CHARACTERIZING EXISTENCE OF A MEASURABLE CARDINAL VIA MODAL LOGIC

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Abstract. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic coincides with the modal logic of the Kripke frame isomorphic to the powerset of a two element set.

1. Introduction

Over the years there have been discovered several intriguing connections between set theory and modal logic. To name a few:

(1) There is an interesting connection between non-well-founded set theory and infinitary modal logic [1, 3, 2].

(2) The modal logic $S4.2$ turns out to be the logic of forcing extensions of $ZFC$ [16].

(3) The only existing proof that the modal logic $S4.1.2$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space $\omega$ requires that each MAD family has cardinality $2^{\omega}$, a principle that is not provable in $ZFC$, and it remains an open problem whether this principle is necessary [8].

To these results we add the following. Let the diamond $\mathfrak{D} = (D, \leq)$ be the partially ordered Kripke frame shown in Figure 1. It is clear that $\mathfrak{D}$ is isomorphic to the powerset of a two element set. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic is the modal logic of $\mathfrak{D}$.

![Figure 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.](image)

We recall that topological semantics generalizes Kripke semantics for the well-known modal logic $S4$. Thus, Kripke completeness implies topological completeness for logics above $S4$. However, topological spaces arising from Kripke frames are usually not even $T_1$. Therefore, it is nontrivial to prove topological completeness results above $S4$ with respect to spaces satisfying higher separation axioms. One such class is the class of Tychonoff spaces. By a celebrated theorem of Tychonoff, these are exactly subspaces of compact Hausdorff spaces. In [5] we initiated the study of modal logics arising from Tychonoff spaces. On the one hand, this yielded a new notion of dimension in topology, called modal Krull dimension. On the
other hand, it provided a new concept of zemanian logics which generalize the well-known modal logic of Zeman.

It is known that extremally disconnected spaces are topological models of the modal logic $S4.2$, and hereditarily extremally disconnected spaces are topological models of the modal logic $S4.3$. In [6] we showed that a modal logic above $S4.3$ is a zemanian logic iff it is the logic of an hereditarily disconnected Tychonoff space. The simplest modal logic above $S4.2$ that is not above $S4.3$ is the logic of $D$. In this paper we show that topological completeness of the logic of $D$ with respect to a normal space is equivalent to the existence of a measurable cardinal. Whether normal can be weakened to Tychonoff remains an open problem.

We conclude the introduction by briefly describing the key ingredients of the proof. If there exists a measurable cardinal $\kappa$, using a countably complete ultrafilter on $\kappa$, we first build a normal $P$-space $Y$. Combining the results of [12] and [13] then allows us to embed $Y$ into the remainder of the Cech-Stone compactification $\beta\mu$ of a cardinal $\mu$ viewed as a discrete space. Letting $Z = Y \cup \mu$ yields a normal space whose logic we prove is the logic of the diamond $D$. This we do by showing that a finite rooted Kripke frame $\mathfrak{F}$ is an interior image of $Z$ iff $\mathfrak{F}$ is an interior image of $D$.

Conversely, suppose there exists a normal space $Z$ whose logic is the logic of the diamond $D$. We first show that $D$ is an interior image of $Z$. We then prove that without loss of generality the inverse image of the root $r$ of $D$ is a singleton $\{a\}$. We next prove that $a$ is a $P$-point of an appropriately chosen subspace of $Z$. This allows us to define a family of subsets of $Z$ whose cardinal is Ulam-measurable. Finally, it is well known that this implies the existence of a measurable cardinal.

2. Preliminaries

In this section we recall the necessary background from modal logic, its topological semantics, and measurable cardinals.

2.1. Modal logic. We use [10] as the main reference for modal logic. Modal formulas are built in the usual way using countably many propositional letters, the classical connectives $\neg$ (negation) and $\rightarrow$ (implication), the modal connective $\Box$ (necessity), and parentheses. We employ the standard abbreviations: $\land$ (conjunction), $\lor$ (disjunction), and $\Diamond$ (possibility).

The well-known modal system $S4$ of Lewis is the least set of formulas containing the classical tautologies, the axioms

$$
\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),
\neg \Box p \rightarrow \neg p,
\Box p \rightarrow \Box \Box p,
$$

and closed under the inference rules of

- Modus Ponens $\varphi, \varphi \rightarrow \psi \vdash \psi$,
- substitution $\varphi(p_1, \ldots, p_n) \vdash \varphi[\psi_1, \ldots, \psi_n]$,
- necessitation $\Box \varphi \vdash \Box \varphi$.

A Kripke frame is a pair $\mathfrak{F} = (W, R)$ where $W$ is a nonempty set and $R$ is a binary relation on $W$. As usual, for $w \in W$ we let

$$
R(w) = \{v \in W \mid wRv\} \quad \text{and} \quad R^{-1}(w) = \{v \in W \mid vRw\};
$$

and for $A \subseteq W$ we let

$$
R(A) = \bigcup\{R(w) \mid w \in A\} \quad \text{and} \quad R^{-1}(A) = \bigcup\{R^{-1}(w) \mid w \in A\}.
$$
Kripke semantics of modal logic recursively assigns to each formula a subset of a Kripke frame $\mathcal{F}$ by interpreting each propositional letter as a subset of $W$, the classical connectives as Boolean operations in the powerset $\varphi(W)$, and $\square$ as the operation $\square_R$ on $\varphi(W)$ defined by

$$\square_R(A) = \{ w \in W \mid R(w) \subseteq A \}.$$ 

Consequently, $\Diamond$ is interpreted as the operation $\Diamond_R$ on $\varphi(W)$ defined by

$$\Diamond_R(A) = R^{-1}(A).$$

Let $\varphi$ be a modal formula and $\mathcal{F} = (W, R)$ a Kripke frame. Call $\varphi$ valid in $\mathcal{F}$, written $\mathcal{F} \models \varphi$, provided $\varphi$ evaluates to $W$ for every assignment of the propositional letters. If $\varphi$ is not valid in $\mathcal{F}$, then we say that $\varphi$ is refuted in $\mathcal{F}$, and write $\mathcal{F} \not\models \varphi$. The logic of $\mathcal{F}$ is the set of modal formulas valid in $\mathcal{F}$; in symbols $L(\mathcal{F}) = \{ \varphi \mid \mathcal{F} \models \varphi \}$.

A Kripke frame $\mathcal{F}$ is called an S4-frame if $R$ is reflexive and transitive. The name is justified by the well-known fact that S4 is sound and complete with respect to S4-frames. In this paper we are mainly interested in the following logic.

**Definition 2.1.** Let $L := L(\Diamond)$ be the logic of the diamond $\Diamond$ shown in Figure 1.

2.2. **Topological semantics.** Topological semantics interprets $\square$ as topological interior (and consequently $\Diamond$ as topological closure). Specifically, for a topological space $X$, the propositional letters are assigned to subsets of $X$, the classical connectives are computed as the Boolean operations in $\varphi(X)$, and $\square$ is interpreted as the interior operator $i : \varphi(X) \to \varphi(X)$, where $iA$ is the greatest open subset of $X$ contained in $A$. Consequently, $\Diamond$ is interpreted as the closure operator $c : \varphi(X) \to \varphi(X)$, where $cA$ is the least closed subset of $X$ containing $A$.

