LOCAL FINITENESS AND COLORINGS FOR VARIETIES OF HEYTING ALGERAS OF BOUNDED WIDTH

THIJS BENJAMINS, NICK BEZHANISHVILI, AND TOMMASO MORASCHINI

Abstract. We give a criterion for local finiteness in the setting of varieties of Heyting algebras of bounded width in terms of the colorability of prime spectra. This improves a known connection between local finiteness and the colorability of prime spectra.

1. Introduction

A variety of algebras is said to be *locally finite* if its finitely generated members are finite. Among Professor Kuznetsov's many research interests, local finiteness in varieties of Heyitng algebras played an important role [11, 12, 13]. Kuznetsov proposed approaching properties such as local finiteness, tabularity and related notions via the concepts of *prelocally finite, pretabular*, and, more generally, *pre-P properties*, where a variety has the pre-P property if it does not have property P, but all its proper subvarieties do. For example, a description of the prelocally finite subvarieties of a variety V often yields a full characterization of the locally finite subvarieties of V (see, e.g., [6, Sec. 12.4 and Thm. 17.7]).

It turns out that there exists a continuum of prelocally finite varieties of Heyting algebras [17], which makes the classification of locally finite varieties of Heyting algebras a very difficult problem. In contrast, in the modal case there exists only one prelocally finite subvariety of the variety of $\mathsf{S4}$-algebras, namely, the variety $\mathsf{Grz.3}$. As a consequence, local finiteness is decidable for (finitely axiomatizable) varieties of $\mathsf{S4}$-algebras [15, 16] (see [6, Sec. 12.4 and Thm. 17.7]. However, for varieties of Heyting algebras the decidability of the local finiteness problem remains open (see [6, Problem 17.2]).

Kuznetsov also showed that every prelocally finite variety of Heyting algebras is a subvariety of the variety KC axiomatized by the *weak law of excluded middle* $\neg p \vee \neg\neg p$ [7]. Bezhanishvili and Grigolia [3], asked whether a variety V of Heyting algebras is locally finite if and only if every 2-generated V-algebra is finite. This problem was instigated by the fact that all the known counter-examples for non-local finiteness in varietes of Heyting algebras were already obtained for 2-generated algebras (for $\mathsf{K4}$ and $\mathsf{S4}$-algebras it is known that 1-generated algebras suffice [16] (see also [6, Sec. 12.4]). This problem was solved in the negative by Hyttinen et al. [10] who, for each $n \in \mathbb{N}$, constructed a variety V_n of Heyting algebras whose n-generated algebras are finite, yet V_n contains an infinite $(n+1)$-generated algebra. On the positive side, Bezhanishvili and Grigolia showed that a nontrivial variety V of Heyting

Date: July 15, 2025.

Key words and phrases. Heyting algebra, intuitionistic logic, Esakia duality, local finiteness, local tabularity, superintuitionistic logic, bounded width axiom.

algebras is locally finite iff it is the variety of Boolean algebra or it contains the three-element $\boldsymbol{A}$ chain and every coproduct of finitely many copies of $\boldsymbol{A}$ in V is finite [3, Thm. 2.3].

In this paper, we focus on local finiteness in varieties of Heyting algebras of bounded width. We recall that a variety of Heyting algebras has *width* $\leqslant n$ when it is generated by algebras whose prime spectra do not feature any antichain of $n+1$ elements with a common lower bound [21]. While undeniably restrictive, the focus on varieties of width $\leqslant n$ is motivated by the fact that some of the simplest and better known failures of local finiteness for varieties of Heyting algebras occur in this context. For instance, the free one-generated Heyting algebra, known as the *Rieger-Nishimura lattice* [18, 20] (see also [2, Sec. 4.1.1]), is infinite and, therefore, generates a non-locally finite variety. Owing to the fact that the prime spectrum of the Rieger-Nishimura lattice contains no three-element antichains [2, Sec. 4.3], this variety is also of width $\leqslant 2$.

It is shown in [2, Sec. 4.6] that for the Kuznetsov and Gerciu variety KG, which is an important variety of Heyting algebras of width $\leqslant 2$ introduced in [14], local finiteness is decidable. Furthermore, a variety of KG-algebras is locally finite if and only if its 2-generated members are finite. The key observation is that the variety generated by the Rieger-Nishimura lattice with a new bottom element (which is a 2-generated Heyting algebra) generates the only prelocally finite variety of KG-algebras. We conjecture that the variety generated by the Rieger-Nishimura lattice with a new bottom element is also the only prelocally finite in all varieties of width $\leqslant 2$ and that the local finiteness problem is decidable for these varieties. We hope that the results presented in this paper will form a basis for addressing this problem, in the spirit and legacy of Professor Kuznetsov's work.

More concretely, we provide a criterion for local finiteness in the setting of varieties of Heyting algebras of width $\leqslant n$ (Theorem 18) with a special emphasis on varieties of width $\leqslant 2$ (Theorem 25). Our main tool is a description of locally finite varieties of Heyting algebras in terms of the "colorability" of infinite sequences of finite posets (Theorem 14). This is made possible by two ingredients: *finite Esakia duality* on the one hand [8] and a characterization of finitely generated Heyting algebras in terms of colorings of their prime spectra on the other [9], see also [2, Sec. 3.1]. Our criterion for local finiteness in varieties of bounded width simplifies the usage of colorings by restricting the attention to colored posets whose colored halves are of negligible size, thus shifting the structural complexity to the colorless parts.

2. Preliminaries

2.1. **Posets.** Let $X = \langle X; \leqslant \rangle$ be a poset. Given $Y \subseteq X$, the sets of minimal (resp. maximal) elements of the subposet of X with universe Y will be denoted by $\min(Y)$ (resp. $\max(Y)$). If X has a minimum, we say that X is *rooted* and call its minimum the *root* of the poset. Given $Y \subseteq X$, let

$$\uparrow^X Y \coloneqq \{x \in X : \text{there exists } y \in Y \text{ such that } y \leqslant x\};$$
$$\downarrow^X Y \coloneqq \{x \in X : \text{there exists } y \in Y \text{ such that } x \leqslant y\}.$$

When the poset X is clear from the context, we will write $\uparrow Y$ and $\downarrow Y$ instead of $\uparrow^X Y$ and $\downarrow^X Y$. Moreover, when $Y = \{x\}$, we will write $\uparrow x$ and $\downarrow x$ instead of $\uparrow\{x\}$ and $\downarrow\{x\}$. A set

$Y \subseteq X$ is said to be an *upset* (resp. *downset*) of X when $Y = {\uparrow}Y$ (resp. $Y = {\downarrow}Y$). The set of upsets of X will be denoted by $\mathsf{Up}(X)$.

