Our approach is closest to that of Balbiani et al. [1] who use similar non-standard rules in the context of region-based theories of space.

**Definition 6.2** (\(\Pi_2\)-rule). A \(\Pi_2\)-rule is a rule of the form

\[
\begin{array}{c}
(\rho) \frac{F(\overline{\varphi}, \overline{\rho}) \rightarrow \chi}{G(\overline{\varphi}) \rightarrow \chi}
\end{array}
\]

where \(F, G\) are formulas, \(\overline{\varphi}\) is a tuple of formulas, \(\chi\) is a formula, and \(\overline{\rho}\) is a tuple of propositional letters which do not occur in \(\overline{\varphi}\) and \(\chi\).
To each \( \Pi_2 \)-rule \( \rho \), we associate the \( \forall \exists \)-statement

\[
\Pi(\rho) := \forall \pi, z \left( G(\pi) \not\rightarrow z \rightarrow \exists y : F(\pi, y) \not\rightarrow z \right).
\]

**Definition 6.3.** We say that a strict implication algebra \((B, \bowtie)\) validates a \( \Pi_2 \)-rule \( \rho \), and write \((B, \bowtie) \models \rho\), provided \((B, \bowtie)\) satisfies \( \Pi(\rho) \).

Consider the \( \Pi_2 \)-rules:

\[
(\rho 6) \quad \frac{\varphi \bowtie p \land (p \bowtie \psi) \rightarrow \chi}{(\varphi \bowtie \psi) \rightarrow \chi};
\]

\[
(\rho 7) \quad \frac{p \land (p \bowtie \varphi) \rightarrow \chi}{\varphi \rightarrow \chi}.
\]

It is easy to see that \( \Pi(\rho 6) = (\Pi 6) \) and \( \Pi(\rho 7) = (\Pi 7) \), so by Lemma 6.1, for each \((B, \bowtie) \in \text{RSub}\), we have:

\[
(B, \bowtie) \models (\rho 6) \iff (B, \bowtie) \models (16);
\]

\[
(B, \bowtie) \models (\rho 7) \iff (B, \bowtie) \models (17).
\]

**Definition 6.4** (Proofs with \( \Pi_2 \)-rules). Let \( \Sigma \) be a set of \( \Pi_2 \)-rules. For a set of formulas \( \Gamma \) and a formula \( \varphi \), we say that \( \varphi \) is derivable from \( \Gamma \) in \( \text{SIC} \) using the \( \Pi_2 \)-rules in \( \Sigma \), and write \( \Gamma \vdash_{\Sigma} \varphi \), provided there is a proof \( \psi_1, \ldots, \psi_n \) such that \( \psi_n = \varphi \) and each \( \psi_i \) is in \( \Gamma \), an instance of an axiom of \( \text{SIC} \), obtained either by (MP) or (N) from some previous \( \psi_j \)'s, or there is \( j < i \) such that \( \psi_i \) is obtained from \( \psi_j \) by an application of one of the \( \Pi_2 \)-rules \( \rho \in \Sigma \); that is, \( \psi_j = F(\xi, \overline{p}) \rightarrow \chi \) and \( \psi_i = G(\overline{\xi}) \rightarrow \chi \), where \( F, G \) are formulas, \( \overline{\xi} \) is a tuple of formulas, \( \chi \) is a formula, and \( \overline{p} \) is a tuple of fresh propositional letters not occurring in any of the formulas from \( \Gamma \) that are involved in the proof as assumptions.

We next show that the deduction theorem remains true when proofs also involve \( \Pi_2 \)-rules.

**Lemma 6.5.** For any set \( \Gamma \) of formulas and for any formulas \( \varphi, \psi \), we have:

\[
\Gamma \cup \{ \varphi \} \vdash_{\Sigma} \psi \iff \Gamma \vdash_{\Sigma} \Box \varphi \rightarrow \psi.
\]

*Proof.* (\( \Leftarrow \)) Same as the corresponding proof of Lemma 6.1.

(\( \Rightarrow \)) The only step that is not covered in the corresponding proof of Lemma 6.1 is the step of applying some \( \Pi_2 \)-rule \( \rho \in \Sigma \): Suppose there is \( j < i \) such that \( \psi_j = F(\xi, \overline{p}) \rightarrow \chi \) and \( \psi_i = G(\overline{\xi}) \rightarrow \chi \) for some formulas \( F, G \), a tuple of formulas \( \overline{\xi} \), a formula \( \chi \), and a tuple \( \overline{p} \) of fresh propositional letters not occurring in any of the formulas involved in the proof. By inductive hypothesis, \( \Gamma \vdash_{\Sigma} \Box \varphi \rightarrow (F(\xi, \overline{p}) \rightarrow \chi) \). Thus, \( \Gamma \vdash_{\Sigma} F(\overline{\xi}, \overline{p}) \rightarrow (\Box \varphi \rightarrow \chi) \). Applying \( \rho \) yields \( \Gamma \vdash_{\Sigma} G(\overline{\xi}) \rightarrow (\Box \varphi \rightarrow \chi) \). From this we conclude that \( \Gamma \vdash_{\Sigma} \Box \varphi \rightarrow (G(\overline{\xi}) \rightarrow \chi) \), as desired. \( \square \)

Let \( S \) be the system obtained by adding the \( \Pi_2 \)-rules \( \{ \rho_n \mid n \in \mathbb{N} \} \) to \( \text{SIC} \). Let also \( U \) be the inductive subclass of \( \text{RSub} \) defined by the \( \forall \exists \)-statements \( \{ \Pi(\rho_n) \mid n \in \mathbb{N} \} \). We next show that \( S \) is strongly sound and complete with respect to \( U \). The proof is a modification of the standard Lindenbaum construction (see, e.g., [20]). The modification follows similar pattern as the one given in [1] Lem. 7.10.
Theorem 6.6. Let $S = \text{SIC} + \{\rho_n \mid n \in \mathbb{N}\}$, let $\mathcal{U}$ be the inductive subclass of $\text{RSub}$ defined by $\{\Pi(\rho_n) \mid n \in \mathbb{N}\}$, and let $\mathcal{V}$ be the variety generated by $\mathcal{U}$. For a set of formulas $\Gamma$ and a formula $\varphi$, we have:

1. $\Gamma \vdash_S \varphi \iff \Gamma \models_{\mathcal{U}} \varphi$.
2. $\vdash_S \varphi \iff \models_{\mathcal{V}} \varphi$.