Let $\varphi$ be a modal formula and $X$ a space. Call $\varphi$ valid in $X$, denoted $X \models \varphi$, provided $\varphi$ evaluates to $X$ for every assignment of the propositional letters. If $\varphi$ is not valid in $X$, then we say that $\varphi$ is refuted in $X$, and write $X \not\models \varphi$. The logic of $X$ is the set of formulas valid in $X$; symbolically, $L(X) = \{ \varphi \mid X \models \varphi \}$. It is well known that S4 is sound and complete with respect to topological spaces.

There is a close connection between topological semantics and Kripke semantics for S4. Let $\mathcal{F} = (W, R)$ be an S4-frame. Call $U \subseteq W$ an $R$-upset of $\mathcal{F}$ if $w \in U$ and $wRv$ imply $v \in U$. The set of $R$-upsets of $\mathcal{F}$ is a topology $\tau_R$ on $W$ in which every point $w$ has a least neighborhood, namely $R(w)$. Such spaces are called Alexandroff spaces. We call $(W, \tau_R)$ the Alexandroff space of $\mathcal{F}$. For a modal formula $\varphi$, we have

$$\mathcal{F} \models \varphi \iff (W, \tau_R) \models \varphi.$$ 

Thus, topological semantics generalizes Kripke semantics for S4, and hence Kripke completeness for logics above S4 implies topological completeness. However, since Alexandroff spaces are usually not even $T_1$-spaces, such topological completeness is not guaranteed with respect to, for example, normal spaces.

We recall that a topological space $X$ is

- **extremally disconnected** (ED) if the closure of each open set is open;
- **resolvable** if $X$ is the union of two disjoint dense subsets of $X$;
- **irresolvable** if $X$ is not resolvable;
- **hereditarily irresolvable** (HI) if every subspace of $X$ is irresolvable.

Let

$$\text{grz} = \square(\square(p \to \square p) \to p) \to p$$

be the Grzegorczyk axiom and

$$\text{ga} = \Diamond \square p \to \square \Diamond p$$
the Geach axiom (see, e.g., [10]). It is well known that

\[
\begin{align*}
X \text{ is ED} & \iff X \models \text{grz;} \\
X \text{ is HI} & \iff X \models \text{ga}.
\end{align*}
\]

We next recall the definition of modal Krull dimension. For this we recall that a subset \( N \) of a space \( X \) is nowhere dense if \( \text{ic} N = \emptyset \).

**Definition 2.2.** ([5, Sec. 3]) Define the modal Krull dimension \( \text{mdim}(X) \) of a topological space \( X \) recursively as follows:

\[
\text{mdim}(X) = -1 \text{ if } X = \emptyset,
\]

\[
\text{mdim}(X) \leq n \text{ if } \text{mdim}(N) \leq n - 1 \text{ for each } N \text{ nowhere dense in } X,
\]

\[
\text{mdim}(X) = n \text{ if } \text{mdim}(X) \leq n \text{ but } \text{mdim}(X) \not\leq n - 1,
\]

\[
\text{mdim}(X) = \infty \text{ if } \text{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \ldots
\]

Let

\[
\begin{align*}
\text{bd}_1 & = \Diamond \Box p_1 \to p_1 \\
\text{bd}_{n+1} & = \Diamond (\Box p_{n+1} \land \neg \text{bd}_n) \to p_{n+1} \text{ for } n \geq 1.
\end{align*}
\]

**Theorem 2.3.** ([5, Thm. 3.6]) Let \( X \) be a nonempty space and \( n \geq 1 \). Then

\[
\text{mdim}(X) \leq n - 1 \iff X \models \text{bd}_n.
\]

For nonempty scattered Hausdorff spaces, there is a close connection between finite modal Krull dimension and Cantor-Bendixson rank. For \( Y \subseteq X \), let \( dY \) be the set of limit points of \( Y \) and for an ordinal \( \alpha \), let \( d^\alpha Y \) be defined recursively as follows:

\[
\begin{align*}
d^0 Y & = Y, \\
d^{\alpha+1} Y & = d(d^\alpha Y), \\
d^\alpha Y & = \bigcap \{d^\beta Y \mid \beta < \alpha \} \text{ if } \alpha \text{ is a limit ordinal.}
\end{align*}
\]

The Cantor-Bendixson rank of \( X \) is the least ordinal \( \gamma \) satisfying \( d^\gamma X = d^{\gamma+1} X \). It is well known that a space \( X \) is scattered iff there is an ordinal \( \alpha \) such that \( d^\alpha X = \emptyset \). Thus, the Cantor-Bendixson rank of a scattered space \( X \) is the least ordinal \( \gamma \) such that \( d^\gamma X = \emptyset \).

Let \( X \) be a nonempty scattered Hausdorff space and \( n \in \omega \). Then the Cantor-Bendixson rank of \( X \) is \( n + 1 \) iff \( d^n X \neq \emptyset \) and \( d^{n+1} X = \emptyset \), which by [7, Thm. 4.9] happens iff \( \text{mdim}(X) = n \).

2.3. **Measurable cardinals.** We use [17, 18] as standard references for set theory, and also rely on [11] as the main reference for measurable cardinals. Let \( S \) be a set and \( p \) a free ultrafilter on \( S \). We denote infinite cardinals by \( \kappa \), the first uncountable cardinal by \( \omega_1 \), and recall that \( p \) is

- **\( \kappa \)-complete** if \( \bigcap K \in p \) for any family \( K \subseteq p \) of cardinality \( < \kappa \);
- **countably complete** if \( p \) is \( \omega_1 \)-complete (that is, \( p \) is closed under countable intersections).

**Definition 2.4.** ([11, Ch. 8]) An uncountable cardinal \( \kappa \) is

- **measurable** if there exists a \( \kappa \)-complete free ultrafilter on \( \kappa \);
- **Ulam-measurable** if there exists a countably complete free ultrafilter on \( \kappa \).

**Remark 2.5.** While in [11] it is not assumed that measurable cardinals are uncountable, it is common to make such an assumption.

It is clear that every measurable cardinal is Ulam-measurable, and it is well known (see, e.g., [11, Thm. 8.31]) that the existence of an Ulam-measurable cardinal implies the existence of a measurable cardinal.
3. Existence of a measurable cardinal is sufficient

In this section we prove that the existence of a measurable cardinal implies that there is a normal space $Z$ such that $\mathcal{L}(Z) = \mathcal{L}$. We build $Z$ in stages. Let $\kappa$ be a measurable cardinal. Then $\kappa$ is Ulam-measurable, and so there is a countably complete free ultrafilter $\mathcal{U}$ on $\kappa$. Let $Y = (\kappa \times \{0,1\}) \cup \{\mathcal{U}\}$. Consider the following family of subsets of $Y$:

$$\tau = \{ U \subseteq Y \mid U \subseteq Y \setminus \{\mathcal{U}\} \text{ or } \exists V, W \in \mathcal{U} : U = (V \times \{0\}) \cup \{\mathcal{U}\} \cup (W \times \{1\}) \}. $$

![Figure 2. The space $Y$ and an open neighborhood of $\mathcal{U}$.](image)