Let $x, y \in X$. We write $x \prec y$ to indicate that x is an *immediate predecessor* of y (or, equivalently, that y is an *immediate successor* of x), i.e., that there exists no $z \in X$ such that $x < z < y$. We say that x and y are *comparable* when either $x \leqslant y$ or $y \leqslant x$. If x and y are not comparable, we call them *incomparable* and write $x \parallel y$. A set $Y \subseteq X$ is said to be an *antichain* (resp. a *chain*) in X when its elements are pairwise incomparable (resp. comparable). We will make us of the following simple observation on finite poset.

Theorem 1. *Let $\{X_n : n \in \mathbb{N}\}$ be a family of posets whose antichains have size $\leqslant m$ for some $m \in \mathbb{N}$. Assume that for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $k \leqslant |X_n|$. Then for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that X_n contains a chain of k elements.*

2.2. **Finite Esakia Duality.** A structure $\langle A; \wedge, \vee, \to, 0, 1\rangle$ is said to be a *Heyting algebra* [1, 6, 8, 19] when $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

We denote the category of finite Heyting algebras with homomorphisms between them by HA^ω.

A map $p \colon X \to Y$ between posets is said to be a *p-morphism* when it is order preserving and for every $x \in X$ and $y \in Y$,

$$\text{if } p(x) \leqslant y, \text{ there exists } z \in X \text{ such that } x \leqslant z \text{ and } y = p(z).$$

We denote the category of finite posets with p-morphisms between them by Pos^ω.

Finite Esakia Duality establishes a dual equivalence between HA^ω and Pos^ω [8]. More precisely, let $\boldsymbol{A}$ be a finite Heyting algebra. A set $F \subseteq A$ is a *prime filter* of $\boldsymbol{A}$ when it is a nonempty proper upset such that for all $a, b \in A$,

$$(a, b \in F \Longrightarrow a \wedge b \in F) \qquad \text{and} \qquad (a \vee b \in F \Longrightarrow a \in F \text{ or } b \in F).$$

We denote the set of prime filters of $\boldsymbol{A}$ by $\mathsf{Pr}(\boldsymbol{A})$. Now, for each $a \in A$ let

$$\gamma_{\boldsymbol{A}}(a) \coloneqq \{F \in \mathsf{Pr}(A) : a \in F\}.$$

Then the triple $\boldsymbol{A}_* \coloneqq \langle \mathsf{Pr}(\boldsymbol{A}); \subseteq\rangle$ is a finite poset. Furthermore, given a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ between finite Heyting algebras, the map $h_* \colon \boldsymbol{B}_* \to \boldsymbol{A}_*$ defined by the rule $h_*(F) \coloneqq h^{-1}[F]$ is a p-morphism between finite posets. The transformation $(-)_* \colon \mathsf{HA}^\omega \to \mathsf{Pos}^\omega$ is a contravariant functor.

On the other hand, given a finite poset X, the structure $X^* \coloneqq \langle \mathsf{Up}(X); \cap, \cup, \to, \varnothing, X\rangle$, where for every $U, V \in \mathsf{Up}(X)$ we have

$$U \to V = \{x \in X : \text{for every } y \in X \text{ if } x \leqslant y \in U, \text{ then } y \in V\},$$

is a finite Heyting algebra. Moreover, given a p-morphism $p \colon X \to Y$ between finite posets, the map $p^* \colon Y^* \to X^*$ defined by the rule $p^*(U) \coloneqq p^{-1}[U]$ is a homomorphism between finite Heyting algebras. Lastly, the transformation $(-)^* \colon \mathsf{Pos}^\omega \to \mathsf{HA}^\omega$ is a contravariant functor.

Finite Esakia Duality 2. *The functors $(-)_*$ and $(-)^*$ witness a dual equivalence between the categories HA^ω and Pos^ω.*

2.3. **Kernels of p-morphisms.** The *kernel* of a function $p\colon X \to Y$ is the set

$$\{\langle x, y\rangle \in X \times X : p(x) = p(y)\}.$$

Kernels of p-morphisms between finite posets can be described as follows. Let X be a finite poset. An *E-partition* on X is an equivalence relation R on X such that for every $x, y, z \in X$,

$$\text{if } \langle x, y\rangle \in R \text{ and } x \leqslant z, \text{ there exists } v \in X \text{ such that } \langle z, y\rangle \in R \text{ and } y \leqslant v.$$

Given an E-partition of X, the quotient X/R can be viewed as a poset whose order relation is defined as follows:

$$x/R \leqslant y/R \iff x' \leqslant y' \text{ for some } x' \in x/R \text{ and } y' \in y/R.$$

Proposition 3 ([2, Lem. 2.3.9])**.** *Let R be a binary relation on a finite poset X. Then R is an E-partition of X iff it is the kernel of a surjective p-morphism $p\colon X \to Y$ for some poset Y, in which case $Y \cong X/R$.*

Example 4. Let X be a finite poset and $U \in \mathsf{Up}(X)$ nonempty. Assume that for every $x \in X$ there exists $y \in U$ such that $x \leqslant y$. Then the relation

$$R = \{\langle x, y\rangle \in X \times X : \text{either } x = y \text{ or } x, y \in U\}$$

is an E-partition of X. Moreover, let Y be the poset obtained by adding a maximum $\top$ to the subposet of X with universe $X - U$. Then the map $p\colon X/R \to Y$ defined by the rule

$$p(x/R) \coloneqq \begin{cases} \top & \text{if } x \in U; \\ x & \text{if } x \notin U \end{cases}$$

is an isomorphism. ⊠

Let X be a poset. A pair of distinct elements $x, y \in X$ is said to be:

(i) *α-reducible* when ${\uparrow}x = \{x\} \cup {\uparrow}y$;
(ii) *β-reducible* when ${\uparrow}x - \{x\} = {\uparrow}y - \{y\}$;
(iii) *reducible* when it is either α-reducible or β-reducible.

