Radboud University Nijmegen

Faculty of Science

General topological semantics for provability logics

Master's Thesis in Mathematics

Author:
Gilan Takapui

Supervisors:
Wim Veldman
Nick Bezhanishvili

July 28, 2024

Contents

Chapter 1

Introduction

Modal logics are extensions of classical logic, obtained by the addition of two unary operators[1], $\Box$ and $\Diamond$, which we call *modal operators*. These modal operators are used to express modalities such as necessity and possibility, provability, knowledge, moral and legal obligation, temporal properties and many more. In this manner, modal logics are more expressive than classical logic, while still being decidable. For a thorough exposition on modal logic, we refer to [10, 13]

While the most popular semantics for modal logics are the relational Kripke semantics, it also admits two, by now classical, topological semantics. One is given by interpreting the modal diamond $\Diamond$ as the closure operator, and the other by interpreting $\Diamond$ as the derived set operator[2]. We refer to [5] for a thorough overview of these semantics. A celebrated result for the closure semantics is the McKinsey and Tarski theorem stating that the modal logic $\mathsf{S4}$ is sound and complete with respect to any dense-in-itself metrizable space, in particular, any Euclidean space [18].

In this thesis we will concentrate on the provability logic GL and derived set semantics. GL, the logic axiomatized by the formula $\Box(\Box p \to p) \to \Box p$, is known as the *provability logic*, as it describes provability for Peano theories when $\Box\varphi$ is interpreted as 'φ is provable in a Peano theory'. We only work with GL as a formal modal logic, for more background on GL as a provability logic we refer to [12] and [3]. A landmark result for the derived set semantics is the Abashidze-Blass theorem stating that GL is sound and complete with respect to any ordinal $\alpha \geq \omega^\omega$ with the standard interval topology [1, 11] (See also: [9]). Earlier Esakia [14] and Simmons [19] showed that GL is sound and complete with respect to the class of scattered spaces. Recall that a topological space is *scattered* if its every non-empty subset contains a point isolated in that subset. It is easy to verify that each ordinal is a scattered space with respect to the order topology.

In modal logic, general Kripke frames constitute an important generalization of Kripke semantics. A *general Kripke frame* consists of a relational Kripke frame for modal logic together with a restricted class of sets one can evaluate

[1] In principle, modal logics can have multiple (even infinitely many) modal operators of any arity. We only consider modal languages with unary $\Box$ and $\Diamond$.

[2] Recall that the *derived set* $d(U)$ of U consists of all points x such that every open neighbourhood of x intersects $U \backslash \{x\}$.

formulas in, which we call the *set of admissible sets*. It is well known that, unlike Kripke semantics, every normal modal logic is sound and complete with respect to its general Kripke frames [13].

Similarly to Kripke frames one can consider general topological spaces for both the closure and derived set semantics. A general (topological) c-space is a pair (X, P), where X is a topological space and P is a set of admissible sets closed under the closure operator. G. Bezhanishvili et al. [8] show that the McKinsey and Tarski theorem can be extended to all connected extensions of $\mathsf{S4}$ by considering general c-spaces. In particular, they showed that for every extension $L \supseteq \mathsf{S4}$ such that L is the logic of a connected $\mathsf{S4}$-algebra, there exists a general c-space $(\mathbb{R}, P)$ over the real line $\mathbb{R}$ such that L is sound and complete for $(\mathbb{R}, P)$. In this manner, it allows one to study logics above $\mathsf{S4}$ while staying in the realm of Euclidean spaces, which are well-understood.

General topological spaces for the derived set semantics have been considered in [16] for provability logics with countably many modal operators and more recently in [21] where it was shown that the bimodal provability logic GLB is sound and complete with respect to general bi-topological spaces. This was recently generalized in [4], where it was shown that the logic GLP, the provability logic with countably many modal operators, is complete with respect to general spaces over the ordinal ϵ_0.

In this thesis we combine these two approaches. We will consider general topological spaces for the derived set semantics over ordinal spaces and we will prove a generalization of the Abashidze-Blass theorem for these spaces, in the same way [8] proved a generalized version of the McKinsey and Tarski theorem for general spaces over the real line. Later on in this thesis, we discuss the classes of logics that can be captured by this semantics and we study completeness results for larger classes of topological spaces.

Outline. Chapter 2 discusses preliminaries required for the rest of the thesis. It discusses the basics of modal logics and Kripke semantics. It also reviews the derived set semantics and provides a proof of the Abashidze-Blass theorem as given in [9].

In chapter 3 we introduce general derived set semantics. Before we generalize the Abashidze-Blass theorem in later chapters, we first note that any topological space X can be equipped with a least general space, that is, a general space that is contained in all other general spaces over X. This can be thought of as the general space where the admissible sets are generated by $\varnothing$. We first set out to describe the structure of least general spaces over scattered spaces and then give a full description of their logics: We introduce the logic $\mathsf{GL.3}$, an extension of GL complete with respect to linear GL-frames, and show that the logic of any least general space is an extension of $\mathsf{GL.3}$. Moreover, we prove that every normal extension of $\mathsf{GL.3}$ is complete for the least general space over some ordinal.

Chapter 4 aims to prove our first generalization of the Abashidze-Blass theorem. We first introduce a tool that we will use often in this thesis, a notion of morphisms that allows us to pull back any logic from one topological space to another. With this tool, we will prove that any logic above GL enjoying the finite model property is also the logic of a general frame over an ordinal $\alpha \leq \omega^\omega$. We prove this by combining our newly introduced tool with the proof in [9] of the Abashidze-Blass theorem, which allows us to pull back structures from finite frames to finite trees, and from finite trees to ordinal numbers.

Chapter 5 contains our second generalization of the Abashidze-Blass theorem. We first introduce a new class of logics which contains the Kripke complete logics, the so-called GL-semicomplete logics, and show that any such logic can be obtained via a general frame over some countable ordinal. The steps of this proof mirror those in the previous chapter, though many additional steps have to be taken as we do not work with finite frames any more. First, we introduce a new class of trees, namely the trees with the finite chain property (FCP-trees, for short). We then show that these trees can be constructed recursively, allowing us to work with proofs by induction on subtrees. We show that we need to only consider countable trees, an approach mirroring work done on S4 in [8, Theorem 4.2]. We then show that we can relate any GL-frame to these trees and show that we may directly link these trees to ordinal spaces.

The results of the previous chapters generate a natural question of how far the aforementioned completeness results can be generalized. Namely, is every GL-logic complete for general spaces over scattered spaces? In chapter 6, we give a negative answer to this question. This chapter first discusses our newly introduced class of logics and a new form of canonicity, and it then shows that there exists a logic above GL that is not complete with respect to general spaces over scattered spaces. Much of this chapter is based on work done in [20] by Vooijs, as it mirrors his proof that GL is not quasicanonical.

In chapter 7 we complement the result from the previous chapter by showing that every GL-logic is complete with respect to general topological semantics if we drop the condition that these topologies are scattered. Moreover, we show that this approach works for any wK4-logic. This is done by showing that any general Kripke wK4-frame can be turned into a general topological space that has the same logic as the original general Kripke frame.

Chapter 2

Preliminaries

2.1 Modal logic

In this chapter, we will closely follow [10].

Definition 2.1. The *basic modal language* is defined using the propositional letters *Prop* with elements usually denoted by p, q, r, etc., the classical logical symbols and two unary modal operators $\Box$ and $\Diamond$. The *well-formed formulas* of the basic modal language are given by the rule

$$\varphi ::= p \mid \bot \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box\varphi$$

Here, p ranges over the elements of *Prop*. We also make use of the classical abbreviations $\top := \neg\bot$, $\varphi \wedge \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and use $\Diamond\varphi$ as an abbreviation for $\neg\Box\neg\varphi$.

From this syntax we can define modal logic. Note that in general we do not speak of a singular modal logic, instead, we speak of many different modal logics.

Definition 2.2. Any collection L of well-formed modal formulas is called a *normal modal logic* if it contains every propositional tautology (i.e., $p \to p$, $p \to (q \to p)$, $p \leftrightarrow \neg\neg p$, etc.) and the following axiom:

- The distribution axiom: $\Box(p \to q) \to (\Box p \to \Box q)$.

Furthermore, L must be closed under the following provability rules:

- Modus Ponens: Given φ and $\varphi \to \psi$, conclude ψ.
- Necessitation: Given φ, conclude $\Box\varphi$.
- Replacement: Given φ, conclude ψ, where ψ is obtained by uniformly replacing propositional letters in φ by arbitrary formulas.

Finally, we require that L is consistent, meaning that $\bot \notin L$.

Note that there is no 'assumptions based rule': We cannot prove from necessitation that $p \to \Box p \in L$, we can only prove that, given $\varphi \in L$, we also have $\Box\varphi \in L$.

From now on, we use the abbreviation NML when talking about a normal modal logic. Note that the arbitrary intersection of NMLs is itself a NML. Thus, for any set of modal formulas Γ, either there is some smallest NML containing Γ or there does not exist a NML containing Γ.

Definition 2.3. Given normal modal logic L, we write $L \oplus \varphi$ for the smallest normal extension of L containing φ. In case that adding the axiom φ to L yields an inconsistent logic we set $L \oplus \varphi$ as the inconsistent logic. We define K to be the intersection of all normal modal logics. Equivalently, K is the smallest NML. It is often called the *basic normal modal logic*. The following is a table of major NMLs we use in this thesis and their axioms:

Name of logic	Axiomatization
K	No additional axioms
$\mathsf{K4}$	$\mathsf{K} \oplus (\Box p \to \Box\Box p)$
$\mathsf{wK4}$	$\mathsf{K} \oplus (\Box p \land p \to \Box\Box p)$
$\mathsf{S4}$	$\mathsf{K4} \oplus (\Box p \to p)$
GL	$\mathsf{K} \oplus (\Box(\Box p \to p) \to \Box p)$

To examine such logics, various types of models are used. We will look at so-called Kripke semantics and topological semantics.

2.2 Kripke semantics for modal logic

Kripke semantics is the most standard semantics for modal logics.

Definition 2.4. A *Kripke frame* is a pair $F = (W, R)$, where W is some nonempty set of points and R is some binary relation on W. An element of W is called a point or a world, and if xRy we often say that x 'sees' y.

1. A *valuation* v on a Kripke frame F is a function from $Prop$ to $\mathcal{P}(W)$.

2. A pair (F, v) where v is a valuation on F is called a *model* on F.

We interpret formulas on these models as follows:

Definition 2.5. Given model $\mathcal{M} = (F, v)$, and point x of F, we inductively define the notion of a formula being true in point x of model $\mathcal{M}$ as follows:

$$
\begin{aligned}
\mathcal{M}, x \vDash \bot &\iff \text{never} \\
\mathcal{M}, x \vDash p &\iff x \in v(p) \\
\mathcal{M}, x \vDash \neg\varphi &\iff \text{not } \mathcal{M}, x \vDash \varphi \\
\mathcal{M}, x \vDash \varphi \lor \psi &\iff \mathcal{M}, x \vDash \varphi \text{ or } \mathcal{M}, x \vDash \psi \\
\mathcal{M}, x \vDash \Box\varphi &\iff \forall_y (xRy \to \mathcal{M}, y \vDash \varphi)
\end{aligned}
$$

Of course, this definition easily extends to $\top, \wedge, \rightarrow$ and $\Diamond$. In that final case, we find

$$\mathcal{M}, x \vDash \Diamond\varphi \iff \exists_y (xRy \wedge \mathcal{M}, y \vDash \varphi).$$

If $\mathcal{M}, x \vDash \varphi$, we say that φ *holds in the point x in the model $\mathcal{M}$*.

Definition 2.6. Given a Kripke frame $F = (W, R)$ and a valuation v on F, we say that φ *holds on the model* (F, v) if $(F, v), x \vDash \varphi$ for each point x of F. We denote this as $(F, v) \vDash \varphi$.

We say that φ *holds on the frame* F, denoted by $F \vDash \varphi$, if φ holds on all models of F. We call the set of modal formulas φ such that $F \vDash \varphi$ the *logic* of F, which we denote by $L(F)$. If C is some class of frames, then $L(C) = \bigcap_{F \in C} L(F)$. This is also the set of formulas that hold on all frames of C.

If Ω is some set of formulas and $\mathcal{M}$ a model, then we write $\mathcal{M} \vDash \Omega$ iff $\mathcal{M} \vDash \varphi$ for every formula $\varphi \in \Omega$. This notation also extends to frames and classes of frames. Finally, if Ω is some set of formulas, we say that a Kripke frame F is a *frame of* Ω if $F \vDash \Omega$.

Note that we can also define $F \vDash \varphi$ in a non-pointwise manner. Any valuation v on F can inductively be extended to all modal formulas as follows:

$$\begin{aligned} v(\bot) &= \varnothing \\ v(p) &\text{ is already given for all propositional letters} \\ v(\neg\varphi) &= v(\varphi)^c \\ v(\varphi \vee \psi) &= v(\varphi) \cup v(\psi) \\ v(\Box\varphi) &= \Box_R(v(\varphi)) := \{x \in W \mid \forall_y [xRy \rightarrow y \in v(\varphi)]\} \end{aligned}$$

Of course, any well-formed modal formula can be rewritten as to consist only of propositional letters, negation, disjunction and $\Box$. We see that for $\Diamond$:

$$v(\Diamond\varphi) = \Diamond_R(v(\varphi)) := \{x \in W \mid \exists_y [xRy \wedge y \in v(\varphi)]\}$$

Lemma 2.7. *Let F be a Kripke frame, v a valuation and x a point. Then $x \in v(\varphi)$ iff $(F, v), x \vDash \varphi$. It follows that $L(F)$ is also the set of formulas φ such that $v(\varphi) = W$ for all valuations v on $F = (W, R)$.*

Proof. This follows easily from the definitions. □

One easily sees that $L(F)$ is a NML for all frames F. As stated before, the intersection of a class of NMLs is also a NML. We deduce that $L(C)$ is a NML for any class of frames C. It also turns out that K, the smallest normal modal logic, is the logic of the class of all Kripke frames.

It is possible to study modal logics by looking at their corresponding frames. For example, the frames respecting the axiom $\Box p \rightarrow p$ turn out to be exactly the reflexive frames. The converse also holds: The logic of all reflexive frames is the smallest logic containing $\Box p \rightarrow p$.

Generally, one can study what kind of frames correspond to some logic, but one can also study whether there is some logic corresponding to some class of frames. It turns out, for example, that there is no logic that characterizes all irreflexive frames: For any NML L, either there exists a frame F containing a reflexive point such that $F \vDash L$, or there exists some irreflexive frame F' such that $F' \nvDash L$.

Definition 2.8. We say that a normal modal logic L is *Kripke complete* if it is the logic of some class of Kripke frames, i.e. there exists a class C of Kripke frames with $L = L(C)$.