**Lemma 3.1.** The family $\tau$ is a topology on $Y$ that is closed under countable intersections.

**Proof.** Clearly $\emptyset, Y \in \tau$. Let $\{U_i \mid i \in I\} \subseteq \tau$ and let $U = \bigcup\{U_i \mid i \in I\}$. If $\mathcal{U} \notin U$, then $U \in \tau$. Suppose $\mathcal{U} \in U$. Then $\mathcal{U} \in U_i$ for some $i \in I$. Since $U_i \in \tau$ and $\mathcal{U} \in U_i$, there are $V_0, V_1 \in \mathcal{U}$ such that $U_i = (V_0 \times \{0\}) \cup \{\mathcal{U}\} \cup (V_1 \times \{1\})$. For $n \in \{0,1\}$, set $W_n = \{ \alpha \in \kappa \mid (\alpha, n) \in U \}$. Let $n \in \{0,1\}$ and $\alpha \in V_n$. Then $(\alpha, n) \in V_n \times \{n\} \subseteq U_i \subseteq U$, giving that $\alpha \in W_n$. Therefore, $V_n \subseteq W_n$. Since $V_n \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, $W_n \in \mathcal{U}$. It follows from the definition of $W_n$ that $W_n \times \{n\} = U \cap (\kappa \times \{n\})$. Thus,

$$U = U \cap Y = U \cap ((\kappa \times \{0\}) \cup \{\mathcal{U}\} \cup (\kappa \times \{1\}))$$

$$= (U \cap (\kappa \times \{0\})) \cup (U \cap \{\mathcal{U}\}) \cup (U \cap (\kappa \times \{1\}))$$

$$= (W_0 \times \{0\}) \cup \{\mathcal{U}\} \cup (W_1 \times \{1\}) \in \tau.$$ 

Consequently, $\tau$ is closed under union.

Let $\{U_i \mid i \in \omega\} \subseteq \tau$ and let $U = \bigcap\{U_i \mid i \in \omega\}$. If $\mathcal{U} \notin U$, then $U \in \tau$. Suppose $\mathcal{U} \in U$. Let $i \in \omega$. Since $\mathcal{U} \in U_i$ and $U_i \in \tau$, there are $V_i, W_i \in \mathcal{U}$ such that $U_i = (V_i \times \{0\}) \cup \{\mathcal{U}\} \cup (W_i \times \{1\})$. Put $V = \bigcap\{V_i \mid i \in \omega\}$ and $W = \bigcap\{W_i \mid i \in \omega\}$. As $\mathcal{U}$ is countably complete, we have that $V, W \in \mathcal{U}$.

**Claim 3.2.** $U = (V \times \{0\}) \cup \{\mathcal{U}\} \cup (W \times \{1\})$.

**Proof.** Let $\alpha \in \kappa$. We have

$$(\alpha, 0) \in U \quad \text{iff} \quad (\alpha, 0) \in U_i \text{ for all } i \in \omega$$

$$\text{iff} \quad \alpha \in V_i \text{ for all } i \in \omega$$

$$\text{iff} \quad \alpha \in V$$

$$\text{iff} \quad (\alpha, 0) \in V \times \{0\}$$

$$\text{iff} \quad (\alpha, 0) \in (V \times \{0\}) \cup \{\mathcal{U}\} \cup (W \times \{1\}).$$

Similarly, $(\alpha, 1) \in U$ iff $(\alpha, 1) \in (V \times \{0\}) \cup \{\mathcal{U}\} \cup (W \times \{1\})$. The claim follows.

We conclude that $\tau$ is a topology on $Y$ that is closed under countable intersections.

**Remark 3.3.** That $\kappa$ is a measurable cardinal is used to see that $\tau$ is closed under countable intersections. In fact, this is the only place where we use that $\kappa$ is a measurable cardinal.

**Definition 3.4.** (See, e.g., [20, p. 37]) A Tychonoff space is a $P$-space if every $G_\delta$-set in $X$ is open.
Lemma 3.5. The space $Y$ is a normal $P$-space.

Proof. It is easy to see that each singleton in $Y$ is closed, so $Y$ is a $T_1$-space. Let $A, B$ be disjoint closed subsets of $Y$. Either $p \notin A$ or $p \notin B$, and we may assume without loss of generality that $p \notin A$. Then $A \subseteq Y \setminus \{p\}$, hence $A$ is open. Therefore, $U := A$ and $V := Y \setminus A$ are disjoint open subsets of $Y$ separating $A$ and $B$. Thus, $Y$ is normal, and hence it follows from Lemma 3.1 that $Y$ is a $P$-space. $\Box$

Since $Y$ is a $P$-space, it follows from [12, Sec. 2] that the Čech-Stone compactification $\beta Y$ of $Y$ can be embedded into a compact Hausdorff ED-space, say $E$. By Efimov’s Theorem [13, Sec. 1], there is a cardinal $\mu$, equipped with the discrete topology, such that the space $E$ can be embedded into $\beta \mu$. It is well known (see, e.g., [14, Exercise 3.6.B.b]) that $\beta \mu$ can be embedded in the remainder $\beta \mu \setminus \mu$. Combining these results yields a sequence of embeddings

$$
Y \hookrightarrow \beta Y \hookrightarrow E \hookrightarrow \beta \mu \hookrightarrow \beta \mu \setminus \mu
$$

that gives an embedding of $Y$ into $\beta \mu \setminus \mu$. We identify $Y$ with its image in $\beta \mu$; see Figure 3.

![Figure 3. $Y$ as a subspace of $\beta \mu$.](image)

Definition 3.6. Let $Z$ be the subspace $\mu \cup Y$ of $\beta \mu$.

Our goal is to show that $Z$ is a normal space such that $L(Z) = L$.

Lemma 3.7. The space $Z$ is a scattered ED-space of Cantor-Bendixson rank 3.

Proof. Since $Z \supseteq \mu$ and $\mu$ is dense in $\beta \mu$, we have that $Z$ is dense in $\beta \mu$. As $\beta \mu$ is an ED-space (see, e.g., [14, Cor. 6.2.28]) and a dense subspace of an ED-space is an ED-space (see, e.g., [14, Exercise 6.2.G.c]), it follows that $Z$ is an ED-space.

We have $d^1Z = d^2Y = d\{p\} = \emptyset$ and $d^2Z = dY = \{p\} \neq \emptyset$. Therefore, $Z$ is scattered and of Cantor-Bendixson rank 3. $\Box$

Lemma 3.8. The space $Z$ is normal.

Proof. Clearly $Z$ is $T_1$ since it is a subspace of a $T_1$-space. Let $A$ and $B$ be disjoint closed subsets of $Z$. Since $\mu$ is the set of isolated points of $Z$, we have that $A \cap \mu$ and $B \cap \mu$ are disjoint open subsets of $Z$. Let $A_0 = c(A \cap \mu)$ and $B_0 = c(B \cap \mu)$. Because $Z$ is ED, $A_0$ and $B_0$ are disjoint clopen subsets of $Z$. Let $A_1 = A \setminus A_0$ and $B_1 = B \setminus B_0$. Then $A_1$ and $B_1$ are disjoint closed subsets of $Y$. Since $Y$ is normal, it follows from [14, Cor. 3.6.4] that $c_{\beta Y}(A_1)$ and $c_{\beta Y}(B_1)$ are disjoint, where $c_{\beta Y}$ is the closure in $\beta Y$. Because $\beta Y$ is (up to homeomorphism) a closed subspace of $\beta \mu$, we have

$$
c_{\beta \mu}(A_1) \cap c_{\beta \mu}(B_1) = c_{\beta Y}(A_1) \cap c_{\beta Y}(B_1) = \emptyset.
$$

Since $\beta \mu$ is normal, there are disjoint open subsets $U_1$ and $V_1$ of $\beta \mu$ such that $c_{\beta \mu}(A_1) \subseteq U_1$ and $c_{\beta \mu}(B_1) \subseteq V_1$.