Notably, if X is a finite poset and $x, y \in X$ a reducible pair, the least equivalence relation R on X containing $\langle x, y\rangle$ is always an E-partition of X. A surjective p-morphism $p\colon X \to Y$ between finite posets is said to be a *reduction* when its kernel is the least equivalence relation on X containing $\langle x, y\rangle$ for some reducible pair $x, y \in X$.

Proposition 5 ([2, Lem. 3.1.7])**.** *Every surjective p-morphism between finite posets is either an isomorphism or a composition of reductions.*

Corollary 6. *Let R be an E-partition of a finite poset X. If R is not the identity relation on X, there exists a reducible pair $x, y \in X$ such that $\langle x, y\rangle \in R$.*

Proof. As R is an E-partition of X, the canonical map $p\colon X \to X/R$ is a surjective p-morphism by Proposition 3. Moreover, p is not an isomorphism because R is not the identity relation on X. Consequently, we can apply Proposition 5, obtaining that p is a composition of reductions. Since R is the kernel of p, we conclude that $\langle x, y\rangle \in R$ for some reducible pair $x, y \in R$. ⊠

A class of algebras is said to be a *variety* when it is closed under homomorphic images, subalgebras, and direct products. In view of Birkhoff's Theorem, varieties coincide with the classes of algebras axiomatized by equations (see, e.g., [5, Thm. II.11.9]). We will rely on the following observation (see, e.g., [2, Thm. 2.3.7]).

Proposition 7. *Let* V *be a variety of Heyting algebras and* X *a finite poset such that* $\mathsf{Up}(X) \in \mathsf{V}$*. Then the following conditions hold:*

(i) *if* $U \in \mathsf{Up}(X)$*, then* $\mathsf{Up}(U) \in \mathsf{V}$*;*
(ii) *if* R *is an E-partition of* X*, then* $\mathsf{Up}(X/R) \in \mathsf{V}$*.*

2.4. **Bounded width.** In the context of Heyting algebras, the notion of width was first introduced in [21].

Definition 8. Let $n \in \mathbb{N}$. A poset X is said to have *width* $\leqslant n$ when for each $x \in X$ the antichains in $\uparrow x$ have size $\leqslant n$.

For every $n \in \mathbb{N}$ let

$$bw_n \coloneqq \bigvee_{m=0}^{n} (x_m \to \bigvee_{k \neq m} x_k).$$

When an equation $\varphi \approx 1$ is valid in a Heyting algebra $\boldsymbol{A}$, we write $\boldsymbol{A} \vDash \varphi$. Finite posets with width $\leqslant n$ can be described as follows [21] (see also [6, Ex. 2.11]).

Theorem 9. *Let* X *be a finite poset and* $n \in \mathbb{N}$*. Then* X *has width* $\leqslant n$ *iff* $\mathsf{Up}(X) \vDash bw_n$*.*

The notion of "having width $\leqslant n$" can be extended to varieties of Heyting algebras. To explain how, given a class K of Heyting algebras, we write $\mathsf{K} \vDash bw_n$ when $\boldsymbol{A} \vDash bw_n$ for every $\boldsymbol{A} \in \mathsf{K}$.

Definition 10. Let V be a variety of Heyting algebras. Given $n \in \mathbb{N}$, we say that V has *width* $\leqslant n$ when $\mathsf{V} \vDash bw_n$. Moreover, V is said to have *bounded width* when it has width $\leqslant n$ for some $n \in \mathbb{N}$.

3. Finite generation and colorability in bounded width

Given a Heyting algebra $\boldsymbol{A}$ and $X \subseteq A$, we denote by $\mathsf{Sg}^{\boldsymbol{A}}(X)$ the subuniverse of $\boldsymbol{A}$ generated by X. We say that $\boldsymbol{A}$ is *n-generated* with $n \in \mathbb{N}$ when there exists $X \subseteq A$ such that $|X| \leqslant n$ and $A = \mathsf{Sg}^{\boldsymbol{A}}(X)$. For Heyting algebras, the property of being n-generated can be described in terms of the colorability of posets, as we proceed to recall. Although we shall restrict our discussion to the finite case, similar results hold for infinite Heyting algebras as well (see, e.g., [2, Sec. 4.1]).

Definition 11. Let X be a finite poset and $n \in \mathbb{N}$. A family $\{U_m : 0 \leqslant m < n\} \subseteq \mathsf{Up}(X)$ is said to be an *n-coloring* of X when there exists no reducible pair $x, y \in X$ such that

$$x \in U_m \iff y \in U_m, \text{ for each } 0 \leqslant m < n. \tag{1}$$

When there exists an n-coloring of X, we say that X is *n-colorable.*

Let X be a finite poset and $U \in \mathsf{Up}(X)$. Notice that a pair $x, y \in U$ is reducible in X iff it is reducible in the subposet of X with universe U. As a consequence, n-colorings can be restricted to upsets in the following sense.

Proposition 12. *Let $\{U_m : 0 \leqslant m < n\}$ be an n-coloring of a finite poset X. For every $V \in \mathsf{Up}(X)$ the family $\{U_m \cap V : 0 \leqslant m < n\}$ is an n-coloring of the subposet of X with universe V.*

The following is a special case of [2, Thm. 3.1.5].

Theorem 13. *Let X be a finite poset and $n \in \mathbb{N}$. Then $\mathsf{Up}(X)$ is n-generated iff X is n-colorable.*

Proof. It is known that $\mathsf{Up}(X)$ is n-generated iff there exist $\{U_m : 0 \leqslant m < n\} \subseteq \mathsf{Up}(X)$ such that for every E-partition R of X other than the identity relation there exist distinct $x, y \in X$ satisfying (1) such that $\langle x, y\rangle \in R$ (see, e.g., [2, Thm. 3.1.5]). The latter is equivalent to the demand that X is n-colorable by Corollary 6 and the fact that the least equivalence relation on X containing $\langle x, y\rangle$ is an E-partition for each reducible pair $x, y \in X$. ⊠

A variety is *locally finite* when its finitely generated members are finite. Locally finite varieties of Heyting algebras admit a transparent description in terms of colorability.

