Quotient Dynamics: the Logic of Abstraction

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Abstract. We propose a Logic of Abstraction, meant to formalize the act of "abstracting away" the irrelevant features of a model. We give complete axiomatizations for a number of variants of this formalism, and explore their expressivity. As a special case, we consider the "logics of filtration".

1 Introduction

In this work, we aim to formalize the process of *abstraction*, in the specific sense of "abstracting away", i.e. disregarding all 'irrelevant' distinctions. Since reality is potentially infinitely complex, abstraction is essential for scientific modeling. In principle, a model should represent *all* the facts, but in practice the model is always tailored to the relevant issues under discussion. In particular, this phenomenon is all-pervasive in the formal epistemology literature: when modeling epistemic scenarios, the modeler focuses on a set of relevant issues, and identifies situations that agree on all these issues, thus reducing the size and complexity of the model to manageable proportions. A well-known example is the Muddy Children puzzle [10]. A standard relational model for the n-children puzzle has 2^n states, but with this we disregard all irrelevant facts (e.g. the color of each kid's clothes, etc.), focusing only on whether each of the n children is dirty or not. However, the same situation may be analyzed at various levels of abstraction, depending on the particular application. Rather than modeling again every new application from scratch, a good modeler develops the art of simplifying older models in order to reuse them in new situations, by again "abstracting away" some of the issues.

We develop technical tools to formalize this concept of abstraction as a dynamic process. We do this in a modal framework based on the standard Kripke models, by introducing *dynamic abstraction modalities*, similar to the update operators of Dynamic Epistemic Logic [17, 1]. The "relevant issues" may be given syntactically, as a set of formulas, inducing an equivalence relation on worlds that satisfy the same relevant formulas; or we may give them semantically, by starting directly with an equivalence relation on possible worlds (the so-called *issue relation* of [5, 9], telling us which worlds agree on all the relevant issues). For most of this paper, we focus on the first (syntactic) option, but we also consider the second option in Section 5. Roughly speaking, we represent the process of abstraction as a *model transformation*, that maps any given model to a *quotient model*. While the *states* of the quotient model can be defined in a natural and canonical way (as *equivalence classes* with respect to the relevant equivalence relation, there are many different ways to define the valuation function, and more interestingly the accessibility relation(s), of the quotient model. This problem is already known from

Modal Logic, where it occurs when an appropriate notion of *filtration* is needed for a given logic. Defining the quotient relations corresponds to *lifting* the relation(s) of the initial model (maybe after first performing a *relational transformation*) to some relation(s) between the induced equivalence classes. Depending on the context, different such liftings can be used. In this work, we focus on what is called the $(\exists, \exists)$-lifting, which corresponds to the so-called *minimal filtration* in Modal Logic.[3]

In Section 2, we start with the (single-agent) basic modal language, then generalize it to all **PDL**-definable relations [7, 11, 14]. Here, **PDL**-programs play a meta-syntactic role: they are used to specify a *relational transformer*. We define a logic for each such transformer, by applying it to the original relation of the model, then applying the $(\exists, \exists)$-lifting to obtain the quotient relation. We investigate the expressivity of these logics, and prove completeness using reduction axioms in the style of Public Announcement Logic (PAL) [15, 12, 16, 4]. In Section 3, we apply this to the special case of modal filtrations. As an added benefit, we show that these logics internalize the so-called Filtration Theorem [7]: while usually stated as meta-logical result, it becomes a plain logical theorem in our proof systems. In Section 4, we move to a "multi-agent" (multi-relational) framework, but also increase the expressivity (by including all **PDL** programs into the syntax), thus obtaining "the Logic of Abstraction": a general logical formalism that can treat and compare various types of quotient-taking operations in a unified formalism. We give a complete axiomatization via reduction axioms. In contrast to PAL (where adding common knowledge operators increases expressivity), the addition of Kleene star (iteration) on programs is innocuous: this logic is co-expressive with a version of **PDL** (with a "universal program" 1). Finally, in Section 5 we discuss two further generalizations and variations of our setting: by considering other relation liftings than the $(\exists, \exists)$-lifting; and by taking the above-mentioned "semantic option", of starting with an issue relation on worlds, and investigating the corresponding logic of quotients.

Due to page restrictions, we omitted the long proofs in Section 4 from this submission. The proofs can be found in the extended version of this paper at *https://sites.google.com /site/ozgunaybuke/publications*.

2 Quotient-taking as a Model Transformer

In this section, we explain the main ideas behind the formalism developed in this paper and fix some notations. In particular, we provide a detailed description of our *quotient models* (defined for a specific modal language through a finite set of formulas), and introduce the so-called *abstraction modalities*. Our quotient models are similar to *filtrations* from modal logic (see [7, Section 2.3] for an overview of filtrations), but our notion is more general[4]. We then introduce our formal dynamic language including the abstraction modalities, and provide sound and complete axiomatizations of a specific family of dynamic abstraction logics.

[3] However, we'll show that, in combination with applying relational transformers described by regular PDL programs, this lifting can capture other filtrations.

[4] In Section 3, we will show precisely how filtrations fit into our framework

We start the section by introducing the static language we work with throughout the section. By $\mathcal{L}_E$ we denote the language of *basic modal logic* enriched with the *universal modality* defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E\varphi \mid \Diamond\varphi,$$

where p is a propositional variable, and E stands for the (dual) of the universal modality. We employ the usual definitions for $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$, and $\Box$. The fragment of $\mathcal{L}_E$ without the modality E is denoted by $\mathcal{L}$. Formulas of $\mathcal{L}_E$ are interpreted on Kripke models $\mathfrak{M} = (W, R, V)$ in a standard way (see, e.g., [7, Chapter 1]). In particular, $\mathfrak{M}, w \models E\varphi$ iff there is $v \in W$ with $\mathfrak{M}, v \models \varphi$.

In the following let a Kripke model $\mathfrak{M} = (W, R, V)$ be fixed. Our aim is to define a *quotient model* $\mathfrak{M}_\Sigma = (W_\Sigma, R_\Sigma, V_\Sigma)$ of $\mathfrak{M}$ wrt a finite[5] set of formulas $\Sigma \subseteq \mathcal{L}_E$.