Clearly $U := U_1 \cap Z$ and $V := V_1 \cap Z$ are disjoint open subsets of $Z$. As both $A_0$ and $B_0$ are clopen in $Z$, it follows that both $U \setminus B_0$ and $V \setminus A_0$ are open in $Z$, and hence $U_0 := A_0 \cup (U \setminus B_0)$ and $V_0 := B_0 \cup (V \setminus A_0)$ are disjoint open subsets of $Z$. It is clear that $A_1 \subseteq U_1 \cap Z = U$. 


Because $A_1$ and $B_0$ are disjoint, $A_1 \subseteq U \setminus B_0$, so $A = A_0 \cup A_1 \subseteq A_0 \cup (U \setminus B_0) = U_0$. Similarly, $B \subseteq V_0$. Thus, $Z$ is normal.

We recall that a map $f : X \to X'$ between spaces is interior if $f$ is both continuous and open. If in addition $f$ is onto, then we call $X'$ an interior image of $X$. If $X'$ is the Alexandroff space of an $S4$-frame $\mathfrak{F}$, then we say that $\mathfrak{F}$ is an interior image of $X$. Finally, if $X$ is the Alexandroff space of an $S4$-frame $\mathfrak{G}$, then we say that $\mathfrak{F}$ is an interior image of $\mathfrak{G}$.

**Remark 3.9.** It is well known that $\mathfrak{F} = (W, R)$ is an interior image of $\mathfrak{G} = (V, S)$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$, where we recall that a p-morphism is a map $f : V \to W$ such that $f^{-1}R^{-1}(w) = S^{-1}f^{-1}(w)$ for each $w \in W$.

**Convention 3.10.** Since the diamond $\mathcal{D} = (D, \leq)$ is a poset (partially ordered set), for $w \in D$ we write $\uparrow w$ and $\downarrow w$ instead of $R(w)$ and $R^{-1}(w)$, respectively.

**Lemma 3.11.** The diamond $\mathcal{D}$ is an interior image of $Z$.

*Proof.* Define $f : Z \to D$ by

$$f(z) = \begin{cases} m & \text{if } z \in \mu \\
w_0 & \text{if } z \in \kappa \times \{0\} \\
w_1 & \text{if } z \in \kappa \times \{1\} \\
r & \text{if } z = p \end{cases}$$

It is clear that $f$ is a well-defined onto mapping. To prove that $f$ is interior, it is sufficient to show that $f^{-1}\downarrow w = c f^{-1}(w)$ for each $w \in D$. Since $\mu$ is dense in $Z$, we have

$$f^{-1}\downarrow m = f^{-1}(D) = Z = c \mu = cf^{-1}(m).$$

Because $Z$ is $T_1$, we have

$$f^{-1}\downarrow r = f^{-1}(r) = \{p\} = c \{p\} = cf^{-1}(r).$$

Since $Y$ is closed in $Z$, we have that $c_Y A = c A$ for any $A \subseteq Y$, where $c_Y A$ is closure in $Y$. Let $n \in \{0, 1\}$. Then $(\kappa \times \{n\}) \cup \{p\}$ is closed in $Y$. Therefore, $p \in c_Y (\kappa \times \{n\})$. Thus, $c (\kappa \times \{n\}) = c_Y (\kappa \times \{n\}) = (\kappa \times \{n\}) \cup \{p\}$. This yields

$$f^{-1}\downarrow w_n = f^{-1}(\{w_n, r\}) = (\kappa \times \{n\}) \cup \{p\} = c (\kappa \times \{n\}) = cf^{-1}(w_n).$$

Consequently, $f$ is interior. \hfill $\Box$

We are ready for the main lemma of this section. For this we recall that an $S4$-frame $\mathfrak{F} = (W, R)$ is rooted if there is $w \in W$ (a root of $\mathfrak{F}$) such that $W = R(w)$.

**Lemma 3.12.** Let $\mathfrak{F} = (W, R)$ be a finite rooted $S4$-frame. If $\mathfrak{F}$ is an interior image of $Z$, then $\mathfrak{F}$ is an interior image of $\mathcal{D}$.

*Proof.* We start by observing some properties of $\mathfrak{F}$. Since $Z$ is scattered, it is HI. Because $Z$ is also of Cantor-Bendixson rank 3, it follows from Section 2.2 that the formulas $grz$ and $bd_3$ are valid in $Z$. As $\mathfrak{F}$ is an interior image of $Z$, these formulas are also valid in $\mathfrak{F}$ (see, e.g., [4, Prop. 2.9(2)]). Therefore, $R$ is a partial order and the $R$-depth of $\mathfrak{F}$ is $\leq 3$ (see, e.g., [10, Props. 3.48 & 3.44]). In addition, since $Z$ is ED, so is $\mathfrak{F}$. Thus, as $\mathfrak{F}$ is rooted, $\mathfrak{F}$ has a maximum (see, e.g., [10, Cor. 3.38]).

We consider three cases based on the depth of $\mathfrak{F}$. First, suppose that the depth of $\mathfrak{F}$ is 1. Then $W$ is a singleton and it is clear that $\mathfrak{F}$ is an interior image of $\mathcal{D}$. Next suppose that the depth of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a rooted poset with a maximum, $\mathfrak{F}$ is isomorphic to the two element chain (see Figure 4). It is easy to see that mapping the root of $\mathcal{D}$ to the root of $\mathfrak{F}$ and all the other points of $\mathcal{D}$ to the maximum of $\mathfrak{F}$ is an onto interior map.
Figure 4. The two element chain.

Finally, suppose that the depth of $\mathfrak{F}$ is 3. Then $\mathfrak{F}$ is isomorphic to the frame depicted in Figure 5 where $W = \{0, v_0, \ldots, v_m, 1\}$ and $m \in \omega$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5.png}
\caption{The poset $\mathfrak{F}$ of depth 3.}
\end{figure}

If $m = 0$, then it is easy to see that mapping the root of $\mathcal{D}$ to the root of $\mathfrak{F}$, the maximum of $\mathcal{D}$ to the maximum of $\mathfrak{F}$, and $w_0, w_1$ to $v_0$ is an onto interior map. If $m = 1$, then $\mathcal{D}$ is isomorphic to $\mathfrak{F}$, so it is obvious that $\mathfrak{F}$ is an interior image of $\mathcal{D}$. Thus, to complete the proof, it suffices to show that $m \neq 2$.

Suppose that $m \geq 2$ and let $f : Z \to W$ be an interior mapping onto $\mathfrak{F}$.

Claim 3.13.

(1) $\mu \subseteq f^{-1}(1)$.