Proof. (1) That $\Gamma \vdash_S \varphi \Rightarrow \Gamma \models_{\mathcal{U}} \varphi$ is a straightforward inductive proof on the length of derivations. We only consider the case of $\Pi_2$-rules.

Suppose $\rho$ is a $\Pi_2$-rule as defined in Definition 6.2. $v$ is a valuation into $(B, \leadsto) \in \text{SIA}$ satisfying $\Pi(\rho)$, and $v(\Gamma) = 1$. We may assume without loss of generality (by re-enumerating all the propositional letters if need be) that the propositional letters $\overline{p}$ do not occur in any of the formulas in $\Gamma$. If $G(v(\overline{x})) \not\models v(\chi)$, then since $(B, \leadsto)$ satisfies $\Pi(\rho)$, there is a tuple $\overline{c}$ in $B$ such that $F(v(\overline{x}), \overline{c}) \not\subseteq v(\chi)$. Consider the valuation $v'$ which coincides with $v$ everywhere, except maps $\overline{p}$ to $\overline{c}$. Then $v'(F(\overline{x}, \overline{p})) = F(v(\overline{x}), \overline{c}) \not\subseteq v'(\chi) = v'(\chi)$, so $v'(F(\overline{x}, \overline{p}) \leadsto \chi) \neq 1$. Since $v'$ coincides with $v$ on all propositional letters except $\overline{p}$ and since we assumed that the $\overline{p}$ do not occur in $\Gamma$, we have $v'(\Gamma) = v(\Gamma) = 1$. So we have found a valuation $v'$ such that $v'(\Gamma) = 1$ and $v'(F(\overline{x}, \overline{p}) \leadsto \chi) \neq 1$, contradicting the inductive hypothesis.

To complete the proof of (1), it remains to show that $\Gamma \not\vdash_{\mathcal{U}} \varphi \Rightarrow \Gamma \not\models_{\mathcal{U}} \varphi$. We do this by slightly modifying the construction of the Lindenbaum algebra. Suppose $\Gamma \not\vdash_{\mathcal{U}} \varphi$. For each rule $\rho_i$, we add a countably infinite set of fresh propositional letters to the set of existing propositional letters, build the Lindenbaum algebra $(B, \leadsto)$ over the expanded set of propositional letters, and construct a maximal $\Box$-filter $M$ of $(B, \leadsto)$ such that $\{[\psi] \mid\psi \in \Gamma\} \cup \{[\neg \Box \varphi]\} \subseteq M$ and for every rule $\rho_i$ and formulas $\overline{p}, \chi$:

(i) if $[G_i(\overline{p}) \leadsto \chi] \not\in M$, then there is a tuple $\overline{p}$ such that $[F_i(\overline{p}, \overline{p}) \leadsto \chi] \not\in M$.

To construct $M$, let $A_0 := \Gamma \cup \{\neg \Box \varphi\}$, a consistent set. We enumerate all formulas $\varphi$ and all tuples $(i, \overline{p}, \chi)$, and we build the sets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ as follows:

- For $n = 2k$, if $A_n \not\vdash \Box \varphi_k$, let $A_{n+1} = A_n \cup \{\neg \Box \varphi_k\}$; otherwise let $A_{n+1} = A_n$.
- For $n = 2k + 1$, let $(i, \overline{p}, \chi)$ be the $k$-th tuple. If $A_n \not\vdash G_i(\overline{p}) \leadsto \chi$, let $A_{n+1} = A_n \cup \{\neg \Box (F_i(\overline{p}, \overline{p}) \leadsto \chi)\}$, where $\overline{p}$ is a tuple of propositional letters for $\rho_i$ not occurring in $\overline{p}, \chi$, and any of $\psi$ with $\psi \in A_n$ (we can take $\overline{p}$ from the countably infinite additional propositional letters which we have reserved for the rule $\rho_i$). Otherwise, let $A_{n+1} = A_n$.

We let $S_A = \{\psi \mid \exists n \in \omega \text{ with } A_n \vdash \psi\}$, and define $M = \{[\psi] \mid \psi \in S_A\}$. It is easy to see that $S_A \vdash \varphi$ implies $\varphi \in S_A$, so $M$ is a $\Box$-filter. Moreover, all $A_n$ are consistent, and hence so is $S_A$. This implies that $M$ is a proper $\Box$-filter. Thus, by the even steps of the construction of the sets $A_n$, and by Lemma 4.10(1), $M$ is a maximal $\Box$-filter.

Because $A \subseteq S_A$, we have $\{[\psi] \mid \psi \in \Gamma\} \cup \{[\neg \Box \varphi]\} \subseteq M$. Finally, the odd steps of the construction of the sets $A_n$ ensure that $M$ satisfies (i). Therefore, we can conclude that $M$ satisfies all the desired properties.

By (i), the quotient of $(B, \leadsto)$ by $M$ satisfies each $\Pi(\rho_i)$. By Lemma 4.10(1), the quotient belongs to $\text{RSub}$. Therefore, the quotient belongs to $\mathcal{U}$. Moreover, since $[\neg \Box \varphi] \in M$, we have
that \([-\Box \varphi]\) maps to 1, so \([\Box \varphi]\) maps to 0 in the quotient. Thus, \([\varphi]\) does not map to 1 in the quotient, while \(\Gamma\) does, and hence \(\Gamma \not\models \varphi\).

(2) Observe that \(\mathcal{U}\) consists of the subdirectly irreducible members of \(\mathcal{V}\), and apply (1).  