The set $\Sigma \subseteq \mathcal{L}_E$ induces an *equivalence relation* $\sim_\Sigma$ on W: for $w, v \in W$

$$w \sim_\Sigma v \quad \text{iff} \quad \text{for all } \varphi \in \Sigma \ (\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}, v \models \varphi)\,. \tag{1}$$

In other words, two worlds are *Σ-equivalent* iff they satisfy the same formulas from Σ. We denote by $|w|_\Sigma$ the equivalence class of w with respect to $\sim_\Sigma$, i.e., $|w|_\Sigma := \{v \in W \mid w \sim_\Sigma v\}$. The domain of our quotient model will be the set of equivalence classes with respect to $\sim_\Sigma$, i.e. $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$.

Concerning the valuation V_Σ, for any propositional letter p, we set

$$V_\Sigma(p) := \{|w|_\Sigma \mid \text{ there is } w' \in |w|_\Sigma \text{ with } w' \in V(p)\}^6.$$

While this generalizes the definition of the valuation used in filtrations, (see, e.g., [7, Chapter 2.3], it also constitutes the minimal valuation that preserves the truth value of *true* propositional letters in each world, in the sense that if $w \models p$ then $|w|_\Sigma \models p$.

Finally, we get to the most important defintion, namely, the definition the relation R_Σ. The relation R_Σ is determined by two factors: the first factor is a prescription on how to transfer a relation on W to a relation on W_Σ. We refer to such a prescription as a *lifting* of the relation R from W to W_Σ (similar to *relation liftings* studied theoretical computer science). As an example consider the definition

$$|w|_\Sigma R_\Sigma |v|_\Sigma \text{ iff } \text{ there exists } w' \in |w|_\Sigma, \text{ and there exists } v' \in |v|_\Sigma \text{ such that } w' R v'.^7 \tag{2}$$

We call this the $(\exists, \exists)$-lifting of R for obvious reasons. In a similar manner, we can also define $(\exists, \forall)$-, $(\forall, \exists)$- and $(\forall, \forall)$-liftings of R. However, in this paper, we work with the $(\exists, \exists)$-lifting, and briefly mention the other options in Section 5.

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[5] The finiteness of Σ is in fact irrelevant for the definition of quotient models, however, this will be required in order to be able to provide reduction axioms for our new dynamic modalities introduced later in this section. This is why we keep the setting simple and work only with finite Σs.

[6] Note that two Σ-equivalent worlds may disagree on the propositional variables that are not in the set Σ.

[7] This definition is known to modal logicians under the name of *smallest filtration* (see, e.g., [7, Chapter 2.3]).

The second factor to characterize R_Σ consists in deciding which relation to lift from W to W_Σ. For example, in (2), the relation R is lifted (as maybe the most obvious choice). In our framework though, we will allow more flexibility by considering liftings of the so-called $\mathbf{PDL}_{-*}$*-definable relations* (à la van Benthem and Liu [6]). More formally, the programs in the language of *star-free Propositional Dynamic Logic* ($\mathbf{PDL}_{-*}$) are defined by the grammar

$$\pi ::= r \mid ?\varphi \mid 1 \mid \pi;\pi \mid \pi \cup \pi,$$

where r is the (only) basic program[8] and φ is a formula in the language $\mathcal{L}_E$. The program 1 stands for the *universal program*. As usual, a program π determines a relation R_π on the model $\mathfrak{M}$ recursively defined as: $R_r := R$, $R_1 := W \times W$, and $R_{?\varphi} := \{(x,x) \mid \mathfrak{M}, x \models \varphi\}$, for some $\varphi \in \mathcal{L}_E$, and for any two programs π and π', we have $R_{\pi;\pi'} := R_\pi; R_{\pi'}$, and $R_{\pi\cup\pi'} := R_\pi \cup R_{\pi'}$, where $R_\pi; R_{\pi'}$ and $R_\pi \cup R_{\pi'}$ are the composition and the union of the relations R_π and $R_{\pi'}$, respectively. A binary relation Q on W is called $\mathbf{PDL}_{-*}$*-definable* iff $Q = R_\pi$ for some program π of $\mathbf{PDL}_{-*}$.

In this section, any $\mathbf{PDL}_{-*}$-definable relation can be used to determine the relations on our quotient models. In detail, in our framework each program π leads to a model transformation function that takes a Kripke model $\mathfrak{M}$ and a finite $\Sigma \subseteq \mathcal{L}_E$, and returns the quotient model $\mathfrak{M}_\Sigma$ *whose relation* R_Σ *is determined by the* $(\exists, \exists)$*-lifting of the relation* R_π. As a consequence, each program π will lead to a π-dependent dynamic logic.

Definition 1 (Quotient model wrt π) *Let $\mathfrak{M} = (W, R, V)$ be a Kripke model. For every finite $\Sigma \subseteq \mathcal{L}_E$, the quotient model of $\mathfrak{M}$ with respect to Σ is $\mathfrak{M}_\Sigma = (W_\Sigma, R_\Sigma, V_\Sigma)$, where $W_\Sigma := \{|w|_\Sigma \mid w \in W\}$, $V_\Sigma(p) := \{|w|_\Sigma \mid$ there is $w' \in |w|_\Sigma$ with $w' \in V(p)\}$, and*

$$|w|_\Sigma R_\Sigma |v|_\Sigma \text{ iff there is } w' \in |w|_\Sigma \text{ and there is } v' \in |v|_\Sigma \text{ such that } w' R_\pi v'.$$

Therefore, each π describes a particular type of model transformation whose arguments vary over finite subsets Σ of the language $\mathcal{L}_E$. As usual in dynamic epistemic logics [17], we introduce dynamic modalities, denoted by $[\Sigma]$, capturing this type of model change and call them the *abstraction modalities*. Before we formally define the dynamic language and the semantics of the abstraction modalities, we point out some observations concerning their expressive power. Unlike e.g. the public announcement operator (see, e.g., [15, 16]), the abstraction modality adds expressivity to the basic modal language $\mathcal{L}$:

Fact 1 *The abstraction modality adds expressivity to the basic modal language $\mathcal{L}$.*

Indeed, let $\pi = r$ be the basic program, i.e. the relation R_Σ on the quotient model $\mathfrak{M}_\Sigma$ is defined as in (2). Using the abstraction modality we can e.g. express the existential statements $\Psi :=$ "$\exists x, y \in W$ with xRy", or $\Psi' :=$ "$\exists x \in W$ with $\mathfrak{M}, x \models p$.", namely by $[\{\top\}]\Diamond\top$, and $[\{\top\}]p$, respectively. It is well-known that neither Ψ nor Ψ' are expressible in the basic modal language $\mathcal{L}$. Note, however, that the statements *are* expressible in $\mathcal{L}_E$, that is, when the universal modality is added to $\mathcal{L}$. On the other hand, the

[8] In this section—since the formalism is based on Kripke models with a single relation—we have only one basic program r in our syntax. In Section 4, we work with multi-relational Kripke models allowing for more than one basic programs, as standard in **PDL**.

universal modality can express statements that are not expressible via the abstraction modality.

Fact 2 *The universal modality and the abstraction modality are not equally expressive.*

For example, the statement χ := "$\exists x \in W$ with $\mathfrak{M}, x \models \neg p$" for some propositional letter p is not expressible with the abstraction modality. To illustrate, consider the two models $\mathfrak{M}$ and $\mathfrak{M}'$.

Then $\mathfrak{M}, x$ satisfies χ but $\mathfrak{M}', x'$ does not satisfy χ. Since x and x' are bisimilar for $\mathcal{L}$, they satisfy the same formulas in the language $\mathcal{L}$. Now for every finite $\Sigma \subseteq \mathcal{L}$, either ($\mathfrak{M}_\Sigma = \mathfrak{M}$ and $\mathfrak{M}'_\Sigma = \mathfrak{M}'$) or $\mathfrak{M}_\Sigma = \mathfrak{M}'_\Sigma = \mathfrak{M}'$. Therefore, x and x' agree on all formulas in the language $\mathcal{L}$ extended by the abstraction modality. Thus, χ is not expressible via $[\Sigma]$. We point out that these examples of course depend on the program π we choose for the quotient model.

The above expressivity results imply that the basic modal language with the abstraction modality is not reducible to basic modal language. This motivates why we work with the language $\mathcal{L}_E$ (but not with the simpler basic modal language $\mathcal{L}$) as our static language. In fact, we will show that $\mathcal{L}_E$ together with the abstraction modality is co-expressive with $\mathcal{L}_E$.

Formally, our dynamic language $\mathcal{L}_{E,[\Sigma]}$ is defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E\varphi \mid \Diamond\varphi \mid [\Sigma]\varphi$$

where Σ is a finite subset of $\mathcal{L}_E$. For a fixed program π, we evaluate formulas of $\mathcal{L}_{E,[\Sigma]}$ as follows:

Definition 2 (Semantics for $[\Sigma]\varphi$ wrt π) *Given a Kripke model $\mathfrak{M} = (X, V, R)$ and a state $w \in W$, the truth of $\mathcal{L}_{E,[\Sigma]}$-formulas is defined for Boolean cases, and the modalities $\Diamond$ and E as usual. The semantics for the abstraction modality $[\Sigma]\varphi$ is given by*

$$\mathfrak{M}, w \models [\Sigma]\varphi \ \textit{ iff } \ \mathfrak{M}_\Sigma, |w|_\Sigma \models \varphi.$$

where $\mathfrak{M}_\Sigma$ is the quotient model built wrt the program π.

In the rest of this section we will define a family of logics $\mathbf{K}_{E,\Sigma}(\pi)$—one for each program π of $\mathbf{PDL}_{-*}$— and show their soundness and completeness wrt to our semantics. While the soundness proof is standard, the completeness is established via reducing the dynamic logic to its underlying static base through a set of so-called *reduction axioms*. The reduction axioms (given in Table 1) describe a recursive rewriting algorithm that converts the formulas in $\mathcal{L}_{E,[\Sigma]}$ to semantically and provably equivalent formulas in $\mathcal{L}_E$. The key property that allows us to obtain reduction axioms in this particular setting is that—by finiteness of Σ and the presence of the universal modality—the equivalence relation $\sim_\Sigma$ becomes *definable* in our language in the sense of Lemma 1.

We fix the following notation: for every finite $\Sigma \subseteq \mathcal{L}_E$, and for every formula $\chi \in \mathcal{L}_{E,[\Sigma]}$ let

$$\langle \sim_\Sigma \rangle \chi := \bigvee_{\Psi \subseteq \Sigma} \left(\hat{\Psi} \wedge E \left(\hat{\Psi} \wedge \chi \right) \right), \tag{3}$$

where $\hat{\Psi} = \bigwedge \Psi \wedge \bigwedge \neg(\Sigma \setminus \Psi)$. The modality $\langle \sim_\Sigma \rangle$ is the diamond modality of the equivalence relation induced by Σ, thus $\sim_\Sigma$ is definable in $\mathcal{L}_{E,[\Sigma]}$:

Lemma 1 *Let $\mathfrak{M} = (W, R, V)$ be a model and let Σ be a finite set of formulas of $\mathcal{L}_{E,[\Sigma]}$. Then $\mathfrak{M}, x \models \langle \sim_\Sigma \rangle \chi$ iff there is $x' \sim_\Sigma x$ with $\mathfrak{M}, x' \models \chi$.*

Proof. Let $\mathfrak{M} = (W, R, V)$ be a Kripke model, Σ a finite subset of $\mathcal{L}_E$ and $\chi \in \mathcal{L}_{E,[\Sigma]}$.