(2) $\{p\} = f^{-1}(0)$.

(3) $f^{-1}(\{v_0, \ldots, v_m\}) \subseteq Y \setminus \{p\}$.

(4) $p \in c(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup c(f^{-1}(v_i) \cap (\kappa \times \{1\}))$ for each $i \in \{0, \ldots, m\}$.

\textbf{Proof.} (1) Since each $z \in \mu$ is isolated and $f$ is interior, we have that $f(z)$ is the maximum of $\mathfrak{F}$. Thus, $f(z) = 1$.

(2) Because $f$ is onto, there is $z \in f^{-1}(0)$. By (1), we have that $z \in Y$. If $z \neq p$, then $z$ is an isolated point of $Y$, so there is an open subset $U$ of $Z$ such that $\{z\} = U \cap Y$. As $f$ is interior and $U$ is open, $f(U)$ is an $R$-upset of $\mathfrak{F}$. Therefore, $f(U) = W$ since $0 = f(z) \in f(U)$.

On the other hand,

$$f(U) = f((U \cap Y) \cup (U \cap \mu)) \subseteq f(\{z\} \cup \mu) = f(\{z\}) \cup f(\mu) = \{0\} \cup \{1\} \neq W.$$  

The obtained contradiction proves that $z = p$. Thus, $f^{-1}(0) = \{p\}$.

(3) Follows immediately from (1) and (2) since $\mu \cup \{p\} \subseteq f^{-1}(\{0, 1\})$.

(4) Let $i \in \{0, \ldots, m\}$. Because $f$ is interior, it follows from (2) and (3) that

$$\{p\} \subseteq f^{-1}(\{0, v_i\}) = f^{-1}R^{-1}(v_i)$$

$$= c f^{-1}(v_i) = c \left( f^{-1}(v_i) \cap (Y \setminus \{p\}) \right)$$

$$= c \left( f^{-1}(v_i) \cap (\kappa \times \{0\}) \cup (\kappa \times \{1\}) \right)$$

$$= c \left( f^{-1}(v_i) \cap (\kappa \times \{0\}) \right) \cup c \left( f^{-1}(v_i) \cap (\kappa \times \{1\}) \right).$$

$\square$
Let 
\[ F_0 = \left\{ f^{-1}(1) \cap (\kappa \times \{0\}), f^{-1}(v_0) \cap (\kappa \times \{0\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{0\}) \right\} \]
and 
\[ F_1 = \left\{ f^{-1}(1) \cap (\kappa \times \{1\}), f^{-1}(v_0) \cap (\kappa \times \{1\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{1\}) \right\}. \]

Then both \( F_0 \) and \( F_1 \) are pairwise disjoint families of sets, \( \bigcup F_0 = \kappa \times \{0\} \), and \( \bigcup F_1 = \kappa \times \{1\} \). We prove that there is a unique \( A_0 \in F_0 \) such that \( p \in cA_0 \). A similar proof yields a unique \( A_1 \in F_1 \) such that \( p \in cA_1 \).

Because \( F_0 \) is finite, we have 
\[ p \in c(\kappa \times \{0\}) = c\left( \bigcup F_0 \right) = \bigcup_{A \in F_0} cA. \]

Therefore, there is \( A_0 \in F_0 \) such that \( p \in cA_0 \). Since \( p \) is an ultrafilter, 
\[ p \not\in c((\kappa \times \{0\}) \setminus A_0) = c\left( \bigcup (F_0 \setminus \{A_0\}) \right) = \bigcup_{A \in F_0 \setminus \{A_0\}} cA. \]

Thus, \( A_0 \) is the unique member \( A \) of \( F_0 \) satisfying the property that \( p \in cA \).

Since \( m \geq 2 \), by the Pigeonhole Principle, there is \( i \in \{0, 1, 2, \ldots, m\} \) such that \( A_0 \neq f^{-1}(v_i) \cap (k \times \{0\}) \) and \( A_1 \neq f^{-1}(v_i) \cap (k \times \{1\}) \). Thus, \( p \not\in c(f^{-1}(v_i) \cap (k \times \{0\})) \) and \( p \not\in c(f^{-1}(v_i) \cap (k \times \{1\})) \), which contradicts Claim 3.13(4). Consequently, \( m \neq 2 \), completing the proof.

Lemma 3.14. The logic of \( Z \) is \( L \).

Proof. By Lemma 3.11, \( Z \) is an interior image of \( Z \). Therefore, \( L(Z) \subseteq L(Z) = L \) (see, e.g., [4, Prop. 2.9(2)]). Conversely, suppose that \( L(Z) \not\models \varphi \). Since \( Z \) is of Cantor-Bendixson rank 3, \( bd_3 \) is a theorem of \( L(Z) \). Therefore, by Segerberg’s theorem (see, e.g., [10, Thm. 8.85]), \( L(Z) \) is complete with respect to finite rooted \( L(Z) \)-frames. Thus, there is a finite rooted \( L(Z) \)-frame \( \mathfrak{F} \) such that \( \mathfrak{F} \not\models \varphi \). As \( \mathfrak{F} \) is an \( L(Z) \)-frame, by [6, Lem 6.2], \( \mathfrak{F} \) is an interior image of an open subspace \( U \) of \( Z \). Let \( f : U \to \mathfrak{F} \) be an interior map, and let \( z \in U \) map to the root of \( \mathfrak{F} \). Since \( Z \) is zero-dimensional, there is a clopen subset \( V \) of \( Z \) such that \( z \in V \) and \( V \subseteq U \). Then the restriction of \( f \) to \( V \) is an interior mapping of \( V \) onto \( \mathfrak{F} \). Because \( \mathfrak{F} \) has a maximum, we have that \( \mathfrak{F} \) is an interior image of \( Z \) by [7, Lem. 5.4]. By Lemma 3.12, \( \mathfrak{F} \) is an interior image of \( Z \). Therefore, \( \mathfrak{F} \not\models \varphi \), and hence \( L(Z) \not\models \varphi \). Thus, \( L(Z) = L(Z) = L \).

As a consequence of Lemmas 3.8 and 3.14 we arrive at the main result of this section.

Theorem 3.15. If there exists a measurable cardinal, then there exists a normal space \( Z \) such that \( L(Z) = L \).

4. Existence of a measurable cardinal is necessary

In this section we prove that the existence of a normal space \( Z \) such that \( L(Z) = L \) implies the existence of a measurable cardinal. Let \( Z \) be a normal space such that \( L(Z) = L \).

Lemma 4.1. The space \( Z \) is an ED-space of modal Krull dimension 2 such that \( \mathfrak{D} \) is an interior image of \( Z \).

Proof. As \( L(Z) = L \), for each modal formula \( \varphi \) we have \( Z \models \varphi \) iff \( \mathfrak{D} \models \varphi \). Since \( \mathfrak{D} \) has a maximum and is of depth 3, we have that

\[ \mathfrak{D} \models ga \]
\[ \mathfrak{D} \models bd_3 \]
\[ \mathfrak{D} \not\models bd_2 \]

Therefore, \( Z \) is an ED-space of modal Krull dimension 2 (see Section 2.2).
Because $\mathfrak{D} \models L(Z)$, [6, Lem. 6.2] yields an open subspace $U$ of $Z$ and an onto interior map $g : U \to D$. Then there is $z \in U$ with $f(z) = r$. Since $Z$ is normal and ED, it is zero-dimensional. Hence, there is clopen $V$ in $Z$ such that $z \in V \subseteq U$. Noting that the restriction of $g$ to $V$ is an interior mapping onto $\mathfrak{D}$, it follows from [7, Lem. 5.4] that $\mathfrak{D}$ is an interior image of $Z$.