Theorem 2.9. *Each of the logics in 2.3 is Kripke complete. Furthermore, for each logic L in 2.3, the following table gives the class C of frames such that $F \in C \iff F \vDash L$.*

Name of logic	Class of Kripke Frames
K	All Kripke Frames
K4	Transitive frames: $\forall_{x,y,z}\ xRy \wedge yRz \rightarrow xRz$
wK4	Weakly transitive frames: $\forall_{x,y,z}\ xRy \wedge yRz \rightarrow xRz \vee x = z$
S4	Transitive and reflexive frames: $\forall_x\ xRx$
GL	Transitive, conversely well-founded* frames.

(: A frame (W, R) is* conversely well-founded *if there is no infinite sequence $\{a_n\}_{n\in\omega} \subseteq W$ such that $a_n R a_{n+1}$ for all n. We will dive more into this in the section on* GL*.)*

Proof. See [13, Corollaries 5.18 and 5.47] for all logics except wK4. For wK4, note that the axiom $\Diamond\Diamond p \rightarrow p \vee \Diamond p$ is equivalent to the wK4 axiom and is a Sahlqvist formula, so that wK4 is Kripke complete, see [10, Definition 3.41, Theorem 4.41]. □

Definition 2.10. We say that L enjoys the *finite model property*, or that L enjoys the FMP, if there exists some class C of finite Kripke frames such that $L = L(C)$. As with Kripke completeness, a logic enjoys the FMP exactly if it is the logic of some class of finite Kripke frames.

Theorem 2.11.

1. *All logics in 2.3 enjoy the FMP.*
2. *There are Kripke complete logics that do not enjoy the FMP.*
3. *There are Kripke incomplete NMLs.*

Proof. For the first statement, see e.g. [13, page 143, Corollary 5.32, Theorem 5.46] and [7, Theorem 3]. For the second, see [13, Theorem 6.2]. For the third, see [13, Theorem 6.16] □

As seen in the previous theorem, not all NMLs are Kripke complete. But by restricting the range of valuations, a semantics can be developed that does fully encapsulate all modal logics, namely so-called general (Kripke) semantics.

Definition 2.12. A *general Kripke frame* is a pair (F, P) such that $F = (W, R)$ is a Kripke frame and P is a subset of $\mathcal{P}(W)$ that contains $\varnothing$ and is closed under taking complements, finite intersections and $\Diamond_R$. We call P the *set of admissible sets.*

In algebraic terms we abbreviate this by saying that $(P, \Diamond_R)$ *is a (modal) subalgebra of* $(\mathcal{P}(W), \Diamond_R)$.

A *valuation* v *on a general frame* (F, P) is a function from *Prop* to P. These can again be extended to include all modal formulas in their domain.

Note that for any modal formula φ, we obtain that $v(\varphi) \in P$. We again say that $\varphi \in L(F, P)$ iff $v(\varphi) = W$ for all valuations v on (F, P). Note that $L(F) \subseteq L(F, P)$ for all general frames P on F. Just as in the 'normal' Kripke semantics, given a class C of general frames, we denote $L(C)$ to be the set of formulas that hold over all general frames of C. It again turns out that $L(C)$ is a NML for all classes of general frames C.

A Kripke frame can be seen as a general frame with $P = \mathcal{P}(W)$, so that general frame semantics are an extension of regular Kripke semantics. We have the following strong result for general frame semantics:

Theorem 2.13. *Let* L *be an NML. Then there exists a class* C *of general frames such that* $L = L(C)$.

Proof. See e.g. [13, Theorem 8.4]. □

We note that the actual frame behind a general frame need not necessarily respect the same axioms as the general frame. For example, there are GL-general frames (F, A), where F itself is not a GL frame, as we will see in 6.3.

2.3 Descriptive and rooted frames

The following is a well-known, useful subclass of general frames. See [10, Definition 5.65 and onwards] and [13, Section 8.4] for more details.

Definition 2.14. We say that a general frame (F, P) is *differentiated* if

$$(\forall_{A \in P}[x \in A \leftrightarrow y \in A]) \implies x = y.$$

We say it is *tight* if

$$(\forall_{A \in P}[y \in A \to x \in \Diamond_R A]) \implies xRy.$$

We say it is *compact* if, for all $\{A_i\}_{i \in I} \subseteq P$ such that $\{A_i\}_{i \in I}$ has the finite intersection property (that is, for all finite $J \subseteq I$: $\bigcap_{i \in J} A_i \neq \varnothing$)

$$\bigcap_{i \in I} A_i \neq \varnothing.$$

We say it is *refined* if it is both differentiated and tight, and we say it is *descriptive* if it is both refined and compact.

The class of descriptive frames are generally well-behaved. We have the following result for descriptive frames.

Theorem 2.15. *Let L be a NML. Then there exists a class C of* descriptive *frames such that $L = L(C)$.*

Proof. See e.g. [13, Theorem 8.36] □

Descriptive frames can also be used to classify well-behaving modal logics:

Definition 2.16. We say that a NML L is *canonical* if for each descriptive frame (F, P):

$$(F, P) \vDash L \implies F \vDash L$$

Remark. This is not the usual definition of canonicity. This requirement is called *d-persistence* and is not currently known to be equivalent to canonicity, though it is known that d-persistance implies canonicity. In the case that a NML is finitely axiomatizable, these notions are equivalent ([10, Proposition 5.58]). We call this property 'canonicity' for clarity as we otherwise overload d later on in this thesis.

The main logic of this thesis, GL, is a well-known example of a non-canonical logic. We will later on show a proof of this well-known statement and discuss why this forms an issue for generalizing the Abashidze-Blass theorem.

Definition 2.17. We say that a frame $F = (W, R)$ is *rooted with root $x \in W$* if xR^*y or $x = y$ for each $y \in W$, where R^* is the transitive closure of R. Given frame $F = (W, R)$ and point $a \in W$, we define the *rooted subframe $F_a = (W_a, R_a)$ of F with root a* as:

$$W_a = \{x \in W \mid aR^*x\}$$
$$R_a = R \cap (W_a \times W_a)$$

Note that, given frame F, for each a the subframe F_a is rooted with root a. We note the following result, which can easily be checked.

Theorem 2.18. *Let $F = (W, R)$ be a Kripke frame. Then*

1. *$L(F) \subseteq L(F_a)$ for each $a \in W$*
2. *For each $\varphi \notin L(F)$ there is an $a \in W$ such that $\varphi \notin L(F_a)$.*

Corollary 2.18.1. *All Kripke complete logic are complete with respect to their rooted Kripke frames.*

Proof. Assume L to be Kripke complete, say $L = L(C)$ with C some class of Kripke frames. Let C' be the class of rooted subframes of C. By the first statement of theorem 2.18, we find $L(C) \subseteq L(C')$ and by the second we find $L(C)^c \subseteq L(C')^c$, so that indeed $L(C) = L(C')$. □

We also have the following strengthening of this statement:

Corollary 2.18.2. *Every normal modal logic is sound and complete with respect to its rooted refined frames.*

Proof. Take any refined frame (F, P) with $F = (W, R)$. Consider any $a \in W$. We define:

$$P_a = \{A \cap W_a \mid A \in P\}$$

We will prove that (F_a, P_a) is again a refined frame.

First, indeed (F_a, P_a) is a general frame: We see that P_a is closed under $\Diamond_R$ as $\Diamond_R(A \cap W_a) = (\Diamond_R A) \cap W_a$. Likewise, P_a is closed under $\wedge$ and $\neg$. As $\varnothing \cap W_a = \varnothing$, we see that $\varnothing \in P_a$. We conclude that (F_a, P_a) is a general frame.

Assume $x, y \in W_a$ such that $x \neq y$. As P is differentiated, there exists some $A \in P$ such that $x \in A$ and $y \notin A$. We see that $x \in A \cap W_a$ and $y \notin A \cap W_a$, so that there is an element of P_a containing x but not y. Indeed, P_a is differentiated.

Assume $x, y \in W_a$ such that $\neg(xRy)$. As P is tight, there exists some A such that $y \in A$ but $x \notin \Diamond_R A$. We deduce that $y \in A \cap W_a$ but $x \notin (\Diamond_R A) \cap W_a = \Diamond_R(A \cap W_a)$. Thus, there exists an element A' of P_a containing y with the property that $x \notin \Diamond_R(A')$. We conclude that P_a is tight.

We see that (F_a, P_a) is tight and differentiated, so that it is refined. We now note that just like before, $L(F, P) = \bigcap_{a \in W} L(F_a, P_a)$. From this, we may conclude that the logic of any refined frame is the logic of some class of rooted refined frames. As noted before, any logic is the logic of some class of descriptive frames, in particular of some class of refined frames. In turn, any logic of some class of refined frames is the logic of some class of rooted refined frames, thereby proving the theorem. □

2.4 GL logics and frames

In this section we address logics with the GL-axiom, as defined in 2.9.

Definition 2.19. GL-logics are NMLs containing the GL-axiom:

$$\Box(\Box p \to p) \to \Box p$$

By contraposition, this axiom is equivalent to $\Diamond p \to \Diamond(p \wedge \neg\Diamond p)$.

Note that this logic is well-known as the Gödel-Löb logic, which is the provability logic of Peano arithmetic. For an in-depth explanation behind this logic we refer to [12] and [3]. In this thesis, we consider GL and its extensions only as a formal modal logic.

As stated in theorem 2.9, GL-frames correspond exactly to the transitive, conversely well-founded frames, where a frame is conversely well-founded iff it contains no infinite ascending chain: There exists no infinite sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n R x_{n+1}$ for all n. This sequence does not need to be strictly increasing. In particular, no GL-frames may contain reflexive points. We see that GL can be thought of as the logic of all frames that are strict partial orders without infinite ascending sequences. We will sometimes write $x < y$ instead of xRy for GL-frames.

Definition 2.20. We say that a Kripke frame $T = (W, <)$ is a *tree* if the following hold:

1. $<$ is a strict partial order on W, that is, an irreflexive, transitive relation.

2. T is *rooted* as in definition 2.17.

3. For each $x \in W$, the *principal downset of* x is linearly ordered by $<$, that is, $\{y \mid y \leq x\}$ is linearly ordered by $<$. Here, $y \leq x$ is an abbreviation of $y < x \vee y = x$.

The *depth* of a tree T is the largest number n such that there exists a linear subset of T consisting of $n + 1$ points. Note that by this convention, the depth of the one-point tree is 0. We say that a tree with chains of any finite length has infinite depth.

Note that GL has the FMP by theorem 2.11. We have the following well-known strengthening of that statement.

Theorem 2.21. GL *is the logic of the finite, irreflexive and transitive trees.*

Proof. See [13, Corollary 5.47.ii]. It uses a notion called 'unraveling', which we use later in this thesis in theorem 4.3. □

We note the following observation: For each finite, transitive, irreflexive tree, each point either has no successors or has a successor that has no successors. From this, we can conclude $\Box\bot \vee \Diamond\Box\bot \in \mathsf{GL}$, equivalently, $\neg\Diamond\top \vee \neg\Box\Diamond\top \in \mathsf{GL}$. This is exactly Gödel's second incompleteness theorem: Any theory capable of providing Peano arithmetic is either inconsistent or is incapable of proving its own consistency.

Within this thesis, we not only use Kripke semantics to study GL, we also use topological semantics. In fact, this is the main semantics studied in this thesis.

2.5 Topological semantics and scattered spaces

The notion of topological semantics is older than the notion of Kripke semantics. We omit the definition of the more well-known closure-semantics and define only d-semantics, as it is much more relevant for this thesis. For details on closure-semantics, see [5].

Definition 2.22. On a topological space (X, τ), the *derived set operator* $d : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as the operator that sends any subset to the set of its limit points. Formally, given $A \subseteq X$:

$$x \in d(A) \iff (x \in U \to \exists_{y \in A \cap U}\ y \neq x) \text{ for every open set } U.$$

We inductively define $d^n(A)$ for $n \in \omega$ as follows[1]:

$$d^0(A) = A$$
$$d^{n+1}(A) = d(d^n(A))$$

Definition 2.23. A *topological frame* is a topological space (X, τ). A *valuation* on a topological frame is a function v from $Prop$ to $\mathcal{P}(X)$.

The valuation v extends to all formulas in the usual manner for $\wedge, \vee, \neg$, etc. For the operators $\Box$ and $\Diamond$, our interpretation depends on the specific type of topological semantics.

d-semantics is the topological semantics in which $\Diamond$ is interpreted as the derived set operator d. The *logic of a topological space* (X, τ) *according to d-semantics* is the set of formulas φ such that $v(\varphi) = X$ for all valuations v. This is denoted by $L_d(X, \tau)$ or by $L_d(X)$.

For all topological spaces X and all $A \subseteq X$, we have that $d(d(A)) \subseteq A \cup d(A)$, so that d-semantics makes sense to consider for logics above wK4. As GL is a logic above K4, it is certainly a logic above wK4 and can be studied with d-semantics. It turns out that we need to restrict our view to a subclass of the topological spaces for GL.

Definition 2.24. A topological space (X, τ) is *scattered* if all its nonempty subsets contain isolated points, i.e. $A \backslash d(A) \neq \varnothing$ for all nonempty $A \subseteq X$.

The following was proven independantly by Simmons [19] and Esakia [14].

Theorem 2.25 (Simmons, Esakia)**.** *Let X be a topological space. Then X is scattered if and only if $d(A) \subseteq d(A \backslash d(A))$ for all $A \subseteq X$.*

From this we see that $L_d(C)$ is a GL-logic if and only if C is a class of scattered spaces. Furthermore, as we will soon see, $L_d(C) = \mathsf{GL}$ when we take C to be the class of all scattered spaces.

We now give a useful theorem that allows us to directly link the scattered spaces to GL Kripke frames.

Theorem 2.26. *Let (W, R) be a GL-frame. Then there exists a scattered space (W, τ) also on W such that $d_\tau(A) = \Diamond_R(A)$ for all $A \subseteq W$.*

Proof. As stated before, any Kripke GL-frame $F = (W, R)$ is a strict partial order without infinite ascending chains. We can equip F with the upset-topology, defined as $A \subseteq W$ is open if $xRy \to y \in A$ for each $x \in A$ and each $y \in W$. We will prove that

$$\Diamond_R A = \Diamond_\tau A \text{ for all } A \subseteq W.$$

where $\Diamond_R A = \{x \mid \exists_{y \in A} xRy\}$ and $\Diamond_\tau A = d(A)$ with the upset topology. Consider any $A \subseteq W$. We first note

$$x \in d(A) \iff \forall_{U \in \tau}[x \in U \to \exists_{y \in U \cap A} y \neq x]$$

[1]It is also possible to define $d^\alpha(A)$ for any ordinal α. In fact, this was the original reason for Cantor to introduce the ordinals.

Take U to be the set $\{z \mid x \leq z\}$. Then U is an upset so that it is open. As $x \in d(A)$, we find that there is some $y \in U \cap A$ with $x \neq y$. Thus, there is some $y \in A$ with $x < y$, so that xRy. We conclude that $x \in \Diamond_R A$.