($\Rightarrow$) Suppose $\mathfrak{M}, x \models \bigvee_{\Psi \subseteq \Sigma} \left(\hat{\Psi} \wedge E \left(\hat{\Psi} \wedge \chi \right) \right)$. This means that $\mathfrak{M}, x \models \hat{\Psi} \wedge E \left(\hat{\Psi} \wedge \chi \right)$ for some $\Psi \subseteq \Sigma$. I.e., we have $\mathfrak{M}, x \models \hat{\Psi}$ and $\mathfrak{M}, x \models E \left(\hat{\Psi} \wedge \chi \right)$. The latter implies that there is $x' \in W$ such that $\mathfrak{M}, x' \models \hat{\Psi} \wedge \chi$. Since $\mathfrak{M}, x' \models \hat{\Psi}$, we obtain $x \sim_\Sigma x'$, therefore the result follows.

($\Leftarrow$) Suppose there is $x' \in W$ such that $x \sim_\Sigma x'$ and $\mathfrak{M}, x' \models \chi$. As $x \sim_\Sigma x'$, the states x and x' make exactly the same formulas in Σ true. Therefore, we obtain that $\mathfrak{M}, x \models \hat{\Psi}$ and $\mathfrak{M}, x' \models \hat{\Psi}$ for some $\Psi \subseteq \Sigma$. The latter together with the assumption $\mathfrak{M}, x' \models \chi$ implies that $\mathfrak{M}, x \models E(\hat{\Psi} \wedge \chi)$. We therefore obtain $\mathfrak{M}, x \models \hat{\Psi} \wedge E(\hat{\Psi} \wedge \chi)$. Thus, $\mathfrak{M}, x \models \bigvee_{\Psi \subseteq \Sigma} \left(\hat{\Psi} \wedge E \left(\hat{\Psi} \wedge \chi \right) \right)$.

Table 1 contains reduction axioms and rules of the logic $\mathbf{K}_{E,\Sigma}(\pi)$. Note that the axiom (Ax-$\Diamond_\pi$) contains the symbol $\langle \pi \rangle$ which is *not* part of the language $\mathcal{L}_{E,[\Sigma]}$. Recall that the programs used to build π do not contain the star-operator. Since the language of star-free-**PDL** (with the universal program) is as expressive as the language $\mathcal{L}_E$, we can legitimately use the axiom (Ax-$\Diamond_\pi$) as an abbreviation for a formula in the language $\mathcal{L}_{E,[\Sigma]}$ (cf. [6]). To be precise, we employ the following abbreviations: $\langle r \rangle \psi := \Diamond \psi$, $\langle 1 \rangle \psi := E\psi$, $\langle ?\varphi \rangle \psi := \psi \wedge \varphi$, $\langle \pi; \pi' \rangle \psi := \langle \pi \rangle \langle \pi' \rangle \psi$, and $\langle \pi \cup \pi' \rangle \psi := \langle \pi \rangle \psi \vee \langle \pi' \rangle \psi$ for formulas $\psi \in \mathcal{L}_{E,[\Sigma]}$, $\varphi \in \mathcal{L}_E$ and programs π, π' of $\mathbf{PDL}_{-*}$.

Table 1: The logic $\mathbf{K}_{E,\Sigma}(\pi)$:

(**K**)	Axioms and rules of the basic modal logic **K**
(**E**)	**S5**-axioms and rules for E, $\Diamond \varphi \rightarrow E\varphi$
(Ax-p)	$[\Sigma]\, p \leftrightarrow \langle \sim_\Sigma \rangle p$
(Ax-$\neg$)	$[\Sigma]\neg\varphi \leftrightarrow \neg[\Sigma]\varphi$
(Ax-$\wedge$)	$[\Sigma](\varphi \wedge \psi) \leftrightarrow [\Sigma]\varphi \wedge [\Sigma]\psi$
(Ax-E)	$[\Sigma]E\varphi \leftrightarrow E[\Sigma]\varphi$
(Ax-$\Diamond_\pi$)	$[\Sigma]\Diamond\varphi \leftrightarrow \bigvee_{\Psi \subseteq \Sigma} \left(\hat{\Psi} \wedge E \left(\hat{\Psi} \wedge \langle \pi \rangle [\Sigma]\varphi \right) \right)$
(Nec$_{[\Sigma]}$)	From φ infer $[\Sigma]\varphi$

Completeness of $\mathbf{K}_{E,\Sigma}(\pi)$ is shown by defining a translation $t_\pi : \mathcal{L}_{E,[\Sigma]} \rightarrow \mathcal{L}_E$ that transforms each formula formula in the language $L_{E,[\Sigma]}$ to a $\mathbf{K}_{E,\Sigma}(\pi)$-provably equivalent formula in the language $\mathcal{L}_E$. We will skip the details of this translation since we will later discuss a similar translation in Section 4. We then obtain:

Theorem 1 (Expressivity) *Let π be a* **PDL**$_{-*}$*-program. For every $\varphi \in \mathcal{L}_{E,[\Sigma]}$, $\vdash_{\mathbf{K}_{E,\Sigma}(\pi)} \varphi \leftrightarrow t_\pi(\varphi)$.*

We can now derive completeness results by standard arguments from the completeness of the basic modal logic with the universal modality $\mathbf{K}_E$ (see [13] for the completeness of $\mathbf{K}_E$) and the soundness of $\mathbf{K}_{E,\Sigma}(\pi)$.

Theorem 2 (Completeness) *Let π be a* **PDL**$_{-*}$*-program. The logic $\mathbf{K}_{E,\Sigma}(\pi)$ is sound and complete wrt to the class of all Kripke models, where the quotient models are taken wrt the program π.*

3 Special Case: Logics of Filtrations

Thinking of quotient models, *filtrations* may be the first thing coming to the mind of a modal logician. Filtrations are used in order to prove the finite model property of some modal logics (see e.g. [7, Section 2.3] and [8, Section 5.3]). Roughly speaking, they turn a (refutation) model into a finite one by forming a quotient. In order to preserve some relational properties of Kripke models (such as transitivity, reflexivity etc.) there are several ways to define quotient models, leading to several notions of filtrations. In this section, we show how some well-known filtrations can be captured by the quotient models given in Definition 1. More precisely, for a filtration f—where f stands for the *smallest*, the *largest*, the *transitive* or the *smallest-transitive* filtration—we will define a program π_f of **PDL**$_{-*}$ such that the quotient model wrt the program π_f corresponds exactly to an f-filtration. In this sense, we can say that the logic of the filtration f is the logic $\mathbf{K}_{E,\Sigma}(\pi_f)$ axiomatized in Table 1. We will also comment on the possibility of adding additional axioms to these logics. Roughly, an axiom χ of the basic modal language $\mathcal{L}$ can be safely added to the logic $\mathbf{K}_{E,\Sigma}(\pi_f)$, whenever the basic modal logic axiomatized by χ *admits f-filtrations* (see, e.g., [8, Chapter 5.3]).