Theorem 14. *A variety V of Heyting algebras is locally finite iff for each $n \in \mathbb{N}$, up to isomorphism, there exist only finitely many n-colorable finite rooted posets X such that $\mathsf{Up}(X) \in \mathsf{V}$.*

Proof. This is a rephrasing of [4, Thm. 4.3], made possible by Finite Esakia Duality 2 and the fact that a finite poset X is rooted iff $\mathsf{Up}(X)$ is finitely subdirectly irreducible [2, Thm. 2.3.16], where the latter means that the identity relation on $\mathsf{Up}(X)$ is meet irreducible in the lattice of congruences of $\mathsf{Up}(X)$ (see, e.g., [5, Thm. II.8.4]). ⊠

Our aim is to present a characterization of locally finite varieties of Heyting algebras of bounded width, which improves Theorem 14 in this more restricted setting. To this end, it is convenient to introduce the following concept.

Definition 15. Let $c = \{U_m : 0 \leqslant m < n\}$ be an n-coloring of a finite poset X. An element $x \in X$ is said to be

(i) *colored* relative to c when $x \in U_m$ for some $0 \leqslant m < n$;
(ii) *colorless* relative to c when it is not colored relative to c.

The set of elements of X that are colored (resp. colorless) relative to c will be called the *colored half* (resp. *colorless half*) of X relative to c. Lastly, for every $M \subseteq \{m : 0 \leqslant m < n\}$ let

$$M_c(X) \coloneqq \{x \in X : \text{for every } 0 \leqslant m < n \text{ we have } x \in U_m \text{ iff } m \in M\}.$$

We will make use the following observations.

Proposition 16. *Let c be an n-coloring of a finite poset X. The set of elements of X that are colorless relative to c is a downset of X.*

Proof. Immediate from the fact that every $U \in c$ is an upset of X. ⊠

Proposition 17. *Let c be an n-coloring of a finite poset X and $x \in X$. If there exists $y \in X$ such that $x < y$ and y is colorless relative to c, then x has at least two immediate successors.*

Proof. Since $x < y$ and X is finite, there exists $z \in X$ such that $x \prec z \leqslant y$. As y is colorless relative to c, the elements x and z are colorless relative to c as well. Therefore, the assumption that c is an n-coloring of X guarantees that the pair $x \prec z$ is not α-reducible. As X is finite, this implies that there exists an immediate successor of X other than z. ⊠

The main result of this section is the following description of locally finite varieties of Heyting algebras of bounded width.

Theorem 18. *Let V be a variety of Heyting algebras and assume V has bounded width. Then V fails to be locally finite iff there exist $n \in \mathbb{N}$ and a sequence $\{X_m : m \in \mathbb{N}\}$ of finite rooted posets satisfying the following conditions:*

(i) *for every $m \in \mathbb{N}$ there exists an n-coloring c_m of X_m and $\mathsf{Up}(X_m) \in \mathsf{V}$;*
(ii) *there exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ the colored half of X_m relative to c_m has size $\leqslant k$;*
(iii) *for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the colorless half of X_m relative to c_m has size $> k$.*

Proof. To prove the implication from right to left, suppose that there exist $n \in \mathbb{N}$ and a sequence $\{X_m : m \in \mathbb{N}\}$ of finite rooted posets satisfying (i), (ii), and (iii). From (iii) it follows that for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $k < |X_m|$. As $\{X_m : m \in \mathbb{N}\}$ is a family of finite posets, this implies that there exists an infinite $I \subseteq \mathbb{N}$ such that the members of $\{X_m : m \in I\}$ are pairwise nonisomorphic. Together with (i) and the assumption that each X_m is finite and rooted, this allows us to apply Theorem 14, obtaining that V is not locally finite.

Next we prove the implication from left to right. Suppose that V fails to be locally finite. In view of Theorem 14, there exist $n \in \mathbb{N}$ and a sequence $\{X_m : m \in \mathbb{N}\}$ of pairwise nonisomorphic finite rooted posets such that for every $m \in \mathbb{N}$ there exists an n-coloring c_m of X_m and $\mathsf{Up}(X_m) \in \mathsf{V}$. For each $m \in \mathbb{N}$ fix an enumeration

$$c_m = \{U_i^m : 0 \leqslant i < n\}.$$

As $\{X_m : m \in \mathbb{N}\}$ is an family of pairwise nonisomorphic posets, there exists no $k \in \mathbb{N}$ such that $|X_m| < k$ for every $m \in \mathbb{N}$. Consequently, there exists $M \subseteq \{i : 0 \leqslant i < n\}$ satisfying the following condition:

(C1) for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $k < |M_{c_m}(X_m)|$.

As $\{i : 0 \leqslant i < n\}$ is a finite set, we may also assume that M is maximal among the subsets of $\{i : 0 \leqslant i < n\}$ be a set satisfying (C1). For every $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$ let Y_m^x be the subposet of X_m with universe $\uparrow x$.

Claim 19. *Let $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$. Then Y_m^x is a finite rooted poset and $\mathsf{Up}(Y_m^x) \in \mathsf{V}$.*

Proof of the Claim. Since Y_m^x is a subposet of the finite poset X_m, we obtain that Y_m^x is finite as well. Moreover, Y_m^x is rooted by definition. Lastly, recall that $\mathsf{Up}(X_m) \in \mathsf{V}$ by assumption. As Y_m^x is an upset of X_m by definition, we can apply Proposition 7(i), obtaining $\mathsf{Up}(Y_m^x) \in \mathsf{V}$. ⊠

Now, let $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$. For each $0 \leqslant k < n$ let

$$V_k^{m,x} \coloneqq (U_k^m \cap Y_m^x) - M_{c_m}(X_m) \ \text{ and } \ d_m^x \coloneqq \{V_i^{m,x} : 0 \leqslant i < n\}.$$

The proof proceeds through a series of claims.