Conversely, take $x \in \Diamond_R A$, so that there is some $y \in A$ with xRy. Note that K is irreflexive so that $x \neq y$. As $x < y$, any upset containing x must necessarily contain y, so that $x \in d_\tau(A)$. We indeed see that $d_\tau(A) = \Diamond_R A$. □

Theorem 2.27. *Let L be a Kripke complete* GL*-logic. Then L is complete with respect to scattered spaces.*

Proof. Let L be a Kripke complete GL-logic with class of Kripke frames C. Note that by the previous theorem, $L(C) = L_d(C)$ so that $L_d(C) = L(C) = L$. □

We see that d-semantics extends the regular Kripke semantics for GL-frames. Of course, as stated before, GL itself is Kripke complete, so by the previous result, it is sound and complete with respect to scattered spaces.

Definition 2.28. Given topological spaces X and Y, we say that $f : X \to Y$ is a *d-morphism* or d-map if it is continuous, open and pointwise discrete, i.e. for any $y \in Y$, $f^{-1}(y)$ is a discrete subset of X. Note that a subset A of X is discrete if $d(A) \cap A = \varnothing$

We have the following result for d-maps by [6], which also holds for non-scattered spaces:

Theorem 2.29. *Let $f : X \to Y$ be a d-morphism and let $A \subseteq Y$ be an open set. Then*

$$f^{-1}(d_Y(A)) = d_X(f^{-1}(A))$$

Furthermore, if f is onto, $L_d(X) \subseteq L_d(Y)$. In other words, for any formula φ we have that $X \vDash \varphi$ implies $Y \vDash \varphi$.

We see that we have a useful way of relating the logics of scattered spaces by using these morphisms. We will use this to prove that GL is sound and complete with respect to the ordinals.

2.6 Ordinal numbers as topological spaces

We assume familiarity with the ordinals and ordinal arithmetic. For an overview of the properties of ordinal numbers, see [17]. In this thesis, we will use ordinals in two different ways. The first is as an index set. The second is as a topological space, with the so-called 'order topology'.

Definition 2.30. The *order topology* on a totally ordered set X is the topology generated by the basis of the form:

$$(a, b) = \{x \in X \mid a < x < b\}$$

As any ordinal is a totally ordered set, we can equip all ordinals with this topology. We note the following, which fully characterizes the order topology on ordinals:

Theorem 2.31. *Let α be an ordinal equipped with the order-topology. For any $\beta \in \alpha$, if β is a successor ordinal then $\{\beta\}$ is an open subset of α. If β is a limit ordinal, then any open set containing β contains $(\gamma, \beta]$ for some $\gamma < \beta$.*

Proof. Consider any such α and any $\beta \in \alpha$. If β is a successor ordinal, say, $\beta = \gamma + 1$, then $\{\beta\} = (\gamma, \gamma + 2)$. If β is a limit ordinal, the result is trivial. $\square$

Definition 2.32. Given a set of ordinals A, we say that α is *cofinal in A* if $sup(A \cap \alpha) = \alpha$. Note that we must have that α is a limit ordinal in this case, as for successor ordinals $\alpha = S(\beta)$:

$$sup(\alpha) = \cup(S(\beta)) = \beta \neq \alpha$$

Lemma 2.33. *Let α be an ordinal and let A be a subset of α. Then for all $\beta \in \alpha$:*

$$\beta \in d(A) \iff \beta \text{ is cofinal in } A$$

Proof. Consider any such α and A. By definition:

$$\beta \in d(A) \iff \forall_{U \in \tau}(\beta \in U \to \exists_{\gamma \in A \cap U} \gamma \neq \beta)$$

If β is a successor ordinal, take $U = \{\beta\}$. We find that there is a $\gamma \in \{\beta\}$ with $\beta \neq \gamma$, which can of course never hold. Likewise, by definition, if β is a successor ordinal it can never be cofinal in A. Thus, we may assume that β is a limit ordinal. In this case:

$$\begin{aligned} \forall_{U \in \tau}(\beta \in U \to \exists_{\gamma \in A \cap U} \gamma \neq \beta) &\iff \forall_{\zeta < \beta} \exists_{\gamma \in A \cap (\zeta, \beta]} \gamma \neq \beta \\ &\iff \forall_{\zeta < \beta} \exists_{\gamma \in A} \zeta < \gamma < \beta \end{aligned}$$

But this means that for each $\zeta < \beta$, there is some $\gamma \in A$ such that $\zeta < \gamma$. Equivalently:

$$sup(A \cap \beta) = sup(A \cap [0, \beta)) = \beta$$

We conclude that $\beta \in d(A) \iff \beta$ is cofinal in A. $\square$

Theorem 2.34. *Let α be an ordinal equipped with the order topology. Then α is a scattered space.*

Proof. Consider any ordinal α and any nonempty $A \subseteq \alpha$. Take β to be the smallest element of A. As β is the smallest element of A, we find $A \cap \beta = \varnothing$ so that β is not cofinal in A, so that $\beta \notin d(A)$. We see that $\beta \in A \backslash d(A)$ so that $A \backslash d(A) \neq \varnothing$.

We conclude that $A \cap d(A)$ is nonempty for every nonempty $A \subseteq \alpha$, so that α is a scattered space. $\square$

2.7 The Abashidze-Blass theorem

Abashidze [1] and Blass [11] independently proved that $\mathsf{GL} = L_d(\alpha)$ for any ordinal α greater than ω^ω. In this section, we will discuss a proof of this theorem by G. Bezhanishvili and P.J. Morandi [9]. It uses the fact that GL is sound and complete w.r.t. the class of finite trees. We give the entire argument as later on in this thesis we will apply similar methods to generalize the Abashidze-Blass theorem. Recall as in definition 2.20 that the depth of a rooted tree T is one less than the length of the largest linear subset of T.

Lemma 2.35. *Let T be a finite, transitive, irreflexive rooted tree of depth $n > 0$. Then there exists an onto d-morphism $f : \omega^n + 1 \to T$. For $n = 0$, there exists an onto d-morphism $f : 1 \to T$*

Proof. We will prove this by induction. For $n = 0$, the result is trivial.

Now assume $n > 0$, and the result holds for all $k < n$. We let r be the root of T. Then r has *immediate successors* $t_1, ..., t_m$, from which 'sprout' subtrees $T_1, ..., T_m$. We view T as the trees $T_1, ..., T_m$, all glued together by the point r. We denote the depth of tree T_i by n_i. As a topological space, we can view T as the union $\biguplus T_i \cup \{r\}$, where any open U either contains the root and is equal to T, or does not contain the root and $U \cap T_i$ is open in each T_i. Likewise, we can split $\omega^n + 1$ as $\biguplus_{i\in\omega} X_i \cup \{\omega^n\}$, where each X_i is isomorphic to $\omega^{n_{i \bmod m}} + 1$ if $n_{i \bmod m} > 0$, and X_i is a singleton if $n_{i \bmod m} = 0$. As there is at least one n_i that equals $n - 1$, we see that $\sum_{i\in\omega} |X_i| = \omega^n$, so that this splitting is indeed well-defined. By the induction hypothesis, for each i there exists an onto d-morphism $g_i : X_i \to T_{i \bmod m}$. We now define $f : \omega^n + 1 \to T$ as follows:

$$f(\omega^n) = r$$
$$f(x) = g_i(x) \text{ for each } x \in X_i$$

Of course, f is onto. We claim it is also a d-morphism:

Consider any open $U \subseteq T$. If $r \in U$, then $U = T$ and $f^{-1}(U) = \omega^n + 1$ which is of course open. If $r \notin U$, then $U \cap T_i$ is open for all i, and:

$$f^{-1}(U) = \bigcup_{i\in\omega} g_i^{-1}(T_{i \bmod m})$$

As each g_i is a continuous map from X_i to $T_{i \bmod m}$ and as each X_i is clopen in $\omega^n + 1$, we deduce that $f^{-1}(U)$ is the union of open sets so that it is an open set itself. We conclude that f is continuous.

f is also open: Consider any open $U \subseteq \omega^n + 1$. If $\omega^n \notin U$, then $U \cap X_i$ is open for all i, so that $f(U) = \bigcup_i g_i(U \cap X_i)$, which is open as the union of open sets. If $\omega^n \in U$, we see that U must contain an co-finite amount of X_i in its entirety, so that $f(U) = T$. We conclude that f is open.

Finally, we prove that f is pointwise discrete. Of course, $f^{-1}(r) = \{\omega^n\}$ which is discrete. Now, for any $t \in T_i$:

$$f^{-1}(t) = \bigcup_{k:\ k \bmod m = i} g_k^{-1}(t)$$

Each such $g_k^{-1}(t)$ is a discrete subspace of X_k, and each X_k is closed. Thus, $f^{-1}(t)$ is discrete as the union of discrete subspaces of disjoint closed sets, so that f is indeed poinwise discrete. We conclude that f is a d-morphism, which proves the lemma. □

Lemma 2.36. *Let α be an ordinal. We consider α as a topological space. Then*

$$d^n(\alpha) = \{\omega^n\gamma \mid \omega^n\gamma \in \alpha\}$$

With d^n defined as in definition 2.22.

Proof. Take any ordinal α. We note that the result is trivial for $n = 0$. For the induction step, by lemma 2.33 we need to only find the set of cofinal points of $\{\omega^n\gamma \mid \omega^n\gamma \in \alpha\}$ that are also points of α. These are of course exactly the sets that can be written as $\omega^{n+1}\gamma$. □

Definition 2.37. We define GL_n to be the logic $\mathsf{GL} \oplus \Box^n\bot$.

We can now finally state and prove the Abashidze-Blass theorem:

Theorem 2.38 (Abashidze-Blass)**.** *Let α be an ordinal number.*

1. *If $1 \leq \alpha \leq \omega$, then $L_d(\alpha) = \mathsf{GL}_1$.*

2. *If $\omega^n + 1 \leq \alpha \leq \omega^{n+1}$, then $L_d(\alpha) = \mathsf{GL}_{n+1}$.*

3. *If $\alpha \geq \omega^\omega$, then $L_d(\alpha) = \mathsf{GL}$.*

Proof. Consider any ordinal number α. If $1 \leq \alpha \leq \omega$, then $d(\alpha) = \varnothing$, so that we trivially see that $L_d(\alpha) = \mathsf{GL}_1$.

Now assume that $\omega^n + 1 \leq \alpha \leq \omega^{n+1}$. As α is a scattered space we see that $\mathsf{GL} \subseteq L_d(\alpha)$. We know

$$d^{n+1}(\alpha) = \{\omega^{n+1}\gamma \mid \omega^{n+1}\gamma \in \alpha\}.$$

From this, we see that $d^{n+1}\alpha = \varnothing$. We conclude $\neg\Diamond^{n+1}\top \in L_d(\alpha)$, so that $\mathsf{GL}_{n+1} \subseteq L_d(\alpha)$.

Now, suppose $\varphi \notin \mathsf{GL}_{n+1}$. As GL_{n+1} is sound and complete w.r.t. the trees of depth at most n, there exists a tree T of depth $m \leq n$ such that $\varphi \notin L_d(T)$. By lemma 2.35, there is an onto d-morphism $f : \omega^m + 1 \to T$. By theorem 2.29, we find that $L_d(\omega^m + 1) \subseteq L_d(T)$, so that $\varphi \notin L_d(\omega^m + 1)$. As $\omega^m + 1$ is a clopen subset of α, we finally deduce that $\varphi \notin L_d(\alpha)$. Thus, $L_d(\alpha) \subseteq \mathsf{GL}_{n+1}$, so that $L_d(\alpha) = \mathsf{GL}_{n+1}$.

Finally, assume $\alpha \geq \omega^\omega$. Again, we see immediately that $\mathsf{GL} \subseteq L_d(\alpha)$. Conversely, GL is complete w.r.t. the finite trees, so that we can use the same argument as before and show that any formula φ not in GL must also not lie in $L_d(\alpha)$. We conclude that indeed $L_d(\alpha) = \mathsf{GL}$. □

We see that the Abashidze-Blass theorem proves that GL is the logic of ordinal spaces by noting that Kripke frames can be related to finite trees, and that finite trees can be related to ordinal spaces in a recursive manner. Our results build upon this approach, showing that logics above GL correspond to general spaces over specific classes of trees, which in turn correspond to general spaces over ordinals.

Chapter 3

General semantics for GL, least general spaces and $\mathsf{GL}.3$

In order to generalize the Abashidze-Blass theorem, we first introduce general semantics for d-semantics. In this chapter we start with a particular d-semantics. We show that for every topological space there must exist a smallest general space over that topological space. We continue with a discription of all logics that may be obtained in such a manner and give a concrete construction of topological spaces for each logic that can be obtained in this way.

3.1 General topological d-semantics

As stated before, we will study general semantics for scattered spaces. Note that these have already been introduced in [16] for the d-semantics and studied more recently in [21] and in [4] for multimodal logics.

Definition 3.1. A *general d-space* is a pair (X, P) with X a topological space and $P \subseteq \mathcal{P}(X)$ a subalgebra of $(\mathcal{P}(X), d)$, i.e.:

$$\varnothing \in P \tag{3.1}$$
$$A \in P \implies X \backslash A \in P \tag{3.2}$$
$$A \in P, B \in P \implies A \cap B \in P \tag{3.3}$$
$$A \in P \implies d(A) \in P \tag{3.4}$$

Note that such P are also closed under finite unions. This allows to restrict any valuation to sets only in P.

Definition 3.2. A *valuation v on a general d-space (X, P) is a function from Prop to P. As before, v naturally extends to include all basic modal formulas in its domain. We again say that $\varphi \in L_d(X, P)$ iff $v(\varphi) = X$ for all valuations v. We call $L_d(X, P)$ the* logic of the general d-space (X, P).

Just as with Kripke semantics, one obtains our original concept of d-semantics by taking $P = \mathcal{P}(X)$.

Theorem 3.3. *Let X be any topological space and let C be a set such that (X, P) is a general space for all $P \in C$. Then $(X, \bigcap C)$ is a general space.*

Proof. This proof is a routine verification. □

Corollary 3.3.1. *Let X be a topological space. Then there exists a general space over X that is contained in all general spaces over X.*

Proof. By the previous theorem the intersection of all general spaces over X is a general space over X, and this general space is automatically a subset of every other general space over X. □

3.2 The structure of least general spaces

Definition 3.4. The *least d-space* of a topological space X is the intersection of all P such that (X, P) is a general d-space. We often denote this by (X, P_{least}), (X, X_{least}) or just by P_{least} or X_{least}. By definition, (P_{least}, d) is the smallest subalgebra of $(\mathcal{P}(X), d)$. In algebraic terms: The least space is the zero-generated subalgebra of $(\mathcal{P}(X), d)$.

In this section, we will give a classification for all logics of classes of least scattered d-spaces, i.e., the least d-spaces of scattered spaces. We will start by describing the structure of least spaces.