□

Remark 4.2.
(1) Since $\mathfrak{D}$ is a finite poset, $\mathfrak{D}$ validates grz. Therefore, so does $Z$, and hence $Z$ is HI.
(2) Observe that $\mathfrak{D}$ is not hereditarily ED since the subspace $\{r, w_0, w_1\}$ is not ED.
(3) Since $Z$ is a Hausdorff ED-space that is not hereditarily ED, $Z$ must be uncountable (see, e.g., [9, Cor. 2.1]).

Definition 4.3. Let $f : Z \to \mathfrak{D}$ be an onto interior mapping. Denote the fibers of $f$ by

$$
M = f^{-1}(m) \\
B_0 = f^{-1}(w_0) \\
B_1 = f^{-1}(w_1) \\
A = f^{-1}(r)
$$

\begin{figure}
\centering
\begin{tikzpicture}
\node (M) at (0,0) {$M$};
\node (B0) at (-1,-1) {$B_0$};
\node (B1) at (1,-1) {$B_1$};
\node (A) at (0,-2) {$A$};
\draw (M) -- (B0);
\draw (M) -- (B1);
\end{tikzpicture}
\caption{Depiction of $Z$ partitioned by the fibers of $f$.}
\end{figure}

Remark 4.4.
(1) Clearly $M$ is an open dense subset of $Z$ (which is infinite as it is a dense subset of an infinite $T_1$-space).
(2) We also have that $A$ is a closed nowhere dense subset of $Z \setminus M$. Therefore, $A$ is discrete. More generally, any nonempty nowhere dense subset $N$ of $Z \setminus M$ is discrete. To see this, since $\text{mdim}(Z) = 2$, the definition of modal Krull dimension gives that $\text{mdim}(Z \setminus M) \leq 1$ and $\text{mdim}(N) \leq 0$. As $N \neq \emptyset$, we have that $\text{mdim}(N) = 0$. Thus, $N$ is discrete by [5, Rem. 4.8 & Thm. 4.9].

Lemma 4.5. There is a normal subspace $U$ of $Z$ such that $U \cap A$ is a singleton and $L(U) = L$.

Proof. Let $a \in A$. Since $A$ is discrete and $Z$ is zero-dimensional, there is a clopen subset $U$ of $Z$ such that $\{a\} = U \cap A$. As $U$ is closed in $Z$, the subspace $U$ is normal. Because $U$ is open in $Z$, the restriction $f|_U$ of $f$ to $U$ is interior. Since $U \cap A \neq \emptyset$, we have that $r \in f(U)$. As $f(U)$ is an upset, $D = \uparrow r \subseteq f(U) \subseteq D$. Therefore, $f|_U$ is onto and $\mathfrak{D}$ is an interior image of $U$. By [4, Prop. 2.9], $L(U) \subseteq L = L(Z) \subseteq L(U)$, so $L(U) = L$, completing the proof. □

By Lemma 4.5, we may assume without loss of generality that $A$ is a singleton, say $\{a\}$, yielding that $Z = B_0 \cup \{a\} \cup B_1 \cup M$ (see Figure 7).
Lemma 4.6. We have that $a \not\in cN$ for any nowhere dense subset $N$ of the subspace $B_0 \cup B_1$.

Proof. We first show that $N \cup A$ is nowhere dense in $Z \setminus M$. Let $U$ be open in $Z \setminus M$ with $U \subseteq c(N \cup A)$. Since $A$ is closed, $U \subseteq c(N) \cup A$. Therefore, $U \setminus A \subseteq c(N) \setminus A = c(N) \cap (B_0 \cup B_1)$, which is the closure of $N$ relative to $B_0 \cup B_1$. Because $U \setminus A$ is open and $N$ is nowhere dense in $B_0 \cup B_1$, we have that $U \setminus A = \emptyset$, so $U \subseteq A$. By Remark 4.4(2), $A$ is a closed nowhere dense subset of $Z \setminus M$, hence $U = \emptyset$. Thus, $N \cup A$ is nowhere dense in $Z \setminus M$. Applying Remark 4.4(2) again yields that $N \cup A$ is discrete. Consequently, there is an open set $V$ in $Z$ such that $\{a\} = V \cap (N \cup A)$. As $V \cap N \subseteq V \cap (N \cup A) = \{a\} \subseteq Z \setminus (B_0 \cup B_1) \subseteq Z \setminus N$, it must be the case that $V \cap N = \emptyset$, so $a \not\in cN$.  

We recall that a normal space $X$ is an $F$-space if any two disjoint open $F_\sigma$-sets in $X$ have disjoint closures in $X$ (see, e.g., [19, Lem. 1.2.2(b)]). Being a normal ED-space, it follows from [15, Exercise 14N.4] that $Z$ is an $F$-space.

Definition 4.7. Let $Y$ denote the subspace $B_0 \cup \{a\} \cup B_1$ of $Z$.

Because $Y = Z \setminus M$ is closed in $Z$, we have that $Y$ is a normal $F$-space by [19, Lem. 1.2.2(d)]. We require the following definition.

Definition 4.8. (See, e.g., [20, p. 37]) A point $x$ of a space $X$ is called a $P$-point provided for any $G_\delta$-set $S$ in $X$ we have that $x \in S$ implies $x \in iS$.

Remark 4.9. By taking complements we obtain that $x \in X$ is a $P$-point iff for each $F_\sigma$-set $S$ in $X$ we have that $x \not\in S$ implies $x \not\in cS$. This will be utilized in Lemma 4.16(5).

Lemma 4.10. Either $a$ is a $P$-point in the subspace $B_0 \cup \{a\}$ or a $P$-point in the subspace $B_1 \cup \{a\}$.