Claim 20. *Let $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$. Then $y \in U_i^m$ for every $y \in Y_m^x$ and $i \in M$.*

Proof of the Claim. To this end, consider $y \in Y_m^x$ and $i \in M$. Recall that $x \in M_{c_m}(X_m)$ by assumption. From $x \in M_{c_m}(X_m)$ and $i \in M$ it follows that $x \in U_i^m$. Moreover, x is the minimum of Y_m^x by the definition of Y_m^x. Together with $y \in Y_m^x$, this yields $x \leqslant y$. Since $x \in U_i^m$ and U_i^m is an upset of X_m, we conclude that $y \in U_i^m$. ⊠

Claim 21. *Let $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$. Then d_m^x is an n-coloring of Y_m^x.*

Proof of the Claim. Recall that $d_m^x = \{V_i^{m,x} : 0 \leqslant i < n\}$. We begin by showing that d_m^x is a family of upsets of Y_m^x. Consider $0 \leqslant i \leqslant n$. The definition of $V_i^{m,x}$ guarantees that $V_i^{m,x} \subseteq Y_m^x$. Then consider $y, z \in Y_m^x$ such that $y \in V_i^{m,x}$ and $y \leqslant z$. By the definition of $V_i^{m,x}$ we have $y \in U_i^m \cap Y_m^x$ and $y \notin M_{c_m}(X_m)$. As $y \leqslant z$ and U_i^m and Y_m^x are upsets of X_m containing y, we obtain $z \in U_i^m \cap Y_m^x$. Therefore, to conclude that z belongs to $V_i^{m,x} = (U_i^m \cap Y_m^x) - M_{c_m}(X_m)$, it only remains to show that $z \notin M_{c_m}(X_m)$. From Claim 20 it follows that $y \in U_k^m$ for each $k \in M$. Together with $y \notin M_{c_m}(X_m)$, this yields that $y \in U_j^m$ for some $0 \leqslant j < n$ such that $j \notin M$. As U_j^m is an upset of X_m and $y \leqslant z$, we obtain $z \in U_j^m$. Since $j \notin M$, we conclude $z \notin M_{c_m}(X_m)$ as desired. Hence, $V_i^{m,x}$ is an upset of Y_m^x.

To prove that the family d_m^x of upsets of Y_m^x is an n-coloring of Y_m^x, consider a reducible pair $y, z \in Y_m^x$. As Y_m^x is an upset of X_m by definition, the pair y, z is also reducible in X_m. Since $c_m = \{U_i^m : 0 \leqslant i < n\}$ is an n-coloring of X_m by assumption, there exists $0 \leqslant i < n$ such that $y \in U_i^m$ iff $z \notin U_i^m$. By symmetry we may assume that $y \in U_i^m$ and $z \notin U_i^m$. To prove that d_m^x is an n-coloring of Y_m^x it suffices to show that

$$y \in V_i^{m,x} \ \text{ and } \ z \notin V_i^{m,x}.$$

From Claim 20 and $z \notin U_i^m$ it follows that $i \notin M$. Since $y \in U_i^m$ and $i \notin M$, we obtain $y \notin M_{c_m}(X_m)$. Thus, $y \in (U_i^m \cap Y_m^x) - M_{c_m}(X_m) = V_i^{m,x}$. On the other hand, from $z \notin U_i^m$ and $V_i^{m,x} \subseteq U_i^m$ it follows that $z \notin V_i^{m,x}$, establishing the above display. ⊠

Claim 22. *There exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$ the colored half of Y_m^x relative to d_m^x has size $\leqslant k$.*

Proof of the Claim. Suppose the contrary, with a view to contradiction. Then for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $x \in M_{c_m}(X_m)$ such that the colored half of Y_m^x relative to d_m^x has size $> k$. As the colored half of each Y_m^x relative to d_m^x is

$$\bigcup \{P_{d_m}(Y_m^x) : \emptyset \neq P \subseteq \{i : 0 \leqslant i < n\}\}$$

and there are only finitely many subsets of $\{i : 0 \leqslant i < n\}$, there exists a nonempty $P \subseteq \{i : 0 \leqslant i < n\}$ such that

$$\text{for every } k \in \mathbb{N} \text{ there exist } m \in \mathbb{N} \text{ and } x \in Y^x_m \text{ such that } k < |P_{d_m}(Y^x_m)|. \tag{2}$$

We will prove that $M \subsetneq P$. Since for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $x \in Y^x_m$ such that $k < |P_{d_m}(Y^x_m)|$, there also exist $m \in \mathbb{N}$ and $x \in Y^x_m$ such that $P_{d_m}(Y^x_m) \neq \emptyset$. Then there exists $y \in P_{d_m}(Y^x_m)$. We begin by showing that $M \subseteq P$. Consider $i \in M$. From Claim 20 it follows that $y \in U^m_i$. Thus, $y \in U^m_i \cap Y^x_m$. We have two cases: either $y \in M_{c_m}(X_m)$ or $y \notin M_{c_m}(X_m)$. We will show that the case where $y \in M_{c_m}(X_m)$ never happens. For suppose that $y \in M_{c_m}(X_m)$. As $M_{c_m}(X_m)$ is disjoint from $V^{m,x}_j$ for every $0 \leqslant j < n$, we obtain $y \notin V^{m,x}_j$ for every $0 \leqslant j < n$. Together with $y \in P_{d_m}(Y^x_m)$, this yields $P = \emptyset$, a contradiction with the assumption that P is nonempty. Then $y \notin M_{c_m}(X_m)$. Consequently, $y \in (U^m_i \cap Y^x_m) - M_{c_m}(X_m) = V^{m,x}_i$. Since $y \in P_{d_m}(Y^x_m)$, we conclude that $i \in P$, whence $M \subseteq P$. Therefore, to prove that $M \subsetneq P$, it only remains to show that $M \neq P$. Suppose, on the contrary, that $M = P$. As $P \neq \emptyset$, there exists $i \in P$. Since $y \in P_{d_m}(Y^x_m)$, we have $y \in V^{m,x}_i \subseteq Y^x_m - M_{c_m}(X_m)$. Thus, $y \notin M_{c_m}(X_m)$. Recall from Claim 20 that $y \in U^m_j$ for each $j \in M$. Together with $y \notin M_{c_m}(X_m)$, this guarantees that $y \in U^m_j$ for some $0 \leqslant j < n$ with $j \notin M$. On the other hand, $j \notin M = P$ and $y \in P_{d_m}(Y^x_m)$ implies $y \notin V^{m,x}_j$. However, we showed that $y \in U^m_j - M_{c_m}(X_m)$ and by assumption we have $y \in P_{d_m}(Y^x_m) \subseteq Y^x_m$. Hence, $y \in (U^m_j \cap Y^x_m) - M_{c_m}(X_m) = V^{m,x}_j$, a contradiction with $y \notin V^{m,x}_j$. Thus, we conclude that $M \neq P$. This establishes that $M \subsetneq P$.