We will refer to the smallest, the largest, the transitive (a.k.a. Lemmon filtrations) and the smallest-transitive filtration, by s, l, t, and st, respectively. For the definitions of the first three filtrations, see e.g. [7, Section 2.3]. The smallest transitive filtration is obtained by first taking the smallest filtration and then replacing the resulting relation Q with its transitive closure Q^+ (see e.g. [8, Chapter 5.3])[9].

We will now define programs π_f in the language of **PDL**$_{-*}$ whose corresponding quotient models coincide with that of the f-filtration for $f \in \{s, l, t, st\}$. Let Σ be a finite set of formulas in the language $\mathcal{L}_E$. For $\Psi \subseteq \Sigma$, we set $\Psi_\Diamond = \bigwedge_{\Diamond\varphi\in\Sigma,\varphi\in\Psi} \Diamond\varphi$, $\Psi_{\Diamond,\vee} = \bigwedge_{\Diamond\varphi\in\Sigma,\varphi\in\Psi}(\Diamond\varphi \vee \varphi)$, and $\neg\Psi = \{\neg\varphi \mid \varphi \in \Psi\}$. We then define the following

[9] The filtrations in the aforementioned sources are defined for a language without the universal modality. However, as observed in [13, Section 5.2], the universal modality does not cause any problems in the theory of filtrations.

programs: let $\pi_\Sigma = \bigcup_{\Psi\subseteq\Sigma}(?\hat{\Psi};1;?\hat{\Psi})$, and for $k \in \mathbb{N}$, let $\pi_1 = r$ and $\pi_{k+1} = r;\pi_\Sigma;\pi_k$, then define

$$\pi_s := r, \quad \pi_l := \bigcup_{\Psi\subseteq\Sigma}(?\Psi_\Diamond;1;?\hat{\Psi}), \quad \pi_t := \bigcup_{\Psi\subseteq\Sigma}(?\Psi_{\Diamond,\vee};1;?\hat{\Psi}), \text{ and } \quad \pi_{st} := \bigcup_{1\leq k\leq 2^{|\Sigma|}} \pi_k.$$

It is easy to see that the quotient model w.r.t the program π_f corresponds exactly to an f-filtration for $f \in \{s, l, t, st\}$. To prove this for the smallest-transitive filtration, observe that by finiteness of Σ, the size of W_Σ is bounded by $2^{|\Sigma|}$. Thus, the transitive closure of a relation on W_Σ is reached by at most $2^{|\Sigma|}$ many iterations.

Proposition 1 *Let $f \in \{s, l, t, st\}$. For every finite and subformula closed*[10] *set $\Sigma \subseteq \mathcal{L}_E$, the model $\mathfrak{M}^{\pi_f}_\Sigma$ is an f-filtration of $\mathfrak{M}$ through Σ.*

The quotient models resulting from the transitive and the smallest-transitive filtrations are always transitive. To a modal logician, these filtrations in fact become interesting only when applied to transitive Kripke models, since otherwise the Filtration Theorem does not hold (see, e.g., [8, Theorem 5.23]). The transitivity of the quotient models implies that the (4)-Axiom ($\Diamond\Diamond\varphi \to \Diamond\varphi$) is valid on these models. Therefore, if the (4)-Axiom is added to the logics $\mathbf{K}_{E,\Sigma}(\pi_t)$ and $\mathbf{K}_{E,\Sigma}(\pi_{st})$, the necessitation rule ($\text{Nec}_{[\Sigma]}$) for $[\Sigma]$ remains sound. We can therefore extend the logics $\mathbf{K}_{E,\Sigma}(\pi_t)$ and $\mathbf{K}_{E,\Sigma}(\pi_{st})$ by the (4)-axiom and obtain sound systems. In more general terms, whenever a quotient model wrt a filtration type f preserves the validity of a certain axiom, this axiom can be safely added to the dynamic logic $\mathbf{K}_{E,\Sigma}(\pi_f)$ without affecting soundness and completeness. In fact, the same is true under slightly weaker assumptions. Let χ be an axiom that characterizes the class $\mathcal{K}$ of Kripke models. Sometimes the validity of χ is not preserved in all quotient models (wrt filtration type f) of the class $\mathcal{K}$, but it is preserved in quotient models of a smaller class of models $\mathcal{K}' \subseteq \mathcal{K}$ (e.g., the filtration *st* preserves the validity of the (.2)-axiom only on *rooted*[11] transitive models, but not on arbitrary transitive models (see [8, Theorem 5.33])). If the smaller class $\mathcal{K}'$ is "big enough", meaning that the logic axiomatized by χ is complete wrt $\mathcal{K}'$, then its dynamic extension is also complete wrt the class $\mathcal{K}'$, where quotient models are taken wrt π_f. To modal logicians such considerations run under the name of *admitting filtration*, see [8, Section 5.2].

For a normal modal logic $\mathbf{L}$ (e.g., $\mathbf{T}$, $\mathbf{KB}$ or $\mathbf{K4}$ etc., see [8] for our notational convention for the normal modal logics.), by $\mathbf{L}_{E,\Sigma}(\pi_f)$, we denote the logic that is obtained from the axioms and rules of $\mathbf{L}$ and from Table 1 for $f \in \{s, l, t, st\}$. Using results explored in [8, Chapter 5.2], Proposition 1 and Theorem 2, we obtain the following:

Corollary 1
1. *For $f \in \{s, l, t, st\}$, the logics $\mathbf{D}_{E,\Sigma}(\pi_f)$ and $\mathbf{T}_{E,\Sigma}(\pi_f)$ are sound and complete wrt the class of serial Kripke models and reflexive Kripke models, respectively (where the quotient models are taken wrt π_f).*
2. *$\mathbf{KB}_{E,\Sigma}(\pi_s)$ is sound and complete wrt symmetric Kripke models.*

[10] Since filtrations are usually only defined for subformula closed sets—the reason being that the Filtration Theorem can only be proved in this case—we add this as an additional condition.