It follows that the class of subdirectly irreducible algebras in \(\mathsf{SIA}\) validating a set of \(\Pi_2\)-rules is an inductive subclass of \(\mathsf{RSub}\). We next show that the converse is also true. Namely, for every inductive subclass \(\mathcal{U}\) of \(\mathsf{RSub}\), there is a set of \(\Pi_2\)-rules \(\{\rho_i \mid i \in I\}\) such that 
\[
\mathcal{S} = \mathsf{SIC} + \{\rho_i \mid i \in I\}
\]

is strongly sound and complete with respect to \(\mathcal{U}\). To obtain such a set of \(\Pi_2\)-rules, it is sufficient to show that every \(\Pi_2\)-statement is equivalent to a statement of the form \(\Pi(\rho)\) for some \(\Pi_2\)-rule \(\rho\). Without loss of generality we may assume that all atomic formulas \(\Phi(\overline{x}, \overline{y})\) are of the form \(t(\overline{x}, \overline{y}) = 1\) for some term \(t\).

**Definition 6.7.** Given a quantifier-free first-order formula \(\Phi(\overline{x}, \overline{y})\), we associate with the tuples of variables \(\overline{x}, \overline{y}\) the tuples of propositional letters \(\overline{p}, \overline{q}\), and define the formula \(\Phi^*(\overline{p}, \overline{q})\) in the language of \(\mathsf{SIC}\) as follows:

\[
(t(\overline{x}, \overline{y}) = 1)^* = 1 \sim t(\overline{p}, \overline{q}) \\
(\neg \Psi)^*(\overline{x}, \overline{y}) = \neg \Psi^*(\overline{p}, \overline{q}) \\
(\Psi_1(\overline{x}, \overline{y}) \land \Psi_2(\overline{x}, \overline{y}))^* = \Psi_1^*(\overline{p}, \overline{q}) \land \Psi_2^*(\overline{p}, \overline{q})
\]

**Lemma 6.8.** Let \((B, \sim) \in \mathsf{RSub}\) and \(\Phi(\overline{x}, \overline{y})\) be a quantifier-free formula.

1. \((B, \sim)\) satisfies \(\Phi(\overline{x}, \overline{y})\) iff \((B, \sim)\) satisfies the formula \(\Phi^*(\overline{p}, \overline{q})\).

2. \((B, \sim)\) satisfies \(\forall \overline{x} \exists \overline{y} \Phi(\overline{x}, \overline{y})\) iff \((B, \sim)\) satisfies \(\forall \overline{x}, \overline{z} (1 \not\preceq \overline{z} \rightarrow \exists \overline{y} : \Phi^*(\overline{x}, \overline{y}) \preceq \overline{z})\).

**Proof.** (1) For each term \(t(\overline{x}, \overline{y})\), we evaluate \(\overline{x}, \overline{p}\) as \(\overline{x}\) and \(\overline{p}, \overline{y}\) as \(\overline{b}\). It is obvious that \(t(\overline{x}, \overline{b}) = 1\) implies \(1 \sim t(\overline{p}, \overline{b}) = 1\), and \(t(\overline{p}, \overline{b}) \not= 1\) implies \(1 \sim t(\overline{p}, \overline{q}) = 0\). This shows the equivalence for atomic formulas, and an easy induction then proves it for all quantifier-free formulas.

2. (\(\Rightarrow\)) Suppose \((B, \sim) \models \forall \overline{x} \exists \overline{y} \Phi(\overline{x}, \overline{y})\). Let \(\bar{a}\) be a tuple of elements of \(B\) and \(c \in B\). By assumption, there exists a tuple \(\overline{b}\) in \(B\) such that \((B, \sim) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]\). Therefore, if \(1 \not\preceq c\), then \(\Phi^*(\overline{a}, \overline{b}) = 1 \not\preceq c\). Thus, \((B, \sim) \models \forall \overline{x}, \overline{z} (1 \not\preceq \overline{z} \rightarrow \exists \overline{y} : \Phi^*(\overline{x}, \overline{y}) \not\preceq \overline{z})\).

(\(\Leftarrow\)) Suppose \((B, \sim) \models \forall \overline{x}, \overline{z} (1 \not\preceq \overline{z} \rightarrow \exists \overline{y} : \Phi^*(\overline{x}, \overline{y}) \not\preceq \overline{z})\). Let \(\bar{a}\) be a tuple of elements of \(B\). Since \(1 \not\preceq 0\), there exists a tuple \(\overline{b}\) in \(B\) such that \(\Phi^*(\overline{a}, \overline{b}) \not= 0\). Therefore, since \(\Phi^*(\overline{a}, \overline{b})\) evaluates only to 0 or 1, we obtain \(\Phi^*(\overline{a}, \overline{b}) = 1\). Thus, \((B, \sim) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]\). This shows that \((B, \sim) \models \forall \overline{x} \exists \overline{y} \Phi(\overline{x}, \overline{y})\). \(\square\)

Consequently, an arbitrary \(\Pi_2\)-statement \(\forall \overline{x} \exists \overline{y} \Phi(\overline{x}, \overline{y})\) is equivalent to the \(\Pi_2\)-statement associated to the \(\Pi_2\)-rule

\[
\begin{array}{c}
\rho \Phi \\
\Phi^*(\overline{p}, \overline{q}) \rightarrow \chi
\end{array}
\]

Thus, by Theorem 6.6 we obtain:

**Theorem 6.9.** If \(T\) is a \(\Pi_2\)-theory of first-order logic axiomatizing an inductive subclass \(\mathcal{U}\) of \(\mathsf{RSub}\), then the system \(\mathcal{S} = \mathsf{SIC} + \{\rho \Phi \mid \Phi \in T\}\) is strongly sound and complete with respect to \(\mathcal{U}\); that is, for a set of formulas \(\Gamma\) and a formula \(\varphi\), we have:

\[
\Gamma \vdash_{\mathcal{S}} \varphi \iff \Gamma \models_{\mathcal{U}} \varphi.
\]
6.2 Admissibility of $\Pi_2$-rules

By Theorem 5.8, $S^2IC$ is strongly sound and complete with respect to $Com$. On the other hand, it follows from Theorem 6.9 that $S^2IC$ together with $(\rho 6)$ and $(\rho 7)$ is also sound and complete with respect to $Com$. Therefore, the rules $(\rho 6)$ and $(\rho 7)$ are admissible in $S^2IC$. We next give a general criterion of admissibility for $\Pi_2$-rules in $SIC$ and $S^2IC$. This yields an alternative proof that $(\rho 6)$ and $(\rho 7)$ are admissible in $S^2IC$.