Lemma 3.5. *Let X be a scattered space and let i, j be any two natural numbers. Then*

$$i \geq j \implies d^i(X) \subseteq d^j(X)$$

Proof. We know trivially that $d(X) \subseteq X$. If $A \subseteq B$, we know that $d(A) \subseteq d(B)$. We see that $d^{i+1}(X) \subseteq d^i(X)$ for all i. The result follows immediately. □

Theorem 3.6. *Let X be a scattered space and let i and j be two different natural numbers. Then*

$$(d^i(X) \backslash d^{i+1}(X)) \cap (d^j(X) \backslash d^{j+1}(X)) = \varnothing$$

Proof. Assume without the loss of generality that $i < j$. Then $i + 1 \leq j$, so by 3.5, we obtain that $d^j(X) \subseteq d^{i+1}(X)$.

We immediately obtain that $(d^i(X) \backslash d^{i+1}(X)) \cap d^j(X) = \varnothing$, so that indeed

$$(d^i(X) \backslash d^{i+1}(X)) \cap (d^j(X) \backslash d^{j+1}(X)) = \varnothing.$$

□

Definition 3.7. Let X be a scattered space. We call the set

$$\{d^i(X) \backslash d^{i+1}(X) \mid i \in \omega\}$$

the set of *atoms* of X, and we call $\bigcap_{i \in \omega} d^i(X)$ the *core* of X.

Note that each atom is pairwise disjoint by the previous corollary and the core is pairwise disjoint with each atom. Furthermore, the union of all atoms and the core is exactly X. Of course, it may be that the core is empty and that there are only finitely many nonempty atoms.

Lemma 3.8. *Let X be a scattered space with corresponding least general space P_{least}. Then P_{least} contains exactly the finite unions of the atoms of X together with the cofinite unions of the atoms of X with its core. Equivalently:*

$$A \in P_{least} \iff \exists_{I \subseteq \omega, I \text{ finite}}[A = \bigcup_{i \in I} d^i(X) \backslash d^{i+1}(X)] \vee$$
$$\exists_{J \subseteq \omega, J \text{ cofinite}}[A = \bigcap_{i \in \omega} d^i(X) \cup \bigcup_{j \in J} d^j(X) \backslash d^{j+1}(X)]$$

Let P denote the set above, so we will prove that $P = P_{least}$. We will prove both inclusions:

Lemma 3.9. *Let X be a scattered space and let P be as defined above. Then each element of P lies in P_{least}.*

Proof. As $\varnothing \in P_{least}$ and P_{least} is closed under complement, we deduce that $X \in P_{least}$. As P_{least} is closed under d, we deduce that $d^i(X) \in P_{least}$ for all $i \in \omega$. As P_{least} is closed under negation and intersection, we conclude that for all i and all j:

$$d^i(X) \backslash d^j(X) \in P_{least}$$

In particular, all atoms are elements of P_{least}. As P_{least} is closed under union, all finite unions of atoms are indeed elements of P_{least}.

Now take any element A of P, such that A is the cofinite union of atoms together with the core. In this case, $X \backslash A$ is the finite union of atoms, so that $X \backslash A \in P_{least}$. We conclude that $A \in P_{least}$. As any element of P is either a finite union of atoms of X or a cofinite union of atoms of X together with the core, we see that $P \subseteq P_{least}$. □

For the converse proof, we need another short lemma.

Lemma 3.10. *Let X be a scattered space and let P be as defined above. For any nonempty $A \in P$, let i be the smallest number such that*

$$d^i(X) \backslash d^{i+1}(X) \subseteq A.$$

Then $d(A) = d^{i+1}(X)$.

Proof. First note that this lemma is indeed well-posed: If $A \in P$ is nonempty it must contain at least one atom of X.

Note that for such A, we must have that $A \subseteq d^i(X)$, so that $d(A) \subseteq d^{i+1}(X)$. But as X is a scattered space, we know that

$$d(A) \subseteq d^{i+1}(X) = d(d^i(X) \backslash d^{i+1}(X)) \subseteq d(A).$$

We conclude that $d(A) = d^{i+1}(X)$. □

From this, we can prove the converse of lemma 3.9.

Lemma 3.11. *Let X be a scattered space and let P be as defined above. Then P itself is a general space, so that $P_{least} \subseteq P$.*

Proof. We will directly work with definition 3.1.

Of course, $\varnothing \in P$, as $\varnothing$ is the empty union of the atoms of X. The complement of a finite union of atoms is a cofinite union of atoms together with the core and vice versa, so that P is closed under taking complements.

We will prove that P is closed under taking intersections. As P is closed under complements, this is equivalent to proving that it is closed under taking unions. Consider any $A, B \in P$. If A and B are both finite unions of atoms, $A \cup B$ is again a finite union of atoms. If at least one of A and B is a cofinite union of atoms together with the core, $A \cup B$ is a cofinite union of atoms together with the core. We conclude that P is closed under unions.

Finally, consider any $A \in P$. If $A = \varnothing$, then $d(A) = \varnothing \in P$. If A is nonempty, it must contain some atom. By lemma 3.10, we deduce that there exists some i such that $d(A) = d^{i+1}(X)$. We note:

$$d^{i+1}(X) = \bigcap_{j \in \omega} d^j(X) \cup \left(\bigcup_{k > i} d^k(X) \setminus d^{k+1}(X)\right)$$

We see that $d(A)$ is a cofinite union of atoms together with the core, so that $d(A) \in P$. We conclude that P is closed under d.

We see that $\varnothing \in P$, and that P is closed under taking complements, intersections and d. We conclude that P is a general space so that $P_{least} \subseteq P$. □

Theorem 3.8 now immediately follows from lemmas 3.9 and 3.11.

3.3 The logic of least general d-spaces

From our full description of the structure of least d-spaces, we can now study the logics of such spaces.

Definition 3.12. We introduce the axiom .3 :

$$\Diamond p \wedge \Diamond q \rightarrow \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$$

We call this axiom the *linearity axiom.* We denote by $\mathsf{GL.3}$ the logic obtained by adding the .3 axiom to GL.

As the name implies, we have the following well-known result for .3:

Theorem 3.13. *All rooted .3-frames are linear.*

Proof. Let $F = (W, R)$ be any Kripke frame. It is enough to show that for any $x, y, z \in W$, if xRy and yRz then either $x = z$, yRz or zRy. But this is easily shown: Take the valuation v on F that sends p to the singleton $\{y\}$ and q to the singleton $\{z\}$. We see that $x \in v(\Diamond p \wedge \Diamond q)$, so that

$$x \in v(\Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)).$$

Indeed, either $y = z$, yRz or zRy. Now, as F is rooted we find it must be linear. □

The .3-axiom is also one of the finite-width axioms, under the name bw_1. The axiom bw_n roughly states for transitive frames that the width of the frame is at most n. We say that a logic is of finite width if it contains at least one finite width axiom. We have the following known result.

Theorem 3.14. *Every finite width logic is Kripke complete.*

Proof. See [13, Theorem 10.42]. □

From this, we can immediately classify the logics above $\mathsf{GL}.3$. Recall that the logics GL_n denote the logics of GL with the added axiom $\Box^n \bot$. We denote by $\mathsf{GL}_n.3$ the GL logic with both the .3-axiom and the axiom $\Box^n \bot$. We call logics containing an axiom of the form $\Box^n \bot$ a *finite depth logic*, and note the following theorem:

Theorem 3.15. *Let L be a finite depth logic. Then L has the FMP.*

Proof. See [13, Theorem 8.85]. □

We define C_n to be the finite irreflexive chain of depth n. Note that C_n contains $n + 1$ points. We denote the highest point of C_n by 0, the second highest point of C_n by 1, etc. Thus, (C_n, R) are the natural numbers up to n with R being the reverse order: $0 > 1 > 2 > ... > n$. The following is already well-known.

Theorem 3.16. $\mathsf{GL}.3$ *has the FMP.*

Proof. See [13, Theorem 11.20] and note that $\mathsf{GL}.3$ is a subframe logic, as discussed in [13, Section 11.3] and [13, Table 9.6]. □

Note that the only rooted finite $\mathsf{GL}.3$-frames are $\{C_n \mid n \in \omega\}$. We see that $\mathsf{GL}.3 = L(\{C_n \mid n \in \omega\})$. We also note that $\mathsf{GL}_{n+1}.3$ has the FMP for each $n \in \omega$ ($\mathsf{GL}_0.3$ is the inconsistent logic and is not considered).

Theorem 3.17. *Let L be a finite depth $\mathsf{GL}.3$-logic. Then $L = \mathsf{GL}_{n+1}.3$ for some $n \in \omega$.*

Proof. Note that $\mathsf{GL}_{n+1}.3$ is a finite depth $\mathsf{GL}.3$-logic for each n by definition. Let L be a finite-depth $\mathsf{GL}.3$-logic. Take n such that $\Box^{n+1}\bot \in L$. As L is a finite-depth logic, it has the FMP. We deduce that L is sound and complete with respect to its finite rooted frames. The only finite rooted $\mathsf{GL}.3$-frames are $\{C_n \mid n \in \omega\}$, so that $L = L(C)$ with C some subset of $\{C_n \mid n \in \omega\}$. As $\Box^{n+1}\bot \in L$ we see that $C_i \notin C$ for $i \geq n+1$. We deduce that $C \subseteq \{C_i \mid i \leq n\}$. Take k to be the largest number such that $C_k \in C$. Note that $L(C_k) \subseteq L(C_i)$ for $i \leq k$. We conclude that $C = \{C_i \mid i \leq k\}$. But note that in that case $L(C) = \mathsf{GL}_{k+1}.3$. As the finite rooted frames of L are exactly C and as L has the FMP, we conclude that $L = \mathsf{GL}_{k+1}.3$. □

Theorem 3.18. *Let L be an infinite-depth $\mathsf{GL}.3$-logic. Then $L = \mathsf{GL}.3$.*

Proof. Let L be an infinite-depth $\mathsf{GL}.3$-logic. As L is a $\mathsf{GL}.3$-logic it is Kripke complete by theorem 3.14. We note that for any $n \in \omega$, there must exist a rooted L-frame of depth at least n. If not, L must be a finite-depth logic as it is Kripke complete. We note that for any rooted $\mathsf{GL}.3$-frame F, if F contains at least $n+1$ points then C_n is an L-frame. Indeed, as F is a linear GL-frame, it has a final point if it contains at least one point, it has a final and a second-to-last point if it contains at least two points, and so forth. As L has rooted frames of depth at least n for each n, we deduce that C_n is an L-frame for each n. But then $\mathsf{GL}.3 = \bigcap_{n\in\omega} L(C_n) \subseteq L \subseteq \mathsf{GL}.3$, so that $L = \mathsf{GL}.3$. □

Theorem 3.19. *Let L be a normal extension of $\mathsf{GL}.3$. Then exactly one of the following must hold:*

1. *$L = \mathsf{GL}.3$*
2. *There exists $n > 0$ such that $L = \mathsf{GL}_{n+1}.3$*

Proof. This follows immediately from the previous two theorems. □

We can now formulate and prove the main theorem of this section.

Theorem 3.20.

1. *Let X be a scattered space. Then the logic of its least space $L_d(X, P_{least})$ is an extension of $\mathsf{GL}.3$.*
2. *Let L be a normal extension of $\mathsf{GL}.3$. Then there exists an ordinal α such that $L_d(\alpha, \alpha_{least}) = L$.*

We will prove this by directly relating least spaces to linear Kripke frames. For this, we first introduce modal algebras and isomorphisms for modal algebras.

Definition 3.21. We say that $(A, \Diamond)$ is a *modal algebra* if A is a Boolean algebra and $\Diamond$ is a unimodal operator on A such that the following hold for all $a, b \in A$:

1. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
2. $\Diamond 0 = 0$

Definition 3.22. Let $(A, \Diamond_A)$ and $(B, \Diamond_B)$ be two modal algebras. Then we say that a function $f : A \to B$ is a *morphisms of algebras* if for each $a, b \in A$:

1. $f(0_A) = 0_B$.

2. $f(\neg a) = \neg f(a)$.

3. $f(a \vee b) = f(a) \vee f(b)$.

4. $f(\Diamond_A a) = \Diamond_B f(a)$.

Furthermore, if f is a bijection we call it an *isomorphism of modal algebras.*

Note that for a general space (X, P), the structure (P, d_X) is a modal algebra when 0_P is interpreted as the empty set, $\Diamond_P$ is interpreted as d_X, $\neg$ is interpreted as the complement and $\vee$ as the union.

Lemma 3.23. *Let (X, P_X) and (Y, P_Y) be two general spaces such that (P_X, d_X) is isomorphic to (P_y, d_Y) as a modal algebra. Then, $L(X, P_X) = L(Y, P_Y)$.*

Proof. This is a routine verification. □

We define the frames C_n as before, and also define C_∞ to be the chain of all natural numbers in reverse order. We take $P_{C_\infty} \subseteq \mathcal{P}(C_\infty)$ to be the general space over C_∞ of all finite and cofinite subsets.

Lemma 3.24. *Let X be a scattered space.*

1. *If $d^{n+1}(X) = \varnothing$ but $d^n(X)$ is nonempty, then (P_{least}, d) is isomorphic to $(\mathcal{P}(C_n), d)$ as a modal algebra.*

2. *If no such n exists, (P_{least}, d) is isomorphic to (P_{C_∞}, d).*

Proof. Here, we use that Kripke semantics is contained in d-semantics for GL-frames by theorem 2.26. Note that all C_n and C_∞ do not have any infinite ascending chains so that they are indeed GL-frames.

Assume $d^{n+1}(X) = \varnothing$ and $d^n(X)$ is nonempty. We define $f : \mathcal{P}(C_n) \to P_{least}$ as follows:

$$f(A) = \bigcup_{i \in A} d^i(X) \backslash d^{i+1}(X)$$

Note that first of all, f is well-defined: For all $A \in \mathcal{P}(C_n)$, A is some finite set of integers, so that $f(A)$ is some finite union of atoms. We conclude that $f(A) \in P_{least}$.

We will prove that f is a bijection. First we will prove that it is one-to-one: Consider $A, B \subseteq C_n$ such that $A \neq B$. Without the loss of generality, we may assume that there exists an $i \leq n$ such that $i \in A$ and $i \notin B$. As $i \leq n$, we see that $d^i(X) \backslash d^{i+1}(X) \neq \varnothing$. We recall that all atoms are disjoint by corollary 3.6, such that indeed

$$d^i(X) \backslash d^{i+1}(X) \subseteq f(A) \text{ but } d^i(X) \backslash d^{i+1}(X) \not\subseteq f(B).$$

We conclude: $f(A) \neq f(B)$.

It is also onto: Consider any $A \in P_{least}$. Then $A = \bigcup_{i \in I} d^i(X) \backslash d^{i+1}(X)$ for some $I \subseteq \mathbb{N}$. But $d^i(X) = \varnothing$ for $i > n$, so that there exists some $I \subseteq C_n$ such that $f(I) = A$. We see that f is onto.