[11] Recall that a transitive Kripke model $\mathfrak{M}$ is called *rooted* if there is $s \in W$ such that sRw for all $w \in \mathfrak{M}$.

3. *For $f \in \{t, st\}$, the logics $\mathbf{K4}_{E,\Sigma}(\pi_f)$, $\mathbf{D4}_{E,\Sigma}(\pi_f)$, and $\mathbf{S4}_{E,\Sigma}(\pi_f)$ are sound and complete wrt transitive, transitive serial, and reflexive transitive models, respectively (where the quotient models are taken wrt π_f).*
4. *$\mathbf{K4.2}_{E,\Sigma}(\pi_{st})$ and $\mathbf{K4.3}_{E,\Sigma}(\pi_{st})$ are sound and complete wrt the class of rooted transitive directed models, and rooted transitive connected models, respectively. Moreover, $\mathbf{S4.2}_{E,\Sigma}(\pi_{st})$ and $\mathbf{S4.3}_{E,\Sigma}(\pi_{st})$ are sound and complete wrt the class of Kripke models based on rooted directed quasi-orders, and rooted linear quasi-orders, respectively.*
5. *For $f \in \{s, l, t, st\}$, the logic $\mathbf{S5}_{E,\Sigma}(\pi_f)$ is sound and complete w.r.t the class of Kripke models based on clusters. In fact, $\mathbf{S5}_{E,\Sigma}(\pi_{st})$ is sound and complete w.r.t the class of Kripke models based on equivalence relations.*

Remark 1: We note that the above corollary can be proved for a larger class of stable and transitive stable logics of [2, 3]. These are logics that are sound and complete with respect to classes of rooted frames closed under graph homomorphisms. In other words, these are the logics admitting all filtrations and all transitive filtrations, respectively. In this respect, stable logics play a similar role to the abstraction modality that subframe logics—logics whose frames are closed under subframes [8, Chapter 11.3]—play for the public announcement operator.

Finally, we comment on the meaning of the Filtration Theorem in our context (see, e.g., [7, Theorem 2.39] for the filtration theorem). Due to the completeness result stated in Theorem 2, the Filtration Theorem can be proven syntactically in our logics $\mathbf{K}_{E,\Sigma}(\pi_f)$, i.e., it can be *internalized* as a theorem of these systems:

Corollary 2 (Internalized Filtration Theorem) *For every finite subformula closed set $\Sigma \subseteq \mathcal{L}_E$ and all $\varphi \in \Sigma$, we have the following:*

1. $\vdash_{\mathbf{K}_{E,\Sigma}(\pi_f)} [\Sigma]\varphi \leftrightarrow \varphi$, *for $f \in \{s, l\}$;*
2. $\vdash_{\mathbf{K4}_{E,\Sigma}(\pi_f)} [\Sigma]\varphi \leftrightarrow \varphi$, *for $f \in \{t, st\}$.*

4 The Logic of Abstraction

This section generalizes the setting presented in Sections 2 and 3 in many ways. To start with, we move to a multi-relational setting, also called multi-agent setting, allowing for many basic programs in a given **PDL**-language. Secondly—and more importantly—we generalise the abstraction modalities in such a way that the **PDL**-programs become a component of these modalities. More precisely, an abstraction modality contains a sequence of programs $\overrightarrow{\pi}$ that are indexed by the set of agents as a parameter. The program π_r corresponding to agent r determines the relation of the same agent in the quotient model. Another generalization over the previous setting is that we allow programs in the (full) **PDL**-language, i.e. the language including the star-operator. In this section, we introduce semantics for this extended langauge on multi-relational Kripke models and provide a sound and complete axiomatization for the logic of abstraction $\mathbf{PDL}_{\mathbf{\Sigma}}$.

Since the star operator properly adds expressivity to the static (multi-)modal language, our resulting dynamic logic $\mathbf{PDL}_{\Sigma}$ will not be reducible to basic modal logic. Instead, we will employ the language of **PDL** as our base language.

We would like to stress the two different uses of the language of propositional dynamic logic: while, in the previous sections, the language of $\mathbf{PDL}_{-*}$ was only used as a "meta-language" for abbreviations of formulas in $\mathcal{L}_E$, the language of **PDL** here becomes an essential part of our logical language. Our dynamic language $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ is defined by extending the language of propositional dynamic logic **PDL** with the abstraction modalities $[\overrightarrow{\pi}/\Sigma]\varphi$. More precisely, $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ is defined by the grammar:

$$\pi ::= r \mid ?\psi \mid 1 \mid \pi;\pi \mid \pi \cup \pi \mid \pi^*, \text{ and } \quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \mid [\overrightarrow{\pi}/\Sigma]\varphi,$$

where r is an element of the set of basic programs Π_0, $\psi \in \mathsf{PDL}$, $\overrightarrow{\pi} = (\pi_r)_{r\in\Pi_0}$ is a sequence of PDL-programs, and Σ is a *finite*[12] subset of PDL (the language $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ without $[\overrightarrow{\pi}/\Sigma]\varphi$).

Given a (multi-relational) Kripke model $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, V)$, we interpret programs as relations on $\mathfrak{M}$ as usual and denote the relation corresponding to the program π by R_π. Recall that the relation R_π is defined recursively on the structure of π. In particular, $R_1 := W \times W$, and $R_{?\psi} := \{(x,x) \mid \mathfrak{M}, x \models \psi\}$, for some $\psi \in \mathsf{PDL}$. Just as in (1) (see Section 2), a finite subset $\Sigma \subseteq \mathsf{PDL}$ induces an equivalence relation $\sim_\Sigma$ on W by relating two worlds that satisfy the same formulas of Σ. We denote by $|w|_\Sigma$ the equivalence class of $w \in W$ with respect to $\sim_\Sigma$.