**Definition 6.10.** A rule $\rho$ is admissible in a system $S$ if for each formula $\varphi$, from $\vdash_{S+\rho} \varphi$ it follows that $\vdash_S \varphi$.

**Lemma 6.11.** If a $\Pi_2$-rule

\[
\begin{array}{c}
(\rho) \\
F(\overline{\varphi}, \overline{p}) \rightarrow \chi \\
G(\overline{\varphi}) \rightarrow \chi
\end{array}
\]

is admissible in $S \supseteq SIC$, then $\Gamma \vdash_{S+\rho} \varphi \iff \Gamma \vdash_S \varphi$.

**Proof.** It suffices to show that for any set of formulas $\Gamma$ and any tuple $\overline{\varphi}, \chi$ of formulas, if $\Gamma \vdash_{SIC} F(\overline{\varphi}, \overline{p}) \rightarrow \chi$ and $\overline{p}$ does not appear in $\Gamma, \overline{\varphi}, \chi$, then $\Gamma \vdash_{SIC} G(\overline{\varphi}) \rightarrow \chi$.

Suppose $\Gamma \vdash_{SIC} F(\overline{\varphi}, \overline{p}) \rightarrow \chi$ and $\overline{p}$ does not appear in $\Gamma, \overline{\varphi}, \chi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{SIC} F(\overline{\varphi}, \overline{p}) \rightarrow \chi$. Let $\psi = \bigwedge \Gamma_0$, so $\{ \psi \} \vdash_{SIC} F(\overline{\varphi}, \overline{p}) \rightarrow \chi$. By Theorem 4.4, $\vdash_{SIC} \Box \psi \rightarrow (F(\overline{\varphi}, \overline{p}) \rightarrow \chi)$, so $\vdash_{SIC} F(\overline{\varphi}, \overline{p}) \rightarrow (\Box \psi \rightarrow \chi)$. Since $\overline{p}$ does not appear in $\overline{\varphi}, \Box \psi \rightarrow \chi$, by admissibility of $\rho$, we have $\vdash_{SIC} G(\overline{\varphi}) \rightarrow (\Box \psi \rightarrow \chi)$. Therefore, $\vdash_{SIC} \Box \psi \rightarrow (G(\overline{\varphi}) \rightarrow \chi)$, and applying Theorem 4.4 again yields $\{ \psi \} \vdash_{SIC} G(\overline{\varphi}) \rightarrow \chi$. Thus, $\Gamma \vdash_{SIC} G(\overline{\varphi}) \rightarrow \chi$. $\square$

**Theorem 6.12** (Admissibility Criterion).

1. A $\Pi_2$-rule $\rho$ is admissible in $SIC$ iff for each $(B, \rightsquigarrow) \in RSub$ there is $(C, \rightsquigarrow) \in RSub$ such that $(B, \rightsquigarrow)$ is a substructure of $(C, \rightsquigarrow)$ and $(C, \rightsquigarrow) \models \Pi(\rho)$.

2. A $\Pi_2$-rule $\rho$ is admissible in $S^2IC$ iff for each $(B, \rightsquigarrow) \in Con$ there is $(C, \rightsquigarrow) \in Con$ such that $(B, \rightsquigarrow)$ is a substructure of $(C, \rightsquigarrow)$ and $(C, \rightsquigarrow) \models \Pi(\rho)$.

**Proof.** (1) ($\Rightarrow$) Suppose $\rho$ is admissible in $SIC$. It is sufficient to show that there exists a model $(C, \rightsquigarrow)$ of the theory

\[ T = Th(RSub) \cup \{ \Pi(\rho) \} \cup \Delta_0(B, \rightsquigarrow) \]

where $\Delta_0(B, \rightsquigarrow)$ is the positive diagram of $(B, \rightsquigarrow)$ ([10] p. 70]. Suppose for a contradiction that $T$ has no models, hence is inconsistent. Then, by compactness, there exists a quantifier-free first-order formula $\Psi(\overline{a})$ and a tuple $\overline{a}$ of propositional letters corresponding to $\overline{a} \in B$ such that

\[ Th(RSub) \cup \{ \Pi(\rho) \} \models \neg \Psi(\overline{a}) \text{ and } (B, \rightsquigarrow) \models \Psi(\overline{a}). \]

We enrich the language of $SIC + \rho$ with $\{ \overline{a} \}$. By Theorem 6.6, $SIC + \rho$ is complete with respect to the algebras in $RSub$ satisfying $\Pi(\rho)$. Therefore, by Lemma 6.8 (1), $\vdash_{SIC + \rho} (\neg \Psi(\overline{a}))^*$ where $(\cdot)^*$ is the translation given in Definition 6.7. By admissibility, $\vdash_{SIC} (\neg \Psi(\overline{a}))^*$. Thus, for each valuation $v$ that maps $p_a$ to $a$, we have $v((\neg \Psi(\overline{a}))^*) = 1$, so $v((\Psi(\overline{a}))^*) = 0$. This
contradicts the fact that \((B, \rightsquigarrow) \models \Psi(\overline{a})\). Consequently, \(T\) must be consistent, and hence it has a model.

\((\Leftarrow)\) Suppose \(\vdash_{\text{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi\) with \(\overline{p}\) not occurring in \(\varphi, \chi\). Let \((B, \rightsquigarrow) \in \text{RSub}\) and let \(v\) be a valuation on \((B, \rightsquigarrow)\). By assumption, there is \((C, \rightsquigarrow) \in \text{RSub}\) such that \((B, \rightsquigarrow)\) is a substructure of \((C, \rightsquigarrow)\) and \((C, \rightsquigarrow) \models \Pi(\rho)\). Let \(i : B \leftrightarrow C\) be the inclusion. Then \(v' := i \circ v\) is a valuation on \((C, \rightsquigarrow)\). For any \(\overline{c} \in C\), let \(v''\) be the valuation \((v')^C\). Since \(\vdash_{\text{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi\), by Theorem 4.11, \(v''(F(\overline{\varphi}, \overline{p}) \to \chi) = 1_C\). This means that for all \(\overline{c} \in C\), we have \(F(v'(\overline{\varphi}), \overline{\rho}) \leq v'(\chi)\). Therefore, \((C, \rightsquigarrow) \models \forall \overline{p} (F(v'(\overline{\varphi}), \overline{\rho}) \leq v'(\chi))\). Since \((C, \rightsquigarrow) \models \Pi(\rho)\), we have \((C, \rightsquigarrow) \models G(v'(\overline{\varphi})) \leq v'(\chi)\). Thus, as \(G(v'(\overline{\varphi})) \leq v(\chi)\) in \(B\), we have \(G(v(\overline{\varphi})) \leq v(\chi)\) in \(B\). Consequently, \(v(G(\overline{\varphi}) \to \chi) = 1_B\). Applying Theorem 4.11 again yields that \(\vdash_{\text{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi\), and hence \(\rho\) is admissible in \(\text{SIC}\).