Now finally, we will show that f is indeed a morphism of modal algebras. The following is easy to verify, as f is a bijection.

1. $f(\varnothing) = \varnothing$.

2. $f(C_n \backslash A) = X \backslash f(A)$ for all $A \subseteq C_n$.

3. For all $A, B \subseteq C_n : f(A) \cap f(B) = f(A \cap B)$.

Now, we only need to check that $f(d(A)) = d(f(A))$ for all $A \subseteq C_n$. For empty A, this is trivial. Now consider any nonempty $A \subseteq C_n$, and let i be the smallest element of A. Now, $d(A)$ consists of all elements pointing to some element of A. This is exactly the set $\{k \mid k \geq i + 1 \wedge k \leq n\}$. Note that this set may be empty. We see that $f(d(A)) = d^{i+1}(X)$. But $d(f(A)) = d^{i+1}(X)$ by lemma 3.5, so that indeed $d(f(X)) = f(d(X))$.

We have now proven that f is indeed an isomorphism between modal algebras. In the case that $d^n(X)$ is nonempty for all n, we again take $f : C_\infty \to X$ to be

$$f(A) = \bigcup_{i \in A} d^i(X) \backslash d^{i+1}(X).$$

The only step to note is that again we have that two sets in P_{least} are different if they differ on at least one atom. The proof from here on out is similar so it is left out. □

Corollary 3.24.1. *Let X be a scattered space.*

1. *If $d^{n+1}(X) = \varnothing$ but $d^n(X)$ is nonempty, then $L_d(X, P_{least}) = \mathsf{GL}_{n+1}.3$.*

2. *If no such n exists, then $L_d(X, P_{least}) = \mathsf{GL}.3$.*

Proof. Via the previous lemma, if $d^{n+1}(X) = \varnothing$ but $d^n(X)$ is nonempty, we deduce that $L_d(X, P_{least}) = L_d(C_n) = \mathsf{GL}_{n+1}.3$. If $d^n(X) \neq \varnothing$ for all n, we deduce that $L_d(X, P_{least}) = L_d(C_\infty, P_{C_\infty})$. We note that $L_d(C_\infty, P_{C_\infty})$ is an infinite-depth $\mathsf{GL}.3$-logic, so that $L_d(X, P_{least}) = \mathsf{GL}.3$ by theorem 3.19. □

Note that for $\omega^n + 1 \leq \alpha \leq \omega^{n+1}$, we have that $d^{n+1}(\alpha) = \varnothing$ but $d^n(\alpha) \neq \varnothing$, and that for $\alpha \geq \omega^\omega$ we have that for all n, $d^n(\alpha) \neq \varnothing$.

Thus, theorem 3.20 now immediately follows from the previous corollary: The logic of a class of least spaces is a logic above $\mathsf{GL}.3$, and there are concrete examples of least spaces for each logic above $\mathsf{GL}.3$. Namely, for $\mathsf{GL}_{n+1}.3$ we have $(\omega^n + 1)_{least}$ and for $\mathsf{GL}.3$ we have $(\omega^\omega)_{least}$.

Note that $\mathsf{GL}.3$ can be thought of as a sort of maximal infinite-depth extension of GL. In this chapter we have proven that this logic can be obtained by restricting the set of valuations as much as possible, so that as many formulae will hold on a frame as possible. $\mathsf{GL}.3$ has also been studied recently in [2], where a different approach was used. Instead of restricting the set of valuations, different topologies on ordinals were considered. The authors connected topological completeness of $\mathsf{GL.3}$ to the existence of sufficiently large cardinal numbers.

Chapter 4

Completeness for logics enjoying the FMP

In this chapter we will generalize the Abashidze-Blass theorem to encompass all logics above GL enjoying the finite model property, or FMP. Recall that the Abashidze-Blass theorem states that $\mathsf{GL} = L_d(\alpha)$ for any ordinal $\alpha \geq \omega^\omega$, and a logic L enjoys the FMP if there exists a class of finite Kripke frames C such that $L = L(C)$. We will prove that for any logic L above GL enjoying the FMP there exists a general space over an ordinal $\alpha \leq \omega^\omega$ giving us the same logic.

4.1 Preservation properties of surjective d-morphisms

Definition 4.1. Given a normal modal logic L, we define the class of *Normal FMP-extensions of* L, or $\mathsf{NExt}_{FMP}(L)$, as the class of logics that extend L and enjoy the FMP.

As a GL Kripke frame is a (strict) partial order without infinite ascending chains, a logic in $\mathsf{NExt}_{FMP}(\mathsf{GL})$ is the logic of a class of finite strict partial orders. As seen in section 2.7, we can easily relate trees to ordinal numbers. We will show that we can relate finite strict partial orders to trees, but we first introduce a tool we will often use in this thesis.

Theorem 4.2. *Let X be a scattered space, let (X', P') be a general d-space and let $f : X \to X'$ be an onto d-morphism as in definition 2.28. Then the set $P = \{f^{-1}(A) \mid A \in P'\}$ is a general space over X, and (P, d) is isomorphic to (P', d) as a modal algebra.*

Proof. We take P as above, and we will prove that it is indeed a general space over X. Note that we often use that P' is a general space.

$$
\begin{aligned}
\varnothing \in P' &\Rightarrow f^{-1}(\varnothing) \in P \\
&\Rightarrow \varnothing \in P \\
\forall_A : A \in P &\Rightarrow \exists_{A' \in P'} A = f^{-1}(A') \\
&\Rightarrow X \backslash A = \{x \in X \mid x \notin f^{-1}(A)\}
\end{aligned}
$$

Note that f is onto. Thus, for every $A \in P$ with $A = f^{-1}(A')$:

$$
\begin{aligned}
\{x \in X \mid x \notin f^{-1}(A)\} &= \{x \in X \mid x \in f^{-1}(X' \backslash A')\} \\
&\Rightarrow X \backslash A = f^{-1}(X' \backslash A') \\
&\Rightarrow X \backslash A \in P
\end{aligned}
$$

We see that P indeed contains $\varnothing$ and is closed under taking complements. As $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ we deduce that P is closed under union (and, by extension, P is also closed under intersection). Finally, as f is a d-morphism, theorem 2.29 gives us that P is closed under d. We conclude that P is indeed a general space.

Note that the proof that (P, d) is isomorphic to (P', d) is already done: The above proof gives us that f^{-1} as seen as a function from P' to P is an isomorphism of algebras. As f is a surjective map from X to X', we also obtain that f^{-1} is a bijection from P' to P: By definition of P, f^{-1} is onto. In order to prove that it is one-to-one, take any $A, B \in P'$ with $A \neq B$. Without the loss of generality, we may assume that there exists some $a \in A$ with $a \notin B$. As f is onto, there exists some $x \in X$ with $f(x) = a$. We deduce that $x \in f^{-1}(A)$ but $x \notin f^{-1}(B)$, so that $f^{-1}(A) \neq f^{-1}(B)$. □

Corollary 4.2.1. *Let X be a scattered space, let (X', P') be a general d-space and let $f : X \to X'$ be an onto d-morphism. Then there exists a general space P over X such that $L_d(X, P) = L_d(X', P')$.*

Proof. Take P as in the above theorem. Then as (P, d) is isomorphic to (P', d) as a modal algebra, we deduce that indeed $L_d(X, P) = L_d(X', P')$. □

4.2 Finite trees and ordinals

From this section onward, we will freely move from the Kripke setting to the setting of scattered spaces as argued in theorem 2.26. We think of trees as Kripke frames, so that in the topological setting they are equipped with the upset topology. The following theorem relates finite partial orders to trees. Note that this specific construction of 'unraveling' transitive frames into trees is already quite well-established, see [13, Theorem 3.20]. Also recall that a GL-tree when viewed as a topological space is a scattered space, by theorem 2.25.

Theorem 4.3. *Let $\mathbb{F}$ be a finite, rooted GL-frame. Then there exist a finite GL-tree T and an onto d-morphism $f : T \to \mathbb{F}$.*

Proof. Consider any finite rooted GL-frame $\mathbb{F} = (W, R)$. We denote the root of $\mathbb{F}$ by r. We will define a tree T and an onto d-morphism $f : T \to \mathbb{F}$. For the points W' of T, we take the finite sequences $\{a_n\}$ of W, such that $a_0 = r$ and such that a_{n+1} is a direct successor of a_n for all $n \in \omega$. For a sequence $(a_i)_{i \leq n}$ and a sequence $(b_j)_{j \leq m}$, we say that $(a_i)R'(b_j)$ if (a_i) is a strict initial part of the sequence (b_j), i.e.:

$$(a_0, ..., a_n)R'(b_0, ..., b_m) \iff (n < m) \wedge \forall_{k \leq n}[a_k = b_k]$$

We now take $T = (W', R')$. We see that R' is irreflexive as no sequence can be its own strict initial part. Likewise, it is easy to reason that R' is transitive. Note that W' is finite as W is finite. As T is a finite, irreflexive and transitive structure, it is certainly a GL-frame. Now, note that $(r) \in W'$, and, as r is the root of $\mathbb{F}$, we notice that T is indeed rooted with root (r). Finally, for any $(a_0, a_1, ..., a_n) \in W'$, we see that the set $\{y \in W' \mid y < x\}$ is equal to the set $\{(a_0, ..., a_k) \mid k < n\}$, which is of course linearly ordered by R'. By definition 2.20, we conclude that T is indeed a finite GL-tree.

Now, define $f : T \to \mathbb{F}$ as $f((a_0, ..., a_n)) = a_n$. As T is rooted, it is easy to see that for any point x in W there exists a finite path of direct successors beginning at r and ending at x, so that f is onto. We will prove that it is a d-morphism.

Consider any open subset A of T, i.e., an upset of T. We will prove that $f(A)$ is open. Consider any $x \in f(A)$. By the definition of f, there exists some sequence $(a_0..., a_n, x) \in W'$. Pick any $y \in W$ with xRy. As xRy, there exists some finite sequence of direct successors beginning at x and ending in y, say $(x, b_0, ..., b_m, y)$. Now note:

$$(a_0, ..., a_n, x, b_0, ..., b_m, y) \in W'$$

and

$$(a_0..., a_n, x)R'(a_0, ..., a_n, x, b_0, ..., b_m, y).$$

As A is open, it is an upset. We see that $(a_0, ..., a_n, x, b_0, ..., b_m, y) \in A$, so that $y \in f(A)$. We deduce that $f(A)$ is an upset, so that $f(A)$ is indeed open. We conclude that f is an open function.

Now, consider any open subset A of $\mathbb{F}$, i.e. an upset. We will prove that $f^{-1}(A)$ is open. Consider any $(a_0, ..., a_n) \in f^{-1}(A)$ and any $(b_0, ..., b_m) \in W'$ with $(a_0, ..., a_n)R'(b_0, ..., b_m)$. We deduce that the second sequence must be in the form of $(a_0, ..., a_n, b_{n+1}, ..., b_m)$. We obtain that $a_n R b_m$. As A is an upset, $b_m \in A$ so that indeed

$$(a_0, ..., a_n, b_{n+1}, ..., b_m) \in f^{-1}(A).$$

We see that $f^{-1}(A)$ is an upset so that it is indeed open. We conclude that f is continuous.

Now, we prove pointwise discreteness. Consider any $b \in W$. We wish to prove that $d(f^{-1}(b)) \cap f^{-1}(b) = \varnothing$. From theorem 2.26, we know that a sequence lies in $d(f^{-1}(b))$ if and only if its final entry is a predecessor of b. Also note that

$f^{-1}(b)$ consists of those sequences ending in b. As b is not a predecessor of itself, this intersection indeed is empty. We deduce that f is pointwise discrete.

We conclude that f is an open, continuous and pointwise discrete map from T to $\mathbb{F}$, and therefore is an onto d-morphism. $\square$

As in definition 2.20, we say that a rooted GL-frame has depth n if the length of its largest linear subset is exactly $n+1$.

Corollary 4.3.1. *Let F be a finite rooted GL-frame of depth $n > 0$. Then there exists a general frame P over $\omega^n + 1$ such that $L(\mathbb{F}) = L_d(\omega^n + 1, P)$*

Proof. Consider a finite rooted GL-frame $\mathbb{F}$ of depth n. By the previous theorem, there exists a finite GL-tree T and an onto d-morphism $f : T \to \mathbb{F}$. Note that T itself has depth n. By Corollary 4.2.1, there exists a general space P' over T such that $L(T, P') = L(\mathbb{F})$. Now, by Lemma 2.35, there exists an onto d-morphism $g : \omega^n + 1 \to T$. Again, by 4.2.1, there exists a general space P such that $L_d(\omega^n + 1, P) = L(T, P') = L(\mathbb{F})$. $\square$

4.3 Gluing and a first generalization of Abashidze-Blass

In the previous section we have related finite trees to ordinals. We now wish to 'glue' such ordinal spaces onto one another. We use the following result.

Theorem 4.4. *Let X be a scattered space and P be a general space over X such that X can be written as a disjoint union $X = \bigsqcup_{a \in A} X_a$ of scattered spaces X_a. Then there exist general spaces $\{P_a\}_{a \in A}$ such that $P_a = \{B \cap X_a \mid B \in P\}$ and we have that $L_d(X, P) = L_d(C)$ for the class $C = \{(X_a, P_a) \mid a \in A\}$.*

Proof. Consider any such pair (X, P). Note that $X = \bigsqcup_{a \in A} X_a$ if and only if each X_a is pairwise disjoint, their union covers X, and each X_a is clopen in X. We now define $P_a = \{B \cap X_a \mid B \in P\}$. We claim that P_a is a general space over X_a for each $a \in X$.

As P is a general space we deduce that each P_a contains $\varnothing$ and is closed under taking complements and intersections. To prove that it is closed under d, one needs to recall that all X_a are clopen in X. From this we see that $d(B \cap X_a) = d(B) \cap X_a$, so that indeed P_a is closed under d.

We now take C to be the set of general spaces $\{(X_a, P_a) \mid a \in K\}$. We will prove that $L(C) = L(X, P)$. Assume $\varphi \notin L(X, P)$. Thus, there exists some valuation v on P such that $v(\varphi) \neq X$. As X is the disjoint union of all X_a, we notice that there exists some a such that $v(\varphi) \cap X_a \neq X_a$. We now define the valuation v_a on X_a for each propositional letter p as follows:

$$v_a(p) = v(p) \cap X_a$$

The valuation v_a extends to all logical formulas in the usual manner. We now claim that $v_a(\psi) = v(\psi) \cap X_a$ for all logical formulas ψ. We will prove this by induction on the complexity of formulas.