Next we define the *(multi-relational) quotient models*. Recall that in Definition 1 we defined quotient models wrt a fixed program π. In the current setting, the sequence of programs $\overrightarrow{\pi}$ becomes a parameter of the quotient models, thus receives a similar status as the set Σ. This is reflected in the shape of the abstraction modalities $[\overrightarrow{\pi}/\Sigma]\varphi$.

Definition 3 (Quotient model) *Let $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, V)$ be a Kripke model. For every finite $\Sigma \subseteq \mathsf{PDL}$ and every sequence $\overrightarrow{\pi} = (\pi_r)_{r\in\Pi_0}$ of programs, the* quotient model $\mathfrak{M}^{\overrightarrow{\pi}}_\Sigma$*, is $\mathfrak{M}^{\overrightarrow{\pi}}_\Sigma = (W_\Sigma, (R^{\pi_r}_\Sigma)_{r\in\Pi_0}, V_\Sigma)$, where $W_\Sigma := \{|w|_\Sigma \mid w \in W\}$, $V_\Sigma(p) := \{|w|_\Sigma \mid$ there is $w' \sim_\Sigma w$ with $w' \in V(p)\}$, and for each $r \in \Pi_0$*

$$|w|_\Sigma R^{\pi_r}_\Sigma |v|_\Sigma \textit{ iff there is } w' \sim_\Sigma w \textit{ and there is } v' \sim_\Sigma v \textit{ with } w' R_{\pi_r} v'.$$

In other words, using the terminology of Section 2, the quotient model $\mathfrak{M}^{\overrightarrow{\pi}}_\Sigma$ arises from $\mathfrak{M}$ by interpreting a basic program $r \in \Pi_0$ via the $(\exists, \exists)$-lifting of the relation R_{π_r} from W to W_Σ.

Definition 4 (Semantics for $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$) *Given a Kripke model $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, V)$ and a state w in W, the truth of $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$-formulas at a world w in $\mathfrak{M}$ is defined recursively as for* PDL *with the additional clause:*

$$\mathfrak{M}, w \models [\overrightarrow{\pi}/\Sigma]\varphi \quad \textit{iff} \quad \mathfrak{M}^{\overrightarrow{\pi}}_\Sigma, |w|_\Sigma \models \varphi$$

[12] Similar to the case in Section 2, the sets Σ being finite is essential in order to obtain reduction axioms for the corresponding dynamic logic.

where $\mathfrak{M}^{\overrightarrow{\pi}}_{\Sigma}$ is as given in Definition 3.

Next we introduce reduction axioms that allow us to convert a formula of $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ to a provably equivalent formula in PDL. In the current setting, there are two key properties that allow us to obtain reduction axioms. Firstly, the equivalence relation $\sim_{\Sigma}$ is *definable* in the language $\mathsf{PDL}_{[\pi/\Sigma]}$ similar to the case in Section 2. Secondly, Σ being finite ensures that the model $\mathfrak{M}^{\pi}_{\Sigma}$ is not only finite but its size is bounded in terms of the size of Σ. In fact, the size of $\mathfrak{M}^{\pi}_{\Sigma}$ is at most $2^{|\Sigma|}$. For this reason we can obtain reduction axioms for the star-operator. As in (3), for every formula $\chi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ and finite $\Sigma \subseteq \mathsf{PDL}$ we fix the following notation:

$$\langle\sim_{\Sigma}\rangle\chi := \bigvee_{\Psi\subseteq\Sigma}\left(\hat{\Psi}\wedge\langle 1\rangle\left(\hat{\Psi}\wedge\chi\right)\right).$$

The modality $\langle\sim_{\Sigma}\rangle$ is the diamond modality of the relation $\sim_{\Sigma}$, as can be shown analogously to Lemma 1.

For an axiomatization of **PDL**, see [7, Section 4.8] or [14]. The universal program 1 requires the **S5** axioms and rules, and $\langle\pi\rangle p \rightarrow \langle 1\rangle p$ for every program π. The logic $\mathbf{PDL_{\Sigma}}$ is defined by the axioms and rules given in Table 2.

Table 2: The logic $\mathbf{PDL_{\Sigma}}$

(**PDL**)	Axiom-schemes and rules of **PDL**	
(Ax-p)	$[\overrightarrow{\pi}/\Sigma]\, p \leftrightarrow \langle\sim_{\Sigma}\rangle p$	
(Ax-$\neg$)	$[\overrightarrow{\pi}/\Sigma]\neg\varphi \leftrightarrow \neg[\overrightarrow{\pi}/\Sigma]\varphi$	
(Ax-$\wedge$)	$[\overrightarrow{\pi}/\Sigma](\varphi\wedge\psi) \leftrightarrow [\overrightarrow{\pi}/\Sigma]\varphi \wedge [\overrightarrow{\pi}/\Sigma]\psi$	
(Ax-$\langle 1\rangle$)	$[\overrightarrow{\pi}/\Sigma]\langle 1\rangle\varphi \leftrightarrow \langle 1\rangle[\overrightarrow{\pi}/\Sigma]\varphi$	
(Ax-$\langle r\rangle$)	$[\overrightarrow{\pi}/\Sigma]\langle r\rangle\varphi \leftrightarrow \langle\sim_{\Sigma}\rangle\langle\pi_r\rangle[\overrightarrow{\pi}/\Sigma]\varphi$	for all $r \in \Pi_0$
(Ax-$*$)	$[\overrightarrow{\pi}/\Sigma]\langle\alpha^*\rangle\varphi \leftrightarrow [\overrightarrow{\pi}/\Sigma]\bigvee_{n\leq 2^{|\Sigma|}}\langle\alpha\rangle^n\varphi$	
(Nec$_{[\overrightarrow{\pi}/\Sigma]}$)	From φ infer $[\overrightarrow{\pi}/\Sigma]\varphi$	

The reduction axioms enables us to show that every formula in $\mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ is provably equivalent (in the system $\mathbf{PDL_{\Sigma}}$) to a formula in the language PDL.