Now, recall that M is maximal among the subsets of $\{i : 0 \leqslant i < n\}$ satisfying (C1). Since $P \subseteq \{i : 0 \leqslant i < n\}$ and $M \subsetneq P$, it follows that P does not satisfy (C1). Consequently, there exists $k \in \mathbb{N}$ such that $|P_{c_m}(X_m)| \leqslant k$ for every $m \in \mathbb{N}$. To conclude the proof, it will be enough to show that $P_{d_m}(Y^x_m) \subseteq P_{c_m}(X_m)$ for every $m \in \mathbb{N}$, for this is a contradiction with (2). To this end, consider $m \in \mathbb{N}$, $x \in P_{d_m}(Y^x_m) \subseteq X_m$, and $0 \leqslant k < n$. We need to prove that

$$k \in P \iff x \in U^m_k.$$

First, suppose that $k \in P$. Since $x \in P_{d_m}(Y^x_m)$, we obtain $x \notin V^{m,x}_k \subseteq U^m_k$, where the inclusion holds by the definition of $V^{m,x}_k$. Then we consider the case where $k \notin P$. As $x \in P_{d_m}(Y^x_m)$, we have $x \notin V^{m,x}_k$. Since $x \in P_{d_m}(Y^x_m) \subseteq Y^m_x$, the definition of $V^{m,x}_k$ yields that either $x \notin U^m_k$ or $x \in M_{c_m}(X_m)$. To conclude the proof, it only remains to show that $x \notin M_{c_m}(X_m)$, for in this case we would obtain $x \notin U^m_k$ as desired. As $M \subsetneq P$, there exists $i \in P - M$. The proof of the implication from left to right in the above display shows that $x \in U^m_i$. Together with $i \notin M$, this yields $x \notin M_{c_m}(X_m)$. ⊠

Claim 23. *For every $k \in \mathbb{N}$ there exist $m_k \in \mathbb{N}$ and $x_k \in M_{c_{m_k}}(X_{m_k})$ such that the colorless half of $Y^{x_k}_{m_k}$ relative to $d^{x_k}_{m_k}$ has size $> k$.*

Proof of the Claim. Recall that V has bounded width by assumption. Therefore, there exists $p \in \mathbb{N}$ such that $\mathsf{V} \vDash bw_p$. Moreover, let $m \in \mathbb{N}$ and recall that $\mathsf{Up}(X_m) \in \mathsf{V}$ by assumption. As $\mathsf{V} \vDash bw_p$, we can apply Theorem 9, obtaining that X_m has width $\leqslant p$. Since X_m is rooted by assumption, this implies that antichains in X_m have size $\leqslant p$. Consequently, antichains in $M_{c_m}(X_m)$ have also size $\leqslant p$.

Now, for each $m \in \mathbb{N}$ we view $M_{c_m}(X_m)$ as a subposet of X_m. Therefore, $\{M_{c_m}(X_m) : m \in \mathbb{N}\}$ is a sequence of poset whose antichains have size $\leqslant p$. Together with (C1) and Theorem 1, this implies that for every $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that $M_{c_m}(X_m)$ contains a chain of $> k$ elements.

Then consider $k \in \mathbb{N}$ and recall that $M_{c_{m_k}}(X_{m_k})$ contains a chain of $k+1$ elements $x_k < y_1 < \cdots < y_k$. Then observe that $x_k, y_1, \ldots, y_n \in {\uparrow} x_k = Y^{x_k}_{m_k}$. Moreover, $V^{m_k, x_k}_i \cap M_{c_{m_k}}(X_{m_k}) = \emptyset$ for each $0 \leqslant i < n$ by the definition of V^{m_k,x_k}_i. Together with $x_k, y_1, \ldots, y_k \in M_{c_{m_k}}(X_{m_k})$, this implies that $x_k, y_1, \ldots, y_k$ belong to the colorless half of $Y^{x_k}_{m_k}$ relative to $d^{x_k}_{m_k}$. Consequently, the colorless half of $Y^{x_k}_{m_k}$ relative to $d^{x_k}_{m_k}$ has size $\geqslant k + 1$. ⊠

Let $\{Y^{x_k}_{m_k} : k \in \mathbb{N}\}$ be the family of posets given by Claim 23. Recall from Claim 19 that each $Y^{x_k}_{m_k}$ is a finite rooted poset such that $\mathsf{Up}(Y^{x_k}_{m_k}) \in \mathsf{V}$. Furthermore, $d^{x_k}_{m_k}$ is an n-coloring of $Y^{x_k}_{m_k}$ by Claim 21. Therefore, the family $\{Y^{x_k}_{m_k} : k \in \mathbb{N}\}$ satisfies condition (i) in the statement. Finally, it also satisfies conditions (ii) and (iii) by Claims 22 and 23, respectively. ⊠

4. Local finiteness in width two

Definition 24. Let c be an n-coloring of a finite poset X. An element $x \in X$ is said to be *strongly colorless* relative to c when there exist distinct $y, z \in X$ colorless relative to c such that $x \prec y, z$. The set of elements of X that are strongly colorless relative to c will be denoted by $S_c(X)$.

Since every $U \in c$ is an upset of X, every element of X that is strongly colorless relative to c is also colorless relative to c. Our aim is to show that, in the setting of varieties of width $\leqslant 2$, Theorem 18 admits the following improvement.