By definition, the statement holds for all propositional letters. Now, assume that the statement holds for ψ_1 and ψ_2. Then

$$\begin{aligned} v_a(\neg\psi_1) &= X_a \backslash v_a(\psi_1) \\ &= X_a \backslash (v(\psi_1) \cap X_a) \\ &= (X \backslash v(\psi_1)) \cap X_a \\ &= v(\neg\psi_1) \cap X_a. \end{aligned}$$

$$\begin{aligned} v_a(\psi_1 \wedge \psi_2) &= v_a(\psi_1) \cap v_a(\psi_2) \\ &= (v(\psi_1) \cap X_a) \cap (v(\psi_2) \cap X_a) \\ &= (v(\psi_1) \cap v(\psi_2)) \cap X_a \\ &= v(\psi_1 \wedge \psi_2) \cap X_a. \end{aligned}$$

$$\begin{aligned} v_a(\Diamond\psi_1) &= d(v_a(\psi_1)) \\ &= d(v(\psi_1) \cap X_a) \\ &= d(v(\psi_1)) \cap X_a \\ &= v(\Diamond\psi_1) \cap X_a. \end{aligned}$$

Here, we used the remark we made earlier, that $d(B \cap X_a) = d(B) \cap X_a$ for all $B \subseteq X$. We indeed obtain by induction that $v_a(\psi) = v(\psi) \cap X_a$ for all formulas ψ. We see that $v_a(\varphi) = v(\varphi) \cap X_a \neq X_a$, implying that $\varphi \notin L(X_a, P_a)$. It follows that $\varphi \notin L(C)$, so that $L(C) \subseteq L(X, P)$.

Conversely, assume $\varphi \notin L_d(C)$. This implies that there exists some $a \in A$ such that $\varphi \notin L_d(X_a, P_a)$. We must have that there exists some valuation v_a on P_a such that $v_a(\varphi) \neq X_a$. We now define a valuation v on X for each propositional letter p as follows:

$$v(p) = v_a(p)$$

Again, v can be extended to all logical formulas. Just as before, we can prove that $v(\psi) \cap X_a = v_a(\psi)$ for all logical modal formulas ψ. As $v_a(\varphi) \neq X_a$, we must have that $v(\varphi) \neq X$, so that $\varphi \notin L_d(X, P)$. We conclude that $L_d(X, P) = L_d(C)$. □

Note that the previous theorem is not trivial, as the general spaces over the disjoint parts may be dependant on one another. The theorem states that we may look at them independently. We now have all the tools necessary to prove the main statement of this chapter.

Theorem 4.5 (Generalized Abashidze-Blass Theorem for logics enjoying the FMP)**.** *Let L be any logic in $\mathsf{NExt}_{FMP}(GL)$. Then there exists an ordinal $\alpha \leq \omega^\omega$ and a general space P over α such that $L = L_d(\alpha, P)$.*

Proof. Consider any logic $L \in \mathsf{NExt}_{FMP}(\mathsf{GL})$. Note that there exists a class C of finite Kripke frames with $L(C) = L$. By corollary 2.18.1, we may assume that

C contains only rooted Kripke frames. Note that we can also assume that C is countable: For each formula $\varphi \notin L$ there exists at least one $\mathbb{F} \in C$ such that $\varphi \notin \mathbb{F}$. As there are countably many modal formulas, we can select a frame in C for each formula not in L by using the Axiom of Choice. We now index the frames of C by $\{\mathbb{F}_n\}_{n \in O}$, where either $O = \omega$ or some natural number.

By corrollary 4.3.1, for each n there exists a pair $(\omega^{k_n} + 1, P_n)$ such that $L(\mathbb{F}_n) = L_d(\omega^{k_n} + 1, P_n)$. We take $\alpha = \sum_{n \in O}(\omega^{k_n} + 1)$. Note that $\alpha \leq \omega^\omega$. As in the proof of theorem 2.38, α can be split into $\bigcup_{n \in O} X_n$ where each X_n is isomorphic to $\omega^{k_n} + 1$. As each X_n is pairwise disjoint, clopen in α and the union of all X_n covers α we may conclude that α is the disjoint union of all X_n.

As X_n is isomorphic to $\omega^{k_n} + 1$ for each n, we may lift P_n to a general space P'_n over X_n so that P'_n is isomorphic to P_n. We then define a general space P over α as follows:

$$A \in P \iff \forall_n \; A \cap X_n \in P_n$$

As α is the disjoint union of all X_n this is indeed a general space, and by theorem 4.4, we see that $L_d(\alpha, P) = \bigcap_{n \in O} L_d(X_n, P'_n) = L$. □

We first note that theorem 4.5 is indeed a generalization of the Abashidze-Blass theorem. We note that GL has the FMP, so that theorem 4.5 states that there exists a general space over ω^ω such that its logic is GL. It follows that GL itself is an extension of the logic of ω^ω. As ω^ω is a scattered space, it follows from theorem 2.25 that $\mathsf{GL} = L_d(\omega^\omega)$. In the following chapter, we will generalize the Abashidze-Blass theorem further to encompass all Kripke complete logics (and possibly more), though we will need to introduce some new tools first to be able to work with non-finite Kripke frames.

Chapter 5

Completeness for GL-semicomplete logics

In the previous chapter we managed to generalize the Abashidze-Blass theorem to logics enjoying the FMP. In this chapter we will further generalize the Abashidze-Blass theorem. A natural class would be the class of Kripke complete logics, but we jump ahead to a new class of logics containing the Kripke complete logics, which we call the GL-semicomplete logics. We will also introduce a new class of trees as tools for such logics.

5.1 Semicompleteness and FCP-trees

Definition 5.1. Let L be a normal modal logic. We say that a normal modal logic L' is *L-semicomplete* if there exists some class C of general Kripke frames (F, P) such that each such F is an L-frame and L' is the logic of the class C. In formal terms:

$$L' = L(C) \wedge \forall_{(K,P)\in C} L \subseteq L(K)$$

We denote the class of L-semicomplete logics by $\mathsf{NExt}_{semi}(L)$. We say that L is *quasicanonical* if all normal extensions of L are L-semicomplete, i.e.

$$\mathsf{NExt}(L) = \mathsf{NExt}_{semi}(L).$$

Here, $\mathsf{NExt}(L)$ denotes the class of all normal modal logics extending L.

We make the following observations. First of all, all L-semicomplete logics are L-logics, as an L-semicomplete logic is the logic of general frames over L-frames. Second of all, all Kripke-complete extensions of L are L-semicomplete, so that $\mathsf{NExt}_{semi}(L)$ contains all Kripke-complete extensions of L. Furthermore, though any logic is sound and complete with respect to its general frames, there may exist non-L-semicomplete extensions of L. We will dive deeper into these notions in a later chapter, where we will show that GL is not quasicanonical.

We will prove that for every GL-semicomplete logic L there exists some general space P over a countable ordinal α such that $L = L_d(\alpha, P)$. From this result we will generalize the Abashidze-Blass theorem to all GL-semicomplete logics, in particular to all Kripke complete GL-logics.

As in the previous section, we will show that all Kripke GL-frames correspond to some class of general frames on trees, which in turn correspond to general frames over ordinal numbers. First, we will introduce a new class of Kripke frames useful for this thesis.

Definition 5.2. We say that a *strictly partially ordered structure* $(W, <)$ enjoys the *finite chain property*, or enjoys the FCP, if there does not exist an *infinite* subset of W linearly ordered by $<$.

Note that all strictly partially ordered Kripke frames enjoying the FCP are GL-frames. Notably, a Kripke frame (W, R) enjoys the FCP if and only if both (W, R) and (W, R^{-1}) are GL-frames. We will restrict our attention to the *trees* enjoying FCP, which we will often call FCP-trees. For the definition of a tree, see definition 2.20. We will prove that classes of general spaces over such trees turn out to be a suitable semantics for GL-semicomplete logics. Even stronger, we will prove that one only needs to consider *countable* FCP-trees, and that these in turn correspond to general spaces over countable ordinals. First, we will make the link between Kripke spaces and FCP-trees. The following theorem extends theorem 4.3 and is proven in a similar manner.

Theorem 5.3. *Let K be a Kripke GL-frame. Then there exist a disjoint union of FCP-trees T and an onto d-morphism $f : T \to K$.*

Proof. Consider any Kripke GL-frame $K = (W, <_K)$. We take the points of $T = (W', <_T)$ to be the finite non-empty strictly increasing sequences in K. Note that these need not be sequences of direct successors. Formally:

$$x \in W' \iff n \geq 0 \wedge x = (a_0, a_1, ..., a_n) \in W^n \wedge \forall_{0 \leq i < n}[a_i <_K a_{i+1}]$$

For a sequence $(a_i)_{i<n}$ and a sequence $(b_j)_{j<m}$, we now define $(a_i) <_T (b_j)$ if and only if (a_i) is a strict inital part of the sequence (b_j). Formally:

$$(a_1, ..., a_n) <_T (b_1, ..., b_m) \iff n < m \wedge \forall_{i \leq n}[a_i = b_i]$$

We first prove that T enjoys the FCP. In the same manner as in the proof of theorem 4.3 we conclude that $<_T$ is irreflexive and transitive. Now, consider any linear subset A of T. Assume that A is infinite. All elements of A must be sequences of unequal length. If not, there are sequences $(a_1, ..., a_n)$ and $(b_1, ..., b_m)$ in A with $n = m$ but $(a_1, ..., a_n) < (b_1, ..., b_m)$ or $(b_1, ..., b_m) < (a_1, ..., a_n)$, which is of course impossible. As A is infinite and each element of A is a sequence of a different length, there must be a sequence of sequences $(x_n)_{n \in \omega}$, with each $x_n \in A$ a sequence, so that if $n < m$, the length of x_n is smaller than the length of x_m. Note that we must have for each n that x_n has length at least n. Now, look at the sequence $\{y_n\}_{n \in \omega} \subseteq K$ defined as $y_n = (x_n)_n$, which is the sequence with at its n'th position the n'th element of the n'th sequence. This is an infinite sequence, and as $x_n <_T x_{n+1}$, we must have that $y_n = (x_n)_n = (x_{n+1})_n <_K (x_{n+1})_{n+1} = y_{n+1}$. Thus, we have found an infinite increasing sequence in K. But then K is not a GL-frame, in contradiction to our assumptions. We deduce that any linear subset of T is finite, so that T indeed enjoys the FCP.

Now we will prove that T is a disjoint union of trees. Indeed, we find that all the disjoint parts of T are rooted with root (a) for some $a \in K$. Also, for any element $(a_0, ..., a_n) \in K$, we see that the downset of $(a_0, ..., a_n)$ is exactly the collection of sets of the form $(a_0, ..., a_i)$ with $i \leq n$, which is a linear set. We conclude that T is the disjoint union of FCP trees.

We will now show that there exists an onto d-morphism $f : T \to K$. We define f as follows:

$$f((a_1, ..., a_n)) = a_n$$

Of course, f is onto, as $f((a)) = a$ for any $a \in W$. We will now prove that it is open. This proof is very similar to the proof of theorem 4.3:

Let $A \subseteq W'$ be such that A is open. We wish to prove that $f(A)$ is open, i.e. that it is an upset. Consider any $x \in f(A)$. This directly implies that there exists some sequence $(a_0, ..., a_n, x) \in A$. Take any $y \in W$ with $x <_K y$. Then

$$(a_0, ..., a_n, x, y) \in T,$$

and

$$(a_0, ..., a_n, x) <_T (a_0, ..., a_n, x, y).$$

As A is open, it is an upset. From this we deduce that $(a_0, ..., a_n, x, y) \in A$. As $f((a_0, ..., a_n, x, y)) = y$ we see that $y \in f(A)$. We conclude that $f(A)$ is an upset so that it is open. Indeed, f is an open function.

We will prove that f is continuous. Let $A \subseteq K$ be an open set. We wish to prove that $f^{-1}(A)$ is open, i.e. an upset. Consider any $(x_0, ..., x_n) \in f^{-1}(A)$, and any $(a_1, ..., a_m)$ with $(x_0, ..., x_n) <_T (a_0, ..., a_m)$. By definition of $<_T$, we deduce that $(a_0, ..., a_m)$ is of the form $(x_0, ..., x_n, a_{n+1}, ..., a_m)$. By definition of $<_T$ we must of course have that $x_n <_K a_m$. As $x_n \in A$, we deduce that $a_m \in A$, so that indeed $(x_1, ..., x_n, a_{n+1}, ..., a_m) \in f^{-1}(A)$. We see that $f^{-1}(A)$ is open, so that f is continuous.

We will finally prove that f is pointwise discrete. Consider any $b \in W$. We wish to prove that $d(f^{-1}(b)) \cap f^{-1}(b) = \varnothing$. From theorem 2.26, we know that a sequence lies in $d(f^{-1}(b))$ if and only if its final entry is a predecessor of b. Also note that $f^{-1}(b)$ consists of those sequences ending in b. As b is not a predecessor of itself, this intersection indeed is empty. We conclude that f is pointwise discrete.

All in all, $f : T \to K$ is onto, open, continuous and pointwise discrete, so that it is an onto d-morphism. □

Corollary 5.3.1. *Let (K, P) be a general frame such that K is a* GL *Kripke frame. Then there exists a class C of general frames over FCP trees such that $L(K, P) = L(C)$.*

Proof. Consider any such pair (K, P). Via theorem 5.3, there exists a disjoint union of FCP trees T and an onto d-morphism $f : T \to K$. Via corollary 4.2.1, there exists a general space P_T such that $L(K,P) = L(T,P_T)$. As in theorem 2.26, we recall that $L(T,P_T) = L_d(T,P_T)$ where we view T as a scattered space. By theorem 4.4, we now see that $L_d(T,P_T) = L_d(C)$, where C is the class of the disjoint trees in T with corresponding general spaces. We again note that we can freely jump back to Kripke semantics, so that $L_d(C) = L(C)$. Our observations together give that $L(K,P) = L(C)$, exactly as required. □

5.2 The tree hierarchy

In the previous section we managed to relate Kripke frames to FCP trees. In this section we provide a recursive construction for FCP trees, which we will use later for induction-based proofs.

Definition 5.4. The *tree hierarchy* is the class $\{H_\alpha\}_{Od(\alpha)}$ indexed by the ordinals such that each H_α is a set of trees, defined as follows:

1. H_0 is the empty set.
2. We take $T \in H_{\alpha+1}$ by taking a root with at most α successors and replacing each successor with a tree in $H_{\alpha+1}$. Formally, $T \in H_{\alpha+1}$ if and only if there exists a function $f : \beta \to H_\alpha$ with $\beta \leq \alpha$ so that T is the tree obtained by taking the tree with root r and direct successors $\{\gamma \mid \gamma < \beta\}$ and replacing each point $\gamma < \beta$ with the tree $f(\gamma)$.
3. For a limit ordinal α, we take $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$.