Theorem 3 (Expressivity) *For every $\varphi \in \mathsf{PDL}_{[\overrightarrow{\pi}/\Sigma]}$ there is a $\psi \in \mathsf{PDL}$ such that $\vdash_{\mathbf{PDL_{\Sigma}}} \varphi \leftrightarrow \psi$.*

Using Theorem 3, the completeness of $\mathbf{PDL_{\Sigma}}$ is a consequence of the completeness theorem for **PDL** and the soundness of the system $\mathbf{PDL_{\Sigma}}$.

Theorem 4 (Completeness) $\mathbf{PDL_{\Sigma}}$ *is sound and complete.*

5 Further Generalizations and Variations

In this final section, we outline some further results and alternatives.

Other Liftings: We used $(\exists, \exists)$-lifting to build the quotient models in Definition 3, but we can use other liftings as discussed in Section 2. However, we conjecture that reduction axioms for the $(\forall, \forall)$- and the $(\exists, \forall)$-lifts are *not available* in our setting.Though such reduction axioms might become available if we extend the base language by nominals as in hybrid logics. On the other hand, the setting using the $(\forall, \exists)$-lift of the relation R_π admits reduction axioms, obtained by replacing Ax-$\langle r\rangle$ from Table 2 by:

$$(\text{Ax-}\langle r\rangle) \quad [\vec{\pi}/\Sigma]\langle r\rangle\varphi \leftrightarrow \bigvee_{\Psi\subseteq\Sigma}\bigvee_{\Phi\subseteq\Sigma}\left(\hat{\Psi}\wedge\langle\pi_r\rangle\left(\hat{\Phi}\wedge[\vec{\pi}/\Sigma]\varphi\right)\wedge[1]\left(\hat{\Psi}\rightarrow\langle\pi_r\rangle\hat{\Phi}\right)\right)$$

The 'Semantic Option': While in this paper we focused on the 'syntactic option' (issues given by a set of formulas), we are also investigating the semantic option: each model comes with its own equivalence "issue" relation Q. In this set-up, models are of the shape $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, Q, V)$, where $(W, (R_r)_{r\in\Pi_0}, V)$ is a Kripke model and Q is an equivalence relation on W. We then define a language $\mathsf{PDL}_{Q,\vec{\pi}/Q}$ as:

$$\pi := r \mid Q \mid ?\psi \mid 1 \mid \pi;\pi \mid \pi\cup\pi \mid \pi^*, \text{ and } \quad \varphi := p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \langle\pi\rangle\varphi \mid [\vec{\pi}/Q]\varphi,$$

where r is an element of the set of the basic programs Π_0 and $\psi \in \mathsf{PDL}$ (the language $\mathsf{PDL}_{Q,\vec{\pi}/Q}$ without $[\vec{\pi}/Q]\varphi$). Note that we add a symbol Q to the basic programs whose intended interpretation is the equivalence relation Q. Its modality $[Q]$ is the so-called *issue modality* from [5]. For a model $\mathfrak{M} = (W, (R_r)_{r\in\Pi_0}, Q, V)$ and a sequence of programs $\vec{\pi}$, we define a model
$\mathfrak{M}^{\vec{\pi}}_Q := (W_Q, (R^{\pi_r}_Q)_{r\in\Pi_0}, \mathsf{Id}, V_Q)$, where $W_Q := \{|w| \mid \text{there is } w'Qw \text{ with } w' \in V(p)\}$, $V_Q(p) := \{|w| \mid w \in V(p)\}$, Id denotes the identity relation, and

$$|w|R^{\pi_r}_Q|v| \text{ iff there is } w'Qw \text{ and there is } v'Qv \text{ such that } w'R_{\pi_r}v',$$

where $|w|$ is the equivalence class of w wrt Q. The crucial step in the semantics is:

$$\mathfrak{M}, x \models [\vec{\pi}/Q]\varphi \quad \text{iff} \quad \mathfrak{M}^{\vec{\pi}}_Q, |x| \models \varphi.$$

To get a convenient representation of the reduction axioms, we define functions $f_{Q,\vec{\pi}}$ on programs by $f_{\vec{\pi},Q}(Q) = ?\top$, $f_{\vec{\pi},Q}(r) = Q;\vec{\pi}$, $f_{\vec{\pi},Q}(\alpha_1\circ\alpha_2) = f_{\vec{\pi},Q}(\alpha_1)\circ f_{\vec{\pi};Q}(\alpha_2)$ for $\circ\in\{\cup, ;\}$ and $f_{\vec{\pi},Q}(\pi^*) = \left(f_{\vec{\pi},Q}(\pi)\right)^*$. Here is the full list of reduction axioms:

Table 3: The logic $\mathbf{PDL}_Q$

(**PDL**)	Axiom-schemes and rules of **PDL**
(**Q**)	**S5**-axioms and rules for Q
(Ax-p)	$[\vec{\pi}/Q]p \leftrightarrow \langle Q\rangle p$
(Ax-$\neg$)	$[\vec{\pi}/Q]\neg\varphi \leftrightarrow \neg[\vec{\pi}/Q]\varphi$
(Ax-$\wedge$)	$[\vec{\pi}/Q](\varphi\wedge\psi) \leftrightarrow [\vec{\pi}/Q]\varphi\wedge[\vec{\pi}/Q]\psi$
(Ax-$\langle\alpha\rangle$)	$[\vec{\pi}/Q]\langle\alpha\rangle\varphi \leftrightarrow \langle f_{Q,\vec{\pi}}(\alpha)\rangle[\vec{\pi}/Q]\varphi$
(Ax-$\langle Q\rangle$)	$[\vec{\pi}/Q]\langle Q\rangle\varphi \leftrightarrow [\vec{\pi}/Q]\varphi$
(DR-Nec)	From φ infer $[\vec{\pi}/Q]\varphi$

Note that in our earlier versions, the analogue of the modality $\langle Q \rangle$ was *definable* in the language $\mathsf{PDL}_{[\vec{\pi}/\Sigma]}$ (cf. Section 2, Lemma 1), thus was not needed in the syntax.

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