(2) The proof is similar to that of (1) and uses the fact that \(\text{SIC}^2\) is strongly sound and complete with respect to \(\text{Con}\). \(\square\)

**Corollary 6.13.**

(1) \((\rho\overline{6})\) is admissible in \(\text{SIC}^2\).

(2) \((\rho\overline{7})\) is admissible in \(\text{SIC}\) and \(\text{SIC}^2\).

**Proof.** (1) Apply Theorem 6.12(2) and Lemmas 6.1(1) and 5.4.

(2) For admissibility of \((\rho\overline{7})\) in \(\text{SIC}\) apply Theorem 6.12(1) and Lemmas 6.1(2) and 5.6.

For admissibility of \((\rho\overline{7})\) in \(\text{SIC}^2\) apply Theorem 6.12(2) and Lemmas 6.1(2) and 5.6. \(\square\)

**Remark 6.14.** It remains an open problem whether \((\rho\overline{6})\) is admissible in \(\text{SIC}\).

### 6.3 Calculi for zero-dimensional and connected compact Hausdorff spaces

In the remainder of this section, we will consider zero-dimensional and connected compact Hausdorff models, and identify their logics. Starting with zero-dimensionality, we consider the following property, studied in [3]:

(S9) \(a \prec b\) implies \(\exists c : c < c \text{ and } a < c \prec b\).

The corresponding \(\forall \exists\)-statement is

(P9) \(\forall x, y, z \big(x \rightsquigarrow y \not\leq z \rightarrow \exists u : (u \rightsquigarrow u) \land (x \rightsquigarrow u) \land (u \rightsquigarrow y) \not\leq z\).\)

**Lemma 6.15.** Let \((B, \rightsquigarrow) \in \text{Com}\). Then \((B, \rightsquigarrow) \models (S9)\) iff \((B, \rightsquigarrow) \models (P9)\).

**Proof.** (\(\Rightarrow\)) Suppose \(a \rightsquigarrow b \not\leq d\). Then \(d \neq 1\) and \(a \rightsquigarrow b \neq 0\), so \(a \rightsquigarrow b = 1\). Therefore, \(a < b\), and so by (S9), there is \(c\) such that \(c < c\) and \(a < c \prec b\). Thus, \((c \rightsquigarrow c) \land (a \rightsquigarrow c) \land (c \rightsquigarrow b) = 1 \not\leq d\). Consequently, \((B, \rightsquigarrow) \models (P9)\).

(\(\Leftarrow\)) Suppose \(a \prec b\). Then \(a \rightsquigarrow b = 1 \not\leq 0\). Therefore, by (P9), there is \(c\) such that \((c \rightsquigarrow c) \land (a \rightsquigarrow c) \land (c \rightsquigarrow b) \not\leq 0\), which implies \((c \rightsquigarrow c) \land (a \rightsquigarrow c) \land (c \rightsquigarrow b) = 1\). Thus, \(c < c\) and \(a \prec c \prec b\). Consequently, \((B, \rightsquigarrow) \models (S9)\). \(\square\)
The \(\Pi_2\)-rule corresponding to \((\Pi 9)\) is

\[
(p \leadsto p) \land (\varphi \leadsto p) \land (p \leadsto \psi) \rightarrow \chi
\]

\[
(\varphi \leadsto \psi) \rightarrow \chi
\]

**Theorem 6.16.** \((\rho 9)\) is admissible in \(S_2^{IC}\).

**Proof.** Apply Theorem 6.12(2), Lemma 6.15, and an analogue of Lemma 5.4 for \((\rho 9)\) since it is easy to see that the \((C, \prec)\) constructed in the proof of Lemma 5.4 satisfies \((S 9)\).

**Remark 6.17.** It remains an open problem whether \((\rho 9)\) is admissible in \(SIC\) (cf. Remark 6.14).

As a consequence of Theorem 6.16 we obtain:

**Corollary 6.18.** \(S_2^{IC}\) is strongly sound and complete with respect to the class of compingent algebras satisfying \((S 9)\).

Following [3, Def. 4.5], we call a de Vries algebra zero-dimensional if it satisfies \((S 9)\), and denote the class of zero-dimensional de Vries algebras by \(zDeV\). Let \((B, \leadsto) \in \text{Com}\) and \(X\) be the de Vries dual of \((B, \leadsto)\). It follows from de Vries duality and [3, Lem. 4.11] that \(X\) is zero-dimensional, and hence \((RO(X), \leadsto)\) is a zero-dimensional de Vries algebra by [3, Lem. 4.1]. We recall from Section 2 that zero-dimensional compact Hausdorff spaces are called Stone spaces, and denote the class of Stone spaces by \(\text{Stone}\). As a consequence of Corollary 6.18 we have:

**Theorem 6.19.**

1. \(S_2^{IC}\) is strongly sound and complete with respect to \(zDeV\).
2. \(S_2^{IC}\) is strongly sound and complete with respect to \(\text{Stone}\).

Turning to connectedness, consider the following property:

\[
(S 10)\ a \prec a \text{ implies } a = 0 \text{ or } a = 1.
\]

Clearly \((B, \leadsto) \in \text{Com}\) satisfies \((S 10)\) iff \(a \leadsto a \leq \square a \lor \square \neg a\) holds in \((B, \leadsto)\). Therefore, \((B, \leadsto)\) satisfies \((S 10)\) iff \((B, \leadsto) \models (C)\), where \((C)\) is the formula

\[
(C) (\varphi \leadsto \varphi) \rightarrow (\square \varphi \lor \square \neg \varphi).
\]

**Definition 6.20.** The connected symmetric strict implication calculus \(CS_2^{IC}\) is the extension of \(S_2^{IC}\) with the axiom \((C)\).