Proof. Suppose not. Then we show that there are disjoint open $F_\sigma$-sets $U_0$ and $U_1$ of $Y$ whose closures have nonempty intersection, which is a contradiction since $Y$ is a normal $F$-space. We only show how to construct $U_0$ because $U_1$ is constructed similarly. Since $a$ is not a $P$-point in $B_0 \cup \{a\}$, for each $n \in \omega$, there is $W_n$ open in $B_0 \cup \{a\}$ such that $a \in \bigcap_{n \in \omega} W_n$ but $a \not\in i\bigcap_{n \in \omega} W_n$, where $i$ is taken in $B_0 \cup \{a\}$. As $Z$ is zero-dimensional, $B_0 \cup \{a\}$ is zero-dimensional. Thus, for each $n \in \omega$, there is $V_n$ clopen in $B_n \cup \{a\}$ such that $a \in V_n \subseteq W_n$. Clearly, $a \in V := \bigcap_{n \in \omega} V_n$ and $V$ is a closed $G_\delta$-set in $B_0 \cup \{a\}$. Moreover, $a \not\in iV$ since $V \subseteq \bigcap_{n \in \omega} W_n$ and $a \not\in i\bigcap_{n \in \omega} W_n$. Put $U_0 = (B_0 \cup \{a\}) \setminus V$. Then $U_0$ is an open $F_\sigma$-set in $B_0 \cup \{a\}$ such that $a \not\in U_0$ and $a \in cU_0$. Clearly $U_0 \subseteq B_0$, and so $U_0$ is open in $B_0$. As $B_0 = Y \cap f^{-1}[w_0]$ is open in $Y$, it follows that $U_0$ is open in $Y$. Because $B_0 \cup \{a\}$ is closed in $Y$ and $U_0$ is an $F_\sigma$-set in $B_0 \cup \{a\}$, we have that $U_0$ is an $F_\sigma$-set in $Y$. Thus, $U_0$ is an open $F_\sigma$-set in $Y$ such that $a \in cU_0$. Analogously, there is an open $F_\sigma$-set $U_1$ in $Y$ such that $a \in cU_1$. By construction, $U_0 \subseteq B_0$ and $U_1 \subseteq B_1$, so $U_0$ and $U_1$ are disjoint. On the other hand, $a \in cU_0 \cap cU_1$, yielding the desired contradiction.  

\[ \begin{tikzpicture}
  \node (A) at (0,0) {$A = \{a\}$};
  \node (B0) at (-2,1) {$B_0$};
  \node (B1) at (2,1) {$B_1$};
  \node (M) at (0,2) {$M$};
  \draw (B0) -- (M) -- (B1);
\end{tikzpicture} \]

Figure 7. Reducing $A$ to a singleton.
Convention 4.11. Without loss of generality we assume that \( a \) is a \( P \)-point in \( X := B_0 \cup \{a\} \).

Remark 4.12. Since \( X \) is closed in \( Z \), the closure in \( X \) of any subset \( S \) of \( X \) coincides with the closure of \( S \) in \( Z \). Therefore, there is no ambiguity in writing \( cS \) whenever \( S \subseteq X \).

The following lemma is an easy consequence of Zorn’s lemma, and we skip its proof.

Lemma 4.13. There is a family \( \mathcal{F} \) of subsets of \( X \) that is maximal with respect to the following two properties:

1. Each \( F \in \mathcal{F} \) is a nonempty clopen in \( X \) such that \( a \notin F \);
2. The family \( \mathcal{F} \) is pairwise disjoint.

Lemma 4.14. Let \( N = B_0 \setminus \bigcup \mathcal{F} \). Then we have:

1. \( \bigcup \mathcal{F} \) is open in both \( X \) and \( B_0 \).
2. \( \bigcup \mathcal{F} \) is dense in both \( B_0 \) and \( X \).
3. \( N \) is closed in \( Z \).
4. There is a clopen subspace \( U \) of \( Z \) such that \( U \cap N = \emptyset \) and \( L(U) = L \).

Proof. (1) Since \( \bigcup \mathcal{F} \) is a union of clopen subsets of \( X \), it is open in \( X \). Also, since \( a \notin F \) for each \( F \in \mathcal{F} \), we have that \( \bigcup \mathcal{F} \subseteq B_0 \), and hence it is also open in \( B_0 \).

(2) Let \( z \in B_0 \). If \( z \notin c(\bigcup \mathcal{F}) \), then as \( X \) is zero-dimensional, there is clopen \( V \) in \( X \) such that \( z \in V \) and \( V \cap \bigcup \mathcal{F} = \emptyset \). Since \( z \neq a \), we may assume that \( a \notin V \) (by shrinking \( V \) further if necessary). But this contradicts the maximality of \( \mathcal{F} \) because the family \( \{V\} \cup \mathcal{F} \) satisfies the conditions of Lemma 4.13. Thus, \( z \in c(\bigcup \mathcal{F}) \), and so \( \bigcup \mathcal{F} \) is dense in \( B_0 \).

(3) It suffices to show that \( N \) is closed in \( X \). For any \( z \in B_0 \setminus N \), we have that \( \bigcup \mathcal{F} \) is open in \( X \) and \( z \in \bigcup \mathcal{F} \). Since \( N \cap \bigcup \mathcal{F} = \emptyset \), it follows that \( z \notin cN \). Because \( \{B_0 \setminus N, N, \{a\}\} \) is a partition of \( X \), it remains to show that \( a \notin cN \). But (1) and (2) imply that \( N \) is nowhere dense in \( B_0 \), hence nowhere dense in \( B_0 \cup B_1 \). This yields that \( a \notin cN \) by Lemma 4.6.

(4) Since \( \{a\} \) and \( N \) are closed in the zero-dimensional normal space \( Z \), there is \( U \) clopen in \( Z \) such that \( a \in U \) and \( U \cap N = \emptyset \). Because \( U \) is open, the restriction of \( f \) as defined in Definition 4.3 is an interior map from \( U \) to \( \mathcal{D} \). To see that it is onto, observe that \( U \cap M \neq \emptyset \) since \( M \) is dense in \( Z \), and both \( U \cap B_0 \) and \( U \cap B_1 \) are nonempty because \( a \in cB_0, cB_1 \) and \( a \in U \). Therefore, \( \mathcal{D} \) is an interior image of \( Z \), and so \( L(U) \subseteq L = L(Z) \subseteq L(U) \) by [4, Prop. 2.9]. Thus, \( L(U) = L \)  

Let \( U \) be the clopen subspace of \( Z \) constructed in the proof of Lemma 4.14(4). Then \( U \) is normal since it is a closed subspace of a normal space. In addition, \( a \) remains a \( P \)-point of \( X \cap U \) because \( X \cap U \) is an open subspace of \( X \) and \( a \) is a \( P \)-point of \( X \). Therefore, without loss of generality we may assume that \( Z = U \). Thus, \( B_0 = \bigcup \mathcal{F} \) and \( N = \emptyset \).

Definition 4.15.

1. Let \( \kappa \) be the cardinality of \( \mathcal{F} \), and let \( \varphi : \kappa \to \mathcal{F} \) be a bijection. Denoting \( \varphi(\alpha) \) by \( F_\alpha \), we may write \( \mathcal{F} = \{F_\alpha \mid \alpha \in \kappa\} \).
2. Let \( \mathcal{G} = \left\{ \Gamma \subseteq \kappa \mid a \in c\left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \right\} \).

We are ready to prove the main lemma of this section.

Lemma 4.16.

1. If \( \Gamma \in \mathcal{G} \) and \( \Gamma \subseteq \Lambda \), then \( \Lambda \in \mathcal{G} \).
2. For any \( \Gamma \subseteq \kappa \), exactly one of \( \Gamma, \kappa \setminus \Gamma \) belongs to \( \mathcal{G} \).
(3) If $\Gamma, \Lambda \in \mathcal{G}$, then $\Gamma \cap \Lambda \in \mathcal{G}$.
(4) $\mathcal{G}$ is a free ultrafilter on $\kappa$.
(5) $\mathcal{G}$ is countably complete.

Proof. (1) Let $\Gamma \in \mathcal{G}$ and $\Gamma \subseteq \Lambda$. Then $\bigcup_{\alpha \in \Gamma} F_\alpha \subseteq \bigcup_{\alpha \in \Lambda} F_\alpha$, yielding
\[ a \in c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \subseteq c \left( \bigcup_{\alpha \in \Lambda} F_\alpha \right). \]
Thus, $\Lambda \in \mathcal{G}$.