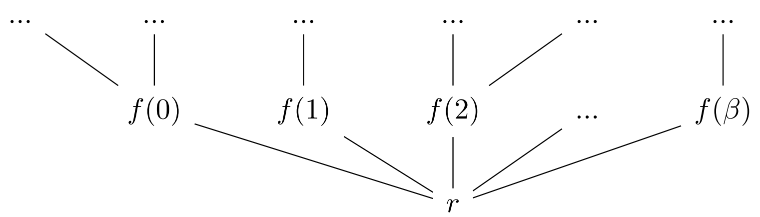

Figure 5.1: Construction of a tree in a successor stage of the tree hierarchy

Note that we allow f to be the empty function in the successor step, thereby obtaining that the one-point tree is in H_α for all $\alpha \geq 1$ as a root with no successors.

We say that a tree T has *tree order* α if α is the smallest ordinal such that there exists a tree $T' \in H_{\alpha+1}$ isomorphic to T.

Note that H_α is a set and not a proper class for each ordinal α. Furthermore, if $\alpha \leq \beta$ we see that $H_\alpha \subseteq H_\beta$. Also note that any tree in the tree hierarchy has a tree order: Any tree is obtained first in some successor stage, as every tree in a limit stage is already obtained in some strictly earlier stage.

Theorem 5.5. *Let T be a tree. Assuming the Axiom of Choice, T enjoys the FCP if and only if it has a tree order.*

Proof. First, from 'right to left', we wish to prove that T enjoys the FCP for each α and each $T \in H_\alpha$. We will prove this via transfinite induction:

First off, for $\alpha = 0$ the statement is trivial as H_0 is empty. Now assume that $\alpha > 0$ and for all $\beta < \alpha$ the statement holds. We will prove it holds for α. If α is a limit ordinal the statement is trivial, as for any $T \in H_\alpha$ there exists some $\beta < \alpha$ such that $T \in H_\beta$.

Now assume α to be a successor ordinal, say $\alpha = \beta + 1$. Consider any $T \in H_{\beta+1}$. We will prove that all linear subsets of T are finite. As $T \in H_{\beta+1}$, we know that T has root r and successors $\{T_\gamma\}_{\gamma \leq \delta}$ with $\delta \leq \beta$ and $T_\gamma \in H_\beta$ for each γ. Consider any linear subset $A \subseteq T$. Note that A may contain r, but $A \backslash \{r\}$ is in its entirety contained within one of the T_γ. But as T_γ enjoys the FCP we know that $A \cap T_\gamma$ must be finite, so that A is finite. We conclude that T has no infinite linear subsets, so that T enjoys the FCP. Via ordinal induction, we have indeed shown that all trees with a tree order enjoys the FCP.

Now, from 'left to right', we will prove that trees without tree order do not enjoy FCP. Assume that T has no tree order but does enjoy FCP. We denote by r the root of T. As T enjoys the FCP, for each $x \in T$ with $x > r$ the set $\{y \mid r < y \leq x\}$ is a non-empty finite set, so that it has a smallest element. We conclude that there exists a set A consisting of only direct successors of r, such that each element of T unequal to r either lies in A or lies above some point in A.

We see that 'chopping off' the root r of T gives us disjoint subtrees T_a for each $a \in A$. We claim that there exists an $a \in A$ such that T_a also has no tree order:

Assume not, so that each T_a has a tree order. Say that tree T_a has order α_a. Then $\{\alpha_a\}_{a \in A}$ is a *set* of ordinals, so that it has a supremum, say α. We see that $H_{\alpha+1}$ contains a tree isomorphic to T_a for each a. Take β to be the ordinal such that there is a bijection $f : \beta \to A$. Via the Axiom of Choice, such a β must exist. We can then construct the function $g : \beta \to H_\alpha$ such that $g(\gamma) = T_{f(\gamma)}$.

Now, take $\gamma = \max(\alpha + 1, \beta)$. Then for each a there exists a tree in H_γ isomorphic to T_a, and g is a function from $\beta \leq \gamma$ to H_γ with $g(\zeta) = T_{f(\zeta)}$ for all ζ. We see that $H_{\gamma+1}$ contains a tree isomorphic to T. But we assumed that T has no tree order. We conclude that there exists a direct successor a from the root r of T such that the subtree T_a as sprouted from a has no tree order.

But now, we know that if a tree has no tree order, the root has some successor a so that the subtree T_a has no tree order. Via the Axiom of Countable Choice we can infinitely repeat this process, obtaining a sequence $\{x_n\}_{n \in \omega}$ so that for each n, x_{n+1} is a direct successor of x_n and the subtree with root x_n has no tree order. But then $\{x_n\}_{n \in \omega}$ is an infinite strictly increasing sequence, so that T has an infinite chain. We deduce that T does not enjoy FCP. We conclude that any tree without tree order does not enjoy FCP. Via contraposition, we obtain our theorem. □

From this theorem we have much more structure to prove statements over the trees enjoying FCP. This will help us prove later on that all FCP-trees are surjective images of ordinal numbers, but first we prove a theorem for the countable FCP-trees.

Theorem 5.6. $H_{\aleph_1}$ *is the set of all countable FCP trees, so that an FCP tree is countable if and only if its tree order is countable.*

Proof. First, we prove that $H_{\aleph_1}$ contains only countable FCP trees. We will prove this via transfinite induction. Note that $H_{\aleph_1}$ is the union of all H_α with α a countable ordinal.

Assume that we have some countable ordinal α, and that H_α contains only countable trees. Then any tree T in $H_{\alpha+1}$ is a root with as successors at most α many trees in H_α. The trees in H_α are all countable and α is countable, so that T is countable as the countable union of countable sets. We deduce that $H_{\alpha+1}$ is countable. For limit ordinals: Assume α to be a countable limit ordinal such that H_β contains only countable trees for all $\beta < \alpha$. As $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, we deduce that H_α itself only contains countable trees. By induction we conclude that $H_{\aleph_1}$ contains only countable trees.

Conversely, we will prove that each countable tree is an element of H_α for some countable α. Assume not. Then there exists a countable tree with uncountable tree order. Take α to be the least ordinal number such that $H_{\alpha+1}$ contains a countable tree with uncountable order, and let T be any such tree in $H_{\alpha+1}$. By definition, there exists an ordinal β and a function $f : \beta \to H_\alpha$ such that T is isomorphic to the tree with root r and successor trees $f(\gamma)$ for $\gamma \in \beta$. As T is countable, β must be countable. Note that each $f(\gamma)$ has lower order than T, so that they all have countable order, say $\{\alpha_\gamma\}_{\gamma\in\beta}$. But take α' to be the supremum of all α_γ. Then α' is countable as the supremum of countably many countable ordinal numbers, and $H_{\alpha'+1}$ contains all $f(\gamma)$. If we now take α'' to be the supremum of α' and β, we find that $T \in H_{\alpha''}$ and α'' is countable. Thus, T has countable tree order. This was in contradiction with our assumption, so that all countable FCP trees are elements of $H_{\aleph_1}$. □

5.3 Countable FCP-trees and the second generalization of Abashidze-Blass

In this section we prove that we need only consider these countable FCP-trees to describe GL-semicomplete logics and use this fact to prove our second generalization of the Abashidze-Blass theorem. The following theorem and its proof mirror [8, Theorem 4.2], where a comparable theorem was proven for logics above $\mathsf{S4}$.

Theorem 5.7. *Let L be a normal modal logic, let T be an FCP tree and let P be a general frame over T such that $L \subseteq L(T,P)$. Let φ be a formula not in $L(T,P)$. Then there exists a* countable *FCP tree T' and a general space P' over T' such that $L \subseteq L(T',P')$ and $\varphi \notin L(T',P')$.*

Proof. Consider any such L, (T,P) and φ. As $\varphi \notin L(T,P)$ there exists a valuation v on P and a $w \in W$ such that $w \notin v(\varphi)$. We inductively define a sequence $\{V_n\}_{n\in\omega}$ of subsets of W. We define $V_0 = \{w\}$, and given V_n define V_{n+1} as follows:

For each normal modal formula ψ and each $u \in V_n$, if $u \in v(\Diamond\psi)$, there exists by definition some $u' \in W$ with $u < u'$ and $u' \in v(\psi)$. For each such pair (ψ, u) we select one such u' and let V_{n+1} be the set of the u' selected in this manner.

We take $T' = (W', R')$, where $W' = \bigcup_{n\in\omega} V_n$ and R' is the order of T restricted on W'. We claim that T' is a countable FCP tree.

First of, note that there exist only countably many normal modal formulas. Thus, V_n is at most countable for each n, so that indeed W' is at most countable as the countable union of countable sets. We prove that T' is indeed a tree. We see that w is the root of T', and, as T was a tree and R' is the restriction of R on W', we find that the set $\{b \in W' \mid b < a\}$ is linear for each $a \in T'$. We conclude that T' is also a tree. Finally, consider any linear $A \subseteq W'$. As A is also some linear subset in T and T enjoys the FCP, A must be finite. We deduce that all linear subsets in T' are finite so that T' enjoys the FCP.

We now define a valuation μ on T' as $\mu(p) = v(p) \cap W'$ for all propositional letters p, where v was as chosen before. We will prove that $\mu(\psi) = v(\psi) \cap W'$ for all modal formulas ψ. We prove this by induction on the complexity for formulas. The only nontrivial step is for formulas of the form $\Diamond\psi$, assuming the statement holds for ψ:

Consider any $x \in W'$ with $x \in \mu(\Diamond\psi)$. By definition there exists some $y \in W'$ so that $y \in \mu(\psi)$ and xRy. Thus, by assumption, $y \in \nu(\psi) \cap W'$ and xRy. But then $x \in v(\Diamond\psi)$ as xRy, and, as $x \in W'$, we conclude that $\mu(\Diamond\psi) \subseteq v(\Diamond\psi) \cap W'$.

Conversely, consider any $x \in v(\Diamond\psi) \cap W'$. As $x \in W'$, there exists some n such that $x \in V_n$. As $x \in V_n$ and $x \in v(\Diamond\psi)$, there exists by construction some $y \in V_{n+1}$ so that $y \in v(\psi)$ and xRy. But then $y \in W'$, so that $y \in \nu(\psi) \cap W'$ and xRy. By assumption, we see that $y \in \mu(\psi)$ and xRy, so that $x \in \mu(\Diamond\psi)$. Indeed, we deduce that $v(\Diamond\psi) \cap W' \subseteq \mu(\Diamond\psi)$. All in all, we see that $\mu(\Diamond\psi) = v(\Diamond\psi) \cap W'$.

By induction, we have shown that $\mu(\psi) = v(\psi) \cap W'$ for all modal formulas ψ. We obtain that $w \notin \mu(\varphi)$. We now define a general space P' on T' as $P' = \{\mu(\psi) \mid \psi \in \Sigma\}$. Here, Σ denotes the set of all modal formulas. Note that P' is trivially a general space on T'. Also note that μ is by definition a valuation over P', so that $\varphi \notin L_d(T', P')$ as $w \notin \mu(\varphi)$.

We will now prove that $L \subseteq L(T', P')$. Consider any valuation λ on P' and any logical formula $\chi(p_1, ..., p_n) \in L$. We will prove that $\lambda(\chi) = W'$. As each $\lambda(p_i)$ is an element of P', there exist logical formulas ψ_i such that $\lambda(p_i) = \mu(\psi_i)$ for all i. Note now that $\lambda(\chi(p_1, ..., p_n)) = \mu(\chi(\psi_1, ..., \psi_n))$. As $\chi(p_1, ..., p_n) \in L$, we deduce via the substitution rule that $\chi(\psi_1, ..., \psi_n) \in L$. Thus, $\nu(\chi(\psi_1, ..., \psi_n)) = W$, so that

$$\mu(\chi(\psi_1, ..., \psi_n)) = \nu(\chi(\psi_1, ..., \psi_n)) \cap W' = W'.$$

We see that $\lambda(\chi(p_1, ..., p_n)) = W'$. We deduce that each theorem of T is valid on (T', P') for all valuations on P'. We conclude that $L \subseteq L(T', P')$, and, as stated before, $\varphi \notin L(T', P')$. This finishes the proof. □

We are now almost ready to prove the main theorem of the thesis. We only need to show that we can relate any countable FCP-tree to an ordinal.

Lemma 5.8. *Let T be a countable FCP tree. Then there exists a countable limit ordinal α and an onto d-morphism $f : \alpha + 1 \to T$.*

Proof. The proof will mirror the proof of theorem 2.35, except that we prove by transfinite induction on the tree order. Consider any countable β and assume that the statement holds for all trees in H_γ for $\gamma < \beta$. Now, consider any $T \in H_\beta$. As T is countable, r has at most countably many direct successors.

In the case that r has infinitely many direct successors, we know that r has exactly ω direct successors. We label them by $\{a_i\}_{i\in\omega}$. We denote the subtree of T with root a_i by T_i, and note that by assumption there exist ordinals $\{\alpha_i\}_{i\in\omega}$ and onto morphisms $\{f_i\}_{i\in\omega}$ such that f_i is an onto d-morphism from α_i to T_i for each i. Now, we take $\{p_i\}_{i\in\omega}$ to be the sequence of prime numbers of ω, so that $p_0 = 2, p_1 = 3$, etc. We define $h : \omega \to \omega$ as follows:

$$h(0) = h(1) = 0$$
$$h(n) = \mu i[p_i \mid n] \text{ for all } n \geq 2$$

With $h(n) = \mu i[p_i \mid n]$ we mean that $h(n)$ equals the least i such that p_i divides n. Note that $h^{-1}(i)$ is an infinite subset of ω for all i, as $h(p_i^n) = i$ for all $n \in \omega$.

Now take $\alpha = \sum_{i\in\omega}(\alpha_{h(i)} + 1)$, which is a limit ordinal. Note that it is countable as the countable sum of countable ordinal numbers. We claim that there exists an onto d-morphism $f : \alpha + 1 \to T$.

Just as in lemma 2.35, we can view T as the union $\{r\} \cup \bigcup_{i\in\omega} T_i$, where $U \subseteq T$ is open if and if only $U = T$, or U does not contain r and $U \cap T_i$ is open in T_i for all i. Likewise, by construction we may split $\alpha+1$ into $(\bigcup_{i\in\omega} X_i) \cup \{\alpha\}$, where each X_i is isomorphic to $\alpha_{h(i)}+1$. Note that each X_i is clopen in $\alpha+1$. As there exist onto d-morphisms $\alpha_{h(i)}+1$ to $T_{h(i)}$, there also exist onto d-morphisms $g_i : X_i \to T_{h(i)}$. Now, define $f : \alpha + 1 \to T$ as follows:

$$f(\alpha) = r,$$
$$f(x) = g_i(x) \text{ for each } x \in X_i.$$

As each g_i is onto and $f(\alpha) = r$, we see that f is indeed onto. We will prove that it is a d-morphism:

Consider any open $U \subseteq T$. If $r \in U$, then $U = T$ so that $f^{-1}(U) = \alpha + 1$ is of course open in $\alpha + 1$. If $r \notin U$, then $U = \bigcup_{i\in\omega}(U \cap T_{h(i)})$ and each $U \cap T_{h(i)}$ is open in $T_{h(i)}$. As X_i is clopen in $\alpha + 1$, each $g_i^{-1}(U \cap T_{h(i)})$ is open in $\alpha + 1$, so that $f^{-1}(U) = \bigcup g_i^{-1}(U \cap T_{h(i)})$ is open as the union of open sets. We deduce that f is continuous.