We call \((A, \leadsto) \in S_2^{IA}\) connected if \((A, \leadsto)\) satisfies \(a \leadsto a \leq \square a \lor \square \neg a\) for each \(a \in A\). Let \(CS_2^{IA}\) be the subvariety of \(S_2^{IA}\) consisting of connected symmetric strict implication algebras. We also call a compingent algebra connected if it satisfies \((S 10)\), and denote the class of connected compingent algebras by \(\text{CCom}\). As a simple consequence of Theorems 5.2 and 6.6 we have:

**Corollary 6.21.**
(1) $\text{CS}^2\text{IC}$ is strongly sound and complete with respect to $\text{CS}^2\text{IA}$.

(2) $\text{CS}^2\text{IC}$ is strongly sound and complete with respect to $\text{CCom}$.

**Lemma 6.22.** A compingent algebra $(B, \prec)$ satisfies (S10) iff its dual compact Hausdorff space $X$ is connected.

**Proof.** If $X$ is connected, then $\emptyset, X$ are the only clopen subsets of $X$. Therefore, for $U \in \mathcal{RO}(X)$, we have $U \prec U$ implies $U = \emptyset$ or $U = X$. Thus, $(\mathcal{RO}(X), \prec)$ satisfies (S10). Since $(B, \prec)$ is isomorphic to a subalgebra of $(\mathcal{RO}(X), \prec)$, we conclude that $(B, \prec)$ satisfies (S10).

Conversely, let $U$ be clopen in $X$. Then, $U = \bigcup \{\beta(a) : \beta(a) \subseteq U\}$. Since $U$ is regular open, the family $\{\beta(a) : \beta(a) \subseteq U\}$ is up-directed. Because $U$ is compact, there is $a \in B$ such that $\beta(a) = U$. As $\beta : B \to \mathcal{RO}(X)$ is an embedding and $(B, \prec)$ satisfies (S10), $U = \beta(0) = \emptyset$ or $U = \beta(1) = X$. Thus, $X$ is connected.

As an immediate consequence we obtain:

**Lemma 6.23.** A de Vries algebra $(B, \prec)$ satisfies (S10) iff its de Vries dual $X$ is connected.

We call a de Vries algebra *connected* if it satisfies (S10), and denote the class of connected de Vries algebras by $\text{cDeV}$. We also let $\text{cKHaus}$ be the class of connected compact Hausdorff spaces. Then Corollary 6.21 and Lemma 6.23 imply:

**Theorem 6.24.**

(1) $\text{CS}^2\text{IC}$ is strongly sound and complete with respect to $\text{cDeV}$.

(2) $\text{CS}^2\text{IC}$ is strongly sound and complete with respect to connected compact Hausdorff models.

The table below summarizes our completeness results.

<table>
<thead>
<tr>
<th>Logic</th>
<th>Complete with respect to</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIC</td>
<td>SIA; RSub</td>
</tr>
<tr>
<td>S^2IC</td>
<td>S^2IA; Con; Com; Dev; zDev KHaus; Stone</td>
</tr>
<tr>
<td>CS^2IC</td>
<td>CS^2IA; CCom; cDeV; cKHaus</td>
</tr>
</tbody>
</table>

**Remark 6.25.** By Theorems 5.2 and 5.10 $\text{S^2IC}$ is strongly sound and complete with respect to Con and Dev. Thus, the logic of contact algebras is the same as the logic of de Vries algebras. On the other hand, the $\Pi_2$-theories (the sets of valid $\Pi_2$-rules) of Con and Dev are obviously different—the $\Pi_2$-rules $(\rho 6)$ and $(\rho 7)$ belong to the latter but not to the former. These two rules capture the very essence of the theory of compact Hausdorff spaces in our language. This generates an interesting methodological question of what the right logical formalism should be to reason about compact Hausdorff spaces. Should we be concerned only with the logics or should we also consider the theories of $\Pi_2$-rules? Although in this paper we are only concerned with logics, our results suggest that a theory of $\Pi_2$-rules may be a more appropriate framework to reason about compact Hausdorff spaces. We leave it as a future work to develop the $\Pi_2$-theory for compact Hausdorff spaces together with the general theory of such calculi.
7 Comparison with relevant work

In this final section we compare our approach to that of Balbiani et al. [1]. Namely, we show how to translate fully and faithfully the language $L(C, \leq)$ of [1] into our language. We start by recalling the concept of contact relation, one of the key concepts of region-based theory of space; see, e.g., [22]. A binary relation $\mathcal{C}$ on a Boolean algebra $B$ is a precontact relation if it satisfies:

\begin{enumerate}
  \item[(C1)] $a \mathcal{C} b \Rightarrow a, b \neq 0$.
  \item[(C2)] $a \mathcal{C} (b \lor c) \Leftrightarrow a \mathcal{C} b$ or $a \mathcal{C} c$.
  \item[(C3)] $(a \lor b) \mathcal{C} c \Leftrightarrow a \mathcal{C} c$ or $b \mathcal{C} c$.
\end{enumerate}

A precontact relation is a contact relation if it satisfies:

\begin{enumerate}
  \item[(C4)] $a \neq 0$ implies $a \mathcal{C} a$.
  \item[(C5)] $a \mathcal{C} b$ implies $b \mathcal{C} a$.
\end{enumerate}

As was pointed out in [5, Sec. 2], there is a one-to-one correspondence between subordinations and precontact relations. If $\mathcal{C}$ is a contact relation, then the relation $\prec_\mathcal{C}$ defined by $a \prec_\mathcal{C} b$ if $a \mathcal{C} \neg b$ is a subordination. Conversely, if $\prec$ is a subordination, then the relation $\mathcal{C}_\prec$ defined by $a \mathcal{C}_\prec b$ if $a \prec \neg b$ is a precontact relation, and this correspondence is one-to-one. Moreover, a subordination $\prec$ satisfies (S5) iff the corresponding precontact relation $\mathcal{C}_\prec$ satisfies (C4), and $\prec$ satisfies (S6) iff $\mathcal{C}_\prec$ satisfies (C5).