(2) Let $\Gamma \subseteq \kappa$. We have that $a \in c B_0 = c \left( \bigcup_{\alpha \in \kappa} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \cup \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \right) \cup c \left( \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \right)$. Therefore, $\Gamma \in \mathcal{G}$ or $\kappa \setminus \Gamma \in \mathcal{G}$. Suppose that both $\Gamma$ and $\kappa \setminus \Gamma$ belong to $\mathcal{G}$. Then the frame $\mathcal{F}$ depicted in Figure 5 with $m = 2$ is an interior image of $Z$ via the mapping $g : Z \to W$ given by
\[
g(z) = \begin{cases} 
1 & \text{if } z \in M \\
v_0 & \text{if } z \in \bigcup_{\alpha \in \Gamma} F_\alpha \\
v_1 & \text{if } z \in \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha \\
v_2 & \text{if } z \in B_1 \\
0 & \text{if } z = a
\end{cases}
\]
The function $g$ is depicted in Figure 8 where each fiber of $g$ is labeled to the right by its image in $W$.

\[
\begin{array}{cccc}
M & \downarrow & \downarrow & \downarrow \\
\bigcup_{\alpha \in \Gamma} F_\alpha & v_0 & \bigcup_{\alpha \in \kappa \setminus \Gamma} F_\alpha & v_1 \\
B_1 & v_2 \\
\{a\} & \bullet & 0
\end{array}
\]

Figure 8. The function $g : Z \to W$.

This yields that $\mathcal{F} \models L(Z) = L$, which is a contradiction since $\mathcal{F} \not\models L$. Thus, exactly one of $\Gamma$ or $\kappa \setminus \Gamma$ is a member of $\mathcal{G}$.

(3) If $\Gamma \cap \Lambda \notin \mathcal{G}$, then $a \notin c \left( \bigcup_{\alpha \in \Gamma \cap \Lambda} F_\alpha \right)$. On the other hand,
\[ a \in c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma} F_\alpha \cup \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right) = c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right) \cup c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right). \]
Therefore, $a \in c \left( \bigcup_{\alpha \in \Gamma \setminus \Lambda} F_\alpha \right)$. Thus, $\Gamma \setminus \Lambda \in \mathcal{G}$. Since $\Gamma \setminus \Lambda \subseteq \kappa \setminus \Lambda$, (1) implies that $\kappa \setminus \Lambda \in \mathcal{G}$. However, as $\Lambda \in \mathcal{G}$, (2) implies that $\kappa \setminus \Lambda \notin \mathcal{G}$. The obtained contradiction proves that $\Gamma \cap \Lambda \in \mathcal{G}$.

(4) That $\mathcal{G}$ is an ultrafilter follows from (1), (2), and (3). To see that $\mathcal{G}$ is free, let $\alpha \in \kappa$. Then $F_\alpha$ is clopen in $X$ and $a \notin F_\alpha$. Therefore, $a \notin c F_\alpha$, yielding that $\{\alpha\} \notin \mathcal{G}$. Thus, $\mathcal{G}$ is a free ultrafilter.
(5) Let $\Lambda_n \in \mathcal{G}$ for each $n \in \omega$ and let $\Gamma := \bigcap_{n \in \omega} \Lambda_n \notin \mathcal{G}$. For $n \in \omega$ set $\Gamma_n = \bigcap_{i=0}^n \Lambda_i$. Then $\Gamma_n \in \mathcal{G}$ by (3), $\Gamma_{n+1} \subseteq \Gamma_n$, and $\Gamma = \bigcap_{n \in \omega} \Gamma_n$. For $n \in \omega$ set $\Delta_n = \Gamma_n \setminus \Gamma_{n+1}$. Since $\mathcal{G}$ is an ultrafilter, $\Delta_n \notin \mathcal{G}$ for each $n \in \omega$.

**Claim 4.17.** The set $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is clopen in $X$.

**Proof.** Clearly $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is open in $X$ since each $F \in \mathcal{F}$ is clopen in $X$. To see that $\bigcup_{\alpha \in \Delta_n} F_\alpha$ is closed in $X$ we show that $c \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right) = \bigcup_{\alpha \in \Delta_n} F_\alpha$. As $X$ is closed in $Z$, we have that $c \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right) \subseteq X$. Let $z \in X \setminus \bigcup_{\alpha \in \Delta_n} F_\alpha$. We show that $z \notin c \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$. Either $z = a$ or $z \in B_0$. The former case is clear since $\Delta_n \notin \mathcal{G}$ implies that $z = a \notin c \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$. Suppose $z \in B_0$. Then there is $\beta \in \kappa$ such that $z \in F_\beta$. Since $z \notin \bigcup_{\alpha \in \Delta_n} F_\alpha$, it follows that $\beta \notin \Delta_n$. Because $F_\beta$ is clopen in $X$, there is $U$ open in $Z$ such that $F_\beta = U \cap X$. Clearly $z \in U$. As $\mathcal{F}$ is pairwise disjoint, we have that

$$U \cap \bigcup_{\alpha \in \Delta_n} F_\alpha = U \cap \bigcup_{\alpha \in \Delta_n} (X \cap F_\alpha) = \bigcup_{\alpha \in \Delta_n} (U \cap X \cap F_\alpha) = \bigcup_{\alpha \in \Delta_n} (F_\beta \cap F_\alpha) = \emptyset.$$ 

Therefore, $z \notin c \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$. 

As $\Gamma_0 \setminus \Gamma = \bigcup_{n \in \omega} \Delta_n$, it follows from Claim 4.17 that

$$\bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha = \bigcup_{n \in \omega} \left( \bigcup_{\alpha \in \Delta_n} F_\alpha \right)$$

is an open $F_\sigma$-set in $X$. Moreover, $a \in c \left( \bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha \right)$ because $\Gamma_0 \setminus \Gamma \in \mathcal{G}$. But $a \notin \bigcup_{\alpha \in \Gamma_0 \setminus \Gamma} F_\alpha$ since $a \notin F_\alpha$ for each $\alpha \in \kappa$. This implies that $a$ is not a $P$-point of $X$ (see Remark 4.9). The obtained contradiction proves that $\mathcal{G}$ is countably complete. 

As a consequence of Lemma 4.16 and Section 2.3, we obtain:

**Lemma 4.18.** The cardinal $\kappa$ is Ulam-measurable, and hence there exists a measurable cardinal.

Consequently, we have proved the following result.

**Theorem 4.19.** If there exists a normal space $Z$ such that $L(Z) = L$, then there exists a measurable cardinal.

Putting Theorems 3.15 and 4.19 together yields the main result of the paper:

**Theorem 4.20.** There exists a measurable cardinal iff there exists a normal space $Z$ such that $L(Z) = L$.

We conclude the paper by the following open problem:

**Problem 4.21.** In Theorem 4.20 can ‘normal’ be replaced by ‘Tychonoff’?

Clearly the interesting implication is to prove that the existence of a Tychonoff space whose logic is $L$ implies the existence of a measurable cardinal.

**References**


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