Theorem 25. *Let V be a variety of Heyting algebras of width $\leqslant 2$. Assume that V is not locally finite. Then there exist $n, t \in \mathbb{N}$ and a family $\{X_m : m \in \mathbb{N}\}$ of finite rooted posets satisfying the following conditions:*

(i) *for every $m \in \mathbb{N}$ there exists an n-coloring c_m of X_m and $\mathsf{Up}(X_m) \in \mathsf{V}$;*
(ii) *for every $k \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $x, y, z \in X_m$ with $x \prec y, z$ and $y \parallel z$ such that $y, z \in S_{c_m}(X_m)$ and $k \leqslant |{\downarrow}\{y, z\}|$ and $|{\uparrow}\{y, z\}| \leqslant t$.*

To this end, we rely on the following observation.

Proposition 26. *Let c be an n-coloring of a finite rooted poset X of width $\leqslant 2$. Then $S_c(X)$ is a downset of X.*

Proof. Consider $x, y \in X$ such that $x < y \in S_c(X)$. Since $y \in S_c(X)$, there exist distinct $u, v \in X$ colorless relative to c such that $y \prec u, v$. As $x < y \in S_c(X)$, we can apply Proposition 17, obtaining that x has at least two immediate successors w and z. As $x < y$ and X is finite, we may assume that $x \prec w \leqslant y$. As y is colorless relative to c and $w \leqslant y$, from Proposition 16 it follows that w is also colorless relative to c. Now, recall that $u \parallel v$ because $y \prec u, v$ and $u \neq v$ by assumption. From $u \parallel v$ and the assumption that X is rooted and of width $\leqslant 2$ it follows that z it comparable with u or v. By symmetry we may assume that z is comparable with u. Since $x < y \prec u$ and $x \prec z$, we have $u \nleqslant z$. As z and u are comparable, this yields $z < u$. Together with Proposition 16 and the assumption that u is colorless relative to c, this implies that so is z. Hence, w and z both colorless relative to c. Since they are distinct and $x \prec z, w$ by assumption, we conclude that $x \in S_c(X)$. ⊠

We are now ready to prove Theorem 25.

Proof. In view of Theorem 18, there exist $n \in \mathbb{N}$ and a family $\{X_m : m \in \mathbb{N}\}$ of finite rooted posets satisfying the following conditions:

(L1) for every $m \in \mathbb{N}$ there exists an n-coloring c_m of X_m and $\mathsf{Up}(X_m) \in \mathsf{V}$;
(L2) there exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ the colored half of X_m relative to c_m has size $\leqslant k$;
(L3) for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the colorless half of X_m relative to c_m has size $> k$.

We will show that the family $\{X_m : m \in \mathbb{N}\}$ and the colorings $\{c_m : m \in \mathbb{N}\}$ given by (L1) satisfy the conditions in the statement. Clearly, (i) holds by (L1). Therefore, it only remains to prove (ii). To this end, for every $m \in \mathbb{N}$ let

$$C_m \coloneqq \{x \in X_m : x \text{ is colorless relative to } c_m \text{ and } x \notin S_{c_m}(X_m)\}.$$

We rely on the following series of observations.

Claim 27. *There exists $k \in \mathbb{N}$ such that C_m has size $\leqslant k$ for every $m \in \mathbb{N}$.*

Proof of the Claim. Let $k \in \mathbb{N}$ be such that for every $m \in \mathbb{N}$ the colored half of X_m relative to c_m has size $\leqslant k$ (such a k exists by (L2)). As each X_m is a rooted poset of width $\leqslant 2$, antichains in C_m have size $\leqslant 2$. Therefore, in view of Theorem 1, it will be enough to show that chains in C_m have size $\leqslant k + 1$ for every $m \in \mathbb{N}$.

Suppose the contrary, with a view to a contradiction. Then there exist $m \in \mathbb{N}$ and $x_1, \dots, x_{k+2} \in C_m$ such that

$$x_{k+2} < x_{k+1} < \cdots < x_1.$$

On the one hand, since $x_1 \in C_m$, we know that x_1 is colorless relative to c_m. From Proposition 16 it follows that so is every member of ${\downarrow} x_1$. On the other hand, since $x_{k+2} \notin S_{c_m}(X_m)$, we can apply Proposition 26, obtaining ${\uparrow} x_{k+2} \cap S_{c_m}(X_m) = \emptyset$. Together with the definition of C_m, this yields ${\downarrow} x_1 \cap {\uparrow} x_{k+2} \subseteq C_m$. By the above display and the fact that X_m is finite, this guarantees the existence of $y_1, \dots, y_{k+2} \in C_m$ such that

$$y_{k+2} \prec y_{k+1} \prec \cdots \prec y_1.$$

We will construct a family $\{z_i : 1 \leqslant i \leqslant k+1\}$ of pairwise distinct elements in the colored half of X_m relative to c_m, a contradiction with the assumption that this half has size $\leqslant k$. To this end, consider $1 \leqslant i \leqslant k + 1$ and recall that $y_{i+1} \prec y_i$ by the above display. As y_i is colorless (because $y_i \in C_m$), we can apply Proposition 17, obtaining that there exists $z_i \in X_m$ such that $y_{i+1} \prec z_i$ and $y_i \parallel z_i$. Therefore, $y_{i+1} \prec y_i, z_i$ with $y_i \neq z_i$ and y_i is colorless relative to c_m (because it belongs to C_m). Since $y_{i+1} \in C_m$, this implies that z_i is colored relative to c_m. Hence, we obtain a family $\{z_i : 1 \leqslant i \leqslant k+1\}$ of elements in the colored half of X_m relative to c_m. It only remains to prove that they are pairwise different. Suppose, on the contrary, that there exist $1 \leqslant i < j \leqslant k+1$ such that $z_i = z_j$. Then $y_{j+1} < y_{i+1} \prec z_i = z_j$, a contradiction with $y_{j+1} \prec z_j$. ⊠

Claim 28. *For every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $S_{c_m}(X_m)$ has size $> k$.*

Proof of the Claim. In view of (L2) and (L3), it suffices to show that there exists $k \in \mathbb{N}$ such that $|C_m| \leqslant k$ for every $m \in \mathbb{N}$. As the latter holds by Claim 27, we are done. ⊠

Claim 29. *There exists $k \in \mathbb{N}$ such that $X_m - S_{c_m}(X_m)$ has size $\leqslant k$ for every $m \in \mathbb{N}$.*