On regular open sets of a compact Hausdorff space $X$ the contact relation is defined by $U \mathcal{C} V$ iff $\text{Cl}(U) \cap \text{Cl}(V) \neq \emptyset$. If $R$ is a reflexive and symmetric relation on a set $X$, then the contact relation $\mathcal{C}_R$ is defined on $\mathcal{P}(X)$ by $U \mathcal{C}_R V$ iff $R[U] \cap V \neq \emptyset$.

We next recall that the formulas of the language $L(C, \leq)$ are built from atomic formulas using Boolean connectives $\neg, \land, \lor, \rightarrow, \bot, \top$; atomic formulas are of the form $t \mathcal{C} s$ and $t \leq s$, where $t, s$ are Boolean terms ($\mathcal{C}$ stands for the contact relation and $\leq$ for the inclusion relation). In turn, Boolean terms are built from Boolean variables using Boolean operations $\cap, \cup, (-)^*$, $0, 1$.

As usual, a Kripke frame is a pair $(W, R)$, where $W$ is a nonempty set and $R$ is a binary relation on $W$, and a valuation is a map $v$ from the set of Boolean variables to the powerset $\mathcal{P}(W)$. It extends to the set of all Boolean terms as follows:

\[
\begin{align*}
    v(t \cap s) &= v(t) \cap v(s), \\
    v(t \cup s) &= v(t) \cup v(s), \\
    v(t^*) &= W \setminus v(t), \\
    v(0) &= \emptyset, \\
    v(1) &= W.
\end{align*}
\]

A Kripke model is a triple $(W, R, v)$ consisting of a Kripke frame $(W, R)$ and a valuation $v$. Atomic formulas are interpreted in $(W, R, v)$ as follows:

\[
\begin{align*}
    (W, R, v) \models (t \leq s) &\Leftrightarrow v(t) \subseteq v(s), \\
    (W, R, v) \models (t \mathcal{C} s) &\Leftrightarrow R[v(t)] \cap v(s) \neq \emptyset.
\end{align*}
\]
Complex formulas are then interpreted by the induction clauses for propositional connectives.

In [1, Sec. 6] the authors define the propositional calculus $PWRCC$ in the language $L(C, \leq)$ and prove that $PWRCC$ is sound and complete with respect to the class of Kripke frames where the binary relation $R$ is reflexive and symmetric. Such Kripke frames are closely related to contact algebras. Namely, as we already pointed out in Section 5, the following lemma holds.

**Lemma 7.1.**

1. Suppose $(W, R)$ is a reflexive and symmetric Kripke frame. Define $\prec_R$ on $\mathcal{P}(W)$ by $U \prec_R V$ iff $R[U] \subseteq V$. Then $(\mathcal{P}(W), \prec_R)$ is a contact algebra.

2. Suppose $(B, \prec)$ is a contact algebra and $(X, R)$ is the dual of $(B, \prec)$. Then $(X, R)$ is a reflexive and symmetric Kripke frame, and the Stone map $\beta : B \to \mathcal{P}(X)$, given by $\beta(a) = \{x \in X \mid a \in x\}$, is an embedding of $(B, \prec)$ into $(\mathcal{P}(X), \prec_R)$.

We next translate $L(C, \leq)$ into our language $L$. We identify the set of Boolean variables of $L(C, \leq)$ with the set of propositional letters of $L$. Then Boolean terms can be translated into formulas of $L$ as follows:

$$a^T = a,$$

for a Boolean variable $a$,

$$(t \land s)^T = t^T \land s^T,$$

$$(t \lor s)^T = t^T \lor s^T,$$

$$(t^*)^T = \neg(t^T),$$

$$0^T = \bot,$$

$$1^T = \top.$$

For atomic formulas, we define:

$$(t \leq s)^T = \Box(t^T \to s^T),$$

$$(tC_s)^T = \neg(t^T \leadsto \neg s^T).$$

Finally, complex formulas are translated inductively as follows:

$$(\neg \varphi)^T = \neg \varphi^T,$$

$$(\varphi \land \psi)^T = \varphi^T \land \psi^T,$$

$$(\varphi \lor \psi)^T = \varphi^T \lor \psi^T,$$

$$(\varphi \rightarrow \psi)^T = \varphi^T \rightarrow \psi^T,$$

$$\bot^T = \bot,$$

$$\top^T = \top.$$

**Theorem 7.2.** For any formula $\varphi$ of $L(C, \leq)$, we have

$$PWRCC \vdash \varphi \iff S^2IC \vdash \varphi^T.$$
Proof. By [1, Cor. 6.1], PWRCC is sound and complete with respect to the class of reflexive and symmetric Kripke frames \((W, R)\); and by Theorem 5.2, \(S^2IC\) is sound and complete with respect to the class of contact algebras. Given a Kripke model \((W, R, v)\), the valuation \(v\) of Boolean variables of \(L(C, \leq)\) into \(\mathcal{P}(W)\) can be seen as a valuation of propositional letters of \(L\) into the algebra \((\mathcal{P}(W), \leadsto_R)\).

Claim. \((W, R, v) \models \varphi\) if and only if \((\mathcal{P}(W), \leadsto_R, v) \models \varphi^T\).