Now, consider any open $U \subseteq \alpha + 1$. If $\alpha \notin U$, then $U = \bigcup_{i\in\omega}(U \cap X_i)$. Note that each $U \cap X_i$ is open in X_i, so that $g_i(U \cap X_i)$ is open in $T_{h(i)}$. We then see that $f(U) = \bigcup_{i\in\omega} g_i(U \cap X_i)$, which is open in T as the union of open sets.

If $\alpha \in U$, then there exists some $\beta < \alpha$ such that $[\beta, \alpha] \subseteq U$. Thus, it must contain some cofinite amount of X_i in their entirety. As $h^{-1}(i)$ is an infinite subset of ω for each i, we deduce that $f(U) = T$, which is of course open in T. We conclude that f is open.

We will prove that f is pointwise discrete. Of course, $f^{-1}(r) = \{\alpha\}$ which is discrete. Now consider any $t \neq r$, so that $t \in T_i$ for some i. We see that $f^{-1}(t) = \bigcup_{h(k)=i} g_k^{-1}(t)$. By assumption, $g_k^{-1}(t)$ is a discrete subset of X_k for each k, and each X_k is disjoint and clopen in $\alpha + 1$. We deduce that $f^{-1}(t)$ is itself discrete, so that f is pointwise discrete.

We see that f is continuous, open and pointwise discrete, so that f is an onto d-morphism.

In the case that r has only finitely many successors, the proof is almost exactly the same as in lemma 2.35, except that we apply some small changes exactly as in the proof for when r has infinitely many successors. Thus, we omit this part of the proof. □

We finally have all the pieces of the puzzle. The following theorem generalizes theorem 4.5.

Theorem 5.9 (Generalized Abashidze-Blass theorem)**.** *Let L be a* GL*-semicomplete logic. Then there exist a countable ordinal α and a general space P over α such that $L_d(\alpha, P) = L$. In particular, every Kripke complete* GL*-logic is complete with respect to some general space over a countable ordinal.*

Proof. Consider any GL-semicomplete logic L. As there are countably many modal formulas, we may enumerate the formulas *not* in L by $\{\varphi_i\}_{i\in\omega}$.

Take any $i \in \omega$. As $\varphi_i \notin L$ and L is GL-semicomplete, there exists a general Kripke frame (K, P_K) such that K is a GL-frame, $L \subseteq L(K, P_K)$ and φ_i is not a theorem of $L(K, P_K)$. By theorem 5.3 and corollary 4.2.1 there exist a disjoint union of FCP trees T and a general space P_T such that $L(T, P_T) = L(K, P_K)$. As $\varphi_i \notin L(T, P_T)$, theorem 4.4 gives us that there exists some FCP tree T_i with general space P_i such that $\varphi_i \notin L(T_i, P_i)$ and $L \subseteq L(T_i, P_i)$. Now, theorem 5.7 gives us that there exists a *countable* tree T_i' and a general space P_i' such that $L \subseteq L(T_i', P_i')$ and $\varphi \notin L(T_i', P_i')$.

For each i, pick exactly one such pair (T_i', P_i'). As $L \subseteq L(T_i', P_i')$ for all $\psi \in L$, we find that $L \subseteq \bigcap_{i\in\omega} L(T_i', P_i')$. But for each $\psi \notin L$, there exists by construction some i such that $\psi \notin L(T_i', P_i')$. We deduce that $L = \bigcap_{i\in\omega} L(T_i', P_i')$. Lemma 5.8 together with corollary 4.2.1 gives us that there exists a sequence $\{(\alpha_i, \mathcal{P}_i)\}$ such that each α_i is countable and so that $L_d(\alpha_i + 1, \mathcal{P}_i) = L(T_i', P_i')$ for all i. We conclude that $L = \bigcap_{i\in\omega} L_d(\alpha_i + 1, \mathcal{P}_i)$.

Now, define $\alpha = \sum_{i\in\omega}(\alpha_i + 1)$. As each α_i is countable, α is countable as the sum of countably many countable ordinal numbers. We write $\alpha = \bigcup_{i\in\omega} X_i$, where each X_i is isomorphic to $\alpha_i + 1$. Note that each X_i is clopen in α, so that α is topologically the disjoint union of all X_i. Note that the general spaces $\mathcal{P}_i$ over α_i can be lifted to general spaces $\mathcal{P}_i'$ over X_i.

We define a general space $\mathcal{P}$ over α as follows:

$$A \in \mathcal{P} \iff \forall_{i\in\omega} A \cap X_i \in \mathcal{P}_i'$$

As each X_i is pairwise disjoint and each $\mathcal{P}_i'$, we indeed find that $\mathcal{P}$ is a general space over α. Now, by theorem 4.4, we finally see that

$$L_d(\alpha, \mathcal{P}) = \bigcap_{i \in \omega} L_d(X_i, \mathcal{P}'_i) = \bigcap_{i \in \omega} L_d(\alpha_i + 1, \mathcal{P}_i) = L.$$

Finally we note that the same construction works for any Kripke complete GL-logic, as all Kripke complete GL-logics are GL-semicomplete. □

Note that the previous theorem is not actually a full generalization of the Abashidze-Blass theorem: For GL, we obtain that there exists a general space over some countable ordinal for which GL is complete. One deduces that there exists some countable ordinal for which GL is complete, but we do not obtain that this works for ω^ω and onwards. If we combine this result with theorem 4.5, we do have a strict generalization.

We have seen that we can extend the Abashidze-Blass theorem to all GL-semicanonical logics. We may now question if much more is possible. In the next chapter we will show that it is impossible to find a general space over an ordinal to obtain any specific GL-logic in general.

Chapter 6

Quasicanonicity & limits of scattered spaces

It is natural to ask if theorem 5.9 classifies all possible GL-logics. If we were working over $\mathsf{S4}$ we could conclude this fact, as $\mathsf{S4}$ is canonical so that it is certainly quasicanonical, as we will show in this chapter. It turns out to be impossible to find similar results for GL. In this chapter we first discuss further the notion of quasticanonicity and then prove that there exists a GL-logic that cannot be developed by any class of general scattered spaces, generalizing a theorem proven by Vooijs [20, Theorem 5.17].

6.1 Quasicanonicity

We recall the definition of semicompleteness and quasicanonicity as in definition 5.1. We will discuss the notion of quasicanonicity in relation to GL. First, some basic results.

Theorem 6.1. *Let L be a canonical logic. Then L is quasicanonical.*

Proof. Consider a canonical logic L and let L' be a normal extension of L. By theorem 2.15, $L' = L(C)$ where C is the class of descriptive frames of L'. Note that any descriptive frame of L' is also a descriptive frame of L. As L is canonical, we deduce that C consists of general frames (F, P) where F are themselves frames of L. As $L' = L(C)$, we deduce that L' is L-semicomplete. We have proven that any extension of L is L-semicanonical, so that L is quasicanonical. □

Theorem 6.2. *Let L be a quasicanonical logic. Then L is Kripke complete.*

Proof. Consider a quasicanonical logic L. We wish to prove that for each $\varphi \notin L$, there exists a Kripke frame $\mathbb{F}$ of L such that $\mathbb{F} \not\models \varphi$. Thus, take any $\varphi \notin L$. As L is an extension of L, we see that there exists a general frame $(\mathbb{F}, P)$ with $(\mathbb{F}, P) \not\models \varphi$ and $\mathbb{F} \models L$. But then we have in particular that $\mathbb{F} \not\models \varphi$ and $\mathbb{F} \models L$. We conclude that L is Kripke complete. □

We will show that the notion of canonicity and quasicanonicity differ. We first note the following well-known fact. We recall the definition of a canonical logic in definition 2.16.

Theorem 6.3. *Both* GL *and* GL.3 *are not canonical.*

Proof. Consider the frame $\mathbb{F} = (\omega + 1, R)$, where aRb if and only if $a = \omega$ or $a > b$. Note that $\omega + 1 = \{0, 1, 2, ..., \omega\}$. We see that this is $\omega + 1$ with the strict reverse order, except that $\omega R \omega$.

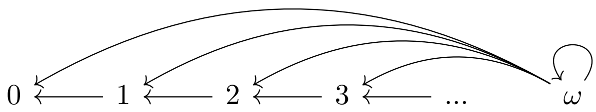

Figure 6.1: The frame $\mathbb{F}$. Note that all arrows are transitive.

As $\mathbb{F}$ contains a reflexive point it is not a GL-frame. We take the general frame P over $\mathbb{F}$ as the finite subsets of ω together with the cofinite subsets also containing ω. Note that $(\mathbb{F}, P)$ is a GL.3 frame, as it is isomorphic as a modal algebra to the least general frame over C_∞ as defined below theorem 3.20.

We claim that $(\mathbb{F}, P)$ is descriptive. We first prove that it is differentiated. Consider any x and y with $x \neq y$. We see that at least one of x or y is unequal to ω. Without the loss of generality, we assume that $x < \omega$. We take $A = \{x\}$. Then $A \in P$, $x \in A$ and $y \notin A$. We conclude that our frame is differentiated.

Now, consider any x and y such that $\neg(xRy)$. By construction of our frame, we must have that yRx and $x < \omega$. We take A to be $\{z \mid zRx\} \cup \{x\}$. As $x \in \omega$ we see that A is a cofinite set containing ω so that $A \in P$. Of course, $y \in P$. We conclude that $\Diamond A = \{z \mid zRx\}$, and as $x \in \omega$ we find that $\neg xRx$ so that $x \notin \Diamond A$. We deduce that $(\mathbb{F}, P)$ is tight.

Finally, consider any $\{A_i\}_{i \in I} \subseteq P$ with the finite intersection property. If all A_i are cofinite, then they all contain ω so that $\bigcap_{i \in I} A_i \neq \varnothing$. Now, assume at least one A_i is finite. We denote this specific index by j. We now see that $\bigcap_{i \in I} A_i \neq \varnothing$. If not, for each $k \in A_j$, there exists an A_{i_k} with $k \notin A_{i_k}$. But then $A_j \cap \bigcap_{k \in A_j} A_{i_k} = \varnothing$. As this is the intersection of finitely many sets in $\{A_i\}_{i \in I}$, we find that that set does not have the finite intersection property. By contraposition, we conclude that $(\mathbb{F}, P)$ is compact.

We see that $(\mathbb{F}, P)$ is a differentiated, tight and compact GL.3-frame (and, by extension, a GL-frame), but $\mathbb{F}$ is not a GL-frame, let alone a GL.3 frame. We conclude that both GL and GL.3 are not canonical. □

One can consider the possibility that GL is quasicanonical. Then indeed, for any GL-logic we could guarantee there to be a general space over some ordinal yielding the same logic. After an informal presentation of this thesis to fellow Master's students in March 2024 at the Radboud University, another Master's student working in modal logic managed to disprove this claim:

Theorem 6.4 ([20, Theorem 5.17])**.** GL *is not quasicanonical.*

For the proof of this statement, we refer to [20, Theorem 5.17], as it uses techniques specific to his work and is out of the scope of this thesis.

A naturally arising question is if quasicanonicity is even different from canonicity itself. It turns out that quasicanonicity is a strictly weaker property. The following is also attributed to Vooijs.

Theorem 6.5 ([20, Proposition 5.5])**.** $\mathsf{GL.3}$ *is quasicanonical.*

Proof. As $\mathsf{GL.3}$ is a finite width logic, it is Kripke complete, as are all its extensions by theorem 3.14. As $\mathsf{GL.3}$ and all its extensions are Kripke complete, they are most certainly $\mathsf{GL.3}$-semicomplete. We conclude that $\mathsf{GL.3}$ is quasicanonical. □

We see that $\mathsf{GL.3}$ is an example of a logic that is quasicanonical but not canonical, so that the notion of quasicanonicity and canonicity differ. Note that $\mathsf{GL.3}$ has the property that it is Kripke complete and all its extensions are Kripke complete. This is certainly a weaker property than being quasicanonical. We leave it open that these notions actually differ: It may the case that all quasicanonical logics are exactly those logics that are Kripke complete and have only Kripke complete extensions.

Going back to GL itself, we may ask ourselves: Are there non-Kripke complete GL-semicomplete logics? This statement is quite nontrivial, and we leave it as an open problem. We note the following equivalence:

Theorem 6.6. *There exist non-Kripke complete* GL*-semicomplete logics if and only if there exists a* GL*-frame* F *and a general frame* P *over* F *such that* $L(F, P) \neq L(C)$ *for all classes* C *of Kripke frames.*

Proof. From 'right to left': If such a pair (F, P) exists, then $L(F, P)$ is a GL-semicomplete logic that is not Kripke complete.

From 'left to right': We prove this via contraposition. Assume that for all GL-frames F and all general frames P over F there exists a class $C_{(F,P)}$ of Kripke frames such that $L(F, P) = L(C_{(F,P)})$. Consider any GL-semicomplete logic L. We see that there exists a class C' of general frames such that $L = L(C')$. We define C'' to be the union of all classes $C_{(F,P)}$ with $(F, P) \in C'$. Indeed:

$$\begin{aligned} L(C'') &= \bigcap_{(F,P)\in C'} L(C_{(F,P)}) \\ &= \bigcap_{(F,P)\in C'} L(F, P) \\ &= L(C') \\ &= L \end{aligned}$$

We see that L is Kripke complete with respect to the class C'' and conclude that all GL-semicomplete logics are Kripke complete in this case. □

Corollary 6.6.1. *All* GL*-semicomplete logics are Kripke complete if and only if for each* GL*-frame F and each general frame P over F, there exists a class C of Kripke frames such that $L(F, P) = L(C)$*

Proof. This is exactly the contraposition of the previous theorem. □

Now, we know that GL is not quasicanonical, so that theorem 5.9 does not classify all GL-logics. Do note, however, that the story need not end there. We noted that all GL Kripke-frames can be viewed as scattered spaces, but the converse need not hold. In general, a scattered space is not isomorphic to a Kripke frame with the upset topology, in fact, any ordinal number greater than ω is not of such form. As we will show later, we cannot extend theorem 5.9, but we could imagine there to be non-GL-semicomplete logics that are complete with respect to general scattered spaces. It may also be possible that any logic sound and complete with respect to a class of general scattered spaces is itself GL-semicomplete. This question, we also leave open.

6.2 A logic incomplete for general scattered spaces

We will now show that it is not possible to classify all possible GL-logics with the general scattered semantics. The following proof will often refer to [20, Chapter 5.4], as it is a 'topologification' of the proof that GL is not quasicanonical. In this proof, we use notation mirroring that of [20].

Theorem 6.7 (Takapui and Vooijs, 2024)**.** *There exists a* GL*-logic Λ such that there exists no class C of general scattered spaces such that $\Lambda = L_d(C)$.*

Proof. We take the logic Λ to be