Proof of the Claim. Immediate from (L2), Claim 27, and the definition of C_m. ⊠

We are now ready to prove (ii). Suppose, with a view to contradiction, that there exists $k_1 \in \mathbb{Z}^+$ such that for every $m \in \mathbb{N}$ and $x, y, z \in X_m$ with $x \prec y, z$ and $y \parallel z$ and $y, z \in S_{c_m}(X_m)$ we have $|{\downarrow}\{y, z\}| \leqslant k_1$. Let also $k_2 \in \mathbb{Z}^+$ be such that $X_m - S_{c_m}(X_m)$ has size $\leqslant k_2$ for every $m \in \mathbb{N}$ (such a k_2 exists by Claim 29). To reach a contradiction, it will be enough to show that for every $m \in \mathbb{N}$ chains in $S_{c_m}(X_m)$ have size $\leqslant k_1 + k_2$. For suppose this is the case. As antichains in each $S_{c_m}(X_m)$ have size $\leqslant 2$ (because X_m is a rooted poset of width $\leqslant 2$), we can apply Theorem 1, obtaining that there exists $k \in \mathbb{N}$ such that $S_{c_m}(X_m)$ has size $\leqslant k$ for every $m \in \mathbb{N}$, a contradiction with Claim 28.

Accordingly, we will prove that for every $m \in \mathbb{N}$ chains in $S_{c_m}(X_m)$ have size $\leqslant k_1 + k_2$. Suppose the contrary, with a view to contradiction. As $S_{c_m}(X_m)$ is a downset of X_m by Proposition 26, there exist $x_1, \dots, x_{k_1+k_2+1} \in S_{c_m}(X_m)$ such that

$$x_1 \prec x_2 \prec \dots \prec x_{k_1+k_2+1}.$$

We will construct a family $\{y_i : k_1 \leqslant i \leqslant k_1 + k_2\}$ of elements of $X_m - S_{c_m}(X_m)$. To this end, consider $k_1 \leqslant i \leqslant k_1 + k_2$. Since k_1 is positive by assumption the element x_i exists. As $x_i \prec x_{i+1}$ and x_{i+1} is colorless relative to c_m (because $x_{i+1} \in S_{c_m}(X_m)$ by assumption), we can apply Proposition 17, obtaining that x_i has an immediate successor y_i such that $y_i \parallel x_{i+1}$. Observe that $x_1, \dots, x_{k_1+1} \leqslant x_{i+1}$, whence $|{\downarrow}\{y_i, x_{i+1}\}| > k_1$. Therefore,

$$x_i \prec y_i, x_{i+1}, \quad y_i \parallel x_{i+1}, \quad x_{i+1} \in S_{c_m}(X_m), \quad \text{and} \quad |{\downarrow}\{y_i, x_{i+1}\}| > k_1.$$

By the choice of k_1 this implies $y_i \notin S_{c_m}(X_m)$. Hence, $\{y_i : k_1 \leqslant i \leqslant k_1 + k_2\}$ of elements of $X_m - S_{c_m}(X_m)$.

Lastly, the choice of k_2 guarantees that $|X_m - S_{c_m}(X_m)| \leqslant k_2$. As k_2 is positive by assumption and

$$|\{y_i : k_1 \leqslant i \leqslant k_1 + k_2\}| \leqslant |X_m - S_{c_m}(X_m)| \leqslant k_2,$$

there exist $k_1 \leqslant i < j \leqslant k_1 + k_2$ such that $y_i = y_j$. Consequently, $x_i < x_j \prec y_j = y_i$, a contradiction with $x_i \prec y_i$.

Hence, we conclude that for every $k \in \mathbb{N}$ there exist $m_k \in \mathbb{N}$ and $x_k, y_k, z_k \in X_{m_k}$ with $x_k \prec y_k, z_k$ and $y_k \parallel z_k$ such that $y_k, z_k \in S_{c_{m_k}}(X_{m_k})$ and $k \leqslant |{\downarrow}\{y_k, z_k\}|$. As X_{m_k} is finite, we may further assume that

$$x_k \in \max(\{v \in X_{m_k} : v \text{ has two immediate successors in } S_{c_{m_k}}(X_{m_k})\}).$$

To complete the proof of (ii), it only remains to find some $t \in \mathbb{N}$ such that $|{\uparrow}\{y_k, z_k\}| \leqslant t$ for every $k \in \mathbb{N}$. In view of Theorem 1, it will be enough to show that the size of chains and antichains in the sequence of posets $\{\uparrow\{y_k, z_k\} : k \in \mathbb{N}\}$ is bounded above by some nonnegative integer. As each X_{m_k} is a rooted poset of width $\leqslant 2$, this is clear for antichains. Therefore, it only remains to prove that there exists $t \in \mathbb{N}$ such that chains in the posets $\{\uparrow\{y_k, z_k\} : k \in \mathbb{N}\}$ have size $\leqslant t$. Recall from Claim 29 that ⊠

Acknowledgements. We are thankful to Alex Citkin and Guram Bezhanishvili for helpful discussions about Prof. Kuznetsov's interests and contributions in connection with locally finite varieties of Heyting algebras. The third author was supported by the proyecto PID2022-141529NB-C21 de investigación financiado por MICIU/AEI/10.13039/501100011033 y por FEDER, UE. He was also supported by the Research Group in Mathematical Logic, 2021SGR00348 funded by the Agency for Management of University and Research Grants of the Government of Catalonia,. The second and third authors acknowledge the support of the MSCA-RISE-Marie Skłodowska-Curie Research and Innovation Staff Exchange (RISE) project MOSAIC 101007627 funded by Horizon 2020 of the European Union.

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Thijs Benjamins: Science Park 107, 1098 XG, Amsterdam, the Netherlands
Email address: xthijs_benjamins@live.nl

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, Science Park 107, 1098 XG, Amsterdam, the Netherlands
Email address: n.bezhanishvili@uva.nl

Tommaso Moraschini: Departament de Filosofia, Facultat de Filosofia, Universitat de Barcelona (UB), Carrer Montalegre, 6, 08001 Barcelona, Spain
Email address: tommaso.moraschini@ub.edu