Proof of Claim. For a Boolean term \(t\), we have \(v(t) = v(t^T) \subseteq W\). If \(\varphi\) is an atomic formula of the form \(t \leq s\), then

\[
(W, R, v) \models \varphi \iff v(t) \subseteq v(s)
\]

if \(v(t^T) \leq v(s^T)\) in \(\mathcal{P}(W)\)

\[
(W, R, v) \models (\mathcal{P}(W), \leadsto_R, v) \models t^T \rightarrow s^T
\]

\[
(W, R, v) \models (\mathcal{P}(W), \leadsto_R, v) \models \square(t^T \rightarrow s^T)
\]

\[
(W, R, v) \models (\mathcal{P}(W), \leadsto_R, v) \models \varphi^T.
\]

If \(\varphi\) is an atomic formula of the form \(tC_s\), then

\[
(W, R, v) \models \varphi \iff R[v(t)] \cap v(s) \neq \emptyset
\]

if \(R[v(t)] \subseteq W \setminus v(s)\)

if \(R[v(t^T)] \subseteq v(\neg s^T)\)

\[
(W, R, v) \models (\mathcal{P}(W), \leadsto_R, v) \models \neg(t^T \leadsto_R \neg s^T)
\]

\[
(W, R, v) \models (\mathcal{P}(W), \leadsto_R, v) \models \varphi^T.
\]

Finally, if \(\varphi\) is a complex formula, then a straightforward induction completes the proof. \(\square\)

Now, if PWRCC \(\not\models \varphi\), then there is a reflexive and symmetric Kripke model \((W, R, v)\) refuting \(\varphi\). By the Claim, \(\varphi^T\) is refuted in \((\mathcal{P}(W), \leadsto_R, v)\). Therefore, \(S^2IC \models \varphi^T\). Conversely, if \(S^2IC \not\models \varphi^T\), then there is a contact algebra \((B, \prec)\) and a valuation \(v\) on \((B, \prec)\) refuting \(\varphi^T\). By Lemma 7.1(2), \(\varphi^T\) is refuted in \((\mathcal{P}(X), \leadsto_R, v)\). By the Claim, \(\varphi\) is refuted in \((X, R, v)\). Thus, PWRCC \(\not\models \varphi\). \(\square\)

As was pointed out to us by D. Vakarelov, our language, like the language of [1], admits a translation \(tr\) into the basic language of modal logic enriched with the universal modality. We conclude the paper by spelling out this connection.

We recall that K denotes the basic modal logic, \(KT := K + (\Box p \rightarrow p)\), \(KTB := KT + (p \rightarrow \Box \Diamond p)\), and \(S5 := KTB + (\Box p \rightarrow \Box \Box p)\) (see, e.g., [6] Sec. 4.1). We denote the universal modality by \(\forall\), and let \(KTU, KTB_U, \text{ and } S5_U\) be the extensions of \(KT, KTB, \text{ and } S5\) with the universal modality (see, e.g., [6] Sec. 7.1).

It is well known that all the above logics are Kripke complete and have the finite model property (see, e.g., [3]). In particular, it is known that \(KTU\) has the finite model property with respect to the finite reflexive Kripke frames enriched with the universal relation, that \(KTB_U\) has the finite model property with respect to the finite reflexive and symmetric frames.
enriched with the universal relation, and that $S5_U$ has the finite model property with respect to the finite reflexive, symmetric, and transitive frames enriched with the universal relation (see, e.g., [17]).

Consider the following translation $tr$ from our language into the modal language enriched with the universal modality:

\[
\begin{align*}
tr(p) &= p, \\
tr(\neg \varphi) &= \neg tr(\varphi), \\
tr(\varphi \land \psi) &= tr(\varphi) \land tr(\psi), \\
tr(\varphi \leadsto \psi) &= [\forall](tr(\varphi) \rightarrow \Box tr(\psi)).
\end{align*}
\]

**Theorem 7.3.** Let $\varphi$ be a formula of $L$.

1. $SIC \vdash \varphi$ iff $KTB_U \vdash tr(\varphi)$.
2. $S^2IC \vdash \varphi$ iff $KTB_U \vdash tr(\varphi)$.
3. $S^2IC \vdash \varphi$ iff $S5_U \vdash tr(\varphi)$.

**Proof.** (1) Let $(B, \prec) \in RSub$ and let $(X, R)$ be the dual of $(B, \prec)$. By Lemma 2.2(1), $R$ is reflexive. Let $\nu$ be a valuation on $\text{Clop}(X)$. By defining the universal relation on $X$, we can view $(X, R, \nu)$ as a model of $KTB_U$. We prove that for each formula $\psi$,

\[
(Clop(X), \prec, \nu) \models \psi \iff (X, R, \nu) \models tr(\psi).
\]

This can be done by induction on the complexity of $\psi$, and the only nontrivial case is $\psi = \chi \leadsto \xi$. If $(Clop(X), \prec, \nu) \not\models \chi \leadsto \xi$, then $R[\nu(\chi)] \not\subseteq \nu(\xi)$. Therefore, there is $x \in X$ such that $x \in \nu(\chi)$ and $R[x] \not\subseteq \nu(\xi)$. By the induction hypothesis, $x \not\models tr(\chi)$ and $x \not\models [\forall]tr(\xi)$. Thus, $x \not\models tr(\chi) \rightarrow [\forall]tr(\xi)$, and so $(X, R, \nu) \not\models [\forall](tr(\chi) \rightarrow \Box tr(\xi))$. The converse implication is proved similarly.

Now, if $SIC \not\vdash \varphi$, then by Theorem 4.11 there is $(B, \prec) \in RSub$ refuting $\varphi$. Therefore, there is a valuation $\nu$ on $\text{Clop}(X)$ such that $(Clop(X), \prec, \nu) \not\models \varphi$. By (1), $(X, R, \nu) \not\models tr(\varphi)$, and hence $KTB_U \not\vdash tr(\varphi)$. Conversely, if $KTB_U \not\vdash tr(\varphi)$, then there is a finite reflexive model $(X, R, \nu)$ with the universal relation such that $(X, R, \nu) \not\models tr(\varphi)$. Since $(X, R)$ is finite, we may view it as the dual of $(P(X), \prec_R)$. By (1), $(X, R, \nu) \not\models tr(\varphi)$ implies $(P(X), \prec_R, \nu) \not\models \varphi$. Since $(P(X), \prec_R) \in RSub$, we conclude that $SIC \not\vdash \varphi$.

The proof of (2) is similar to (1) but uses Theorem 5.2.

The proof of (3) is similar to (1) and (2) but uses Theorem 5.8.

The theorem above also indicates that unlike the classical modal case, where reflexivity, symmetry, and transitivity are all expressible, in our system we can only express reflexivity and symmetry, but not transitivity as $S^2IC$ is complete with respect to $(X, R)$ where $R$ is an equivalence relation. Thus, our language cannot distinguish between $KTB_U$ and $S5_U$. This is yet another motivation to investigate non-standard rules and inductive classes of subordination algebras further.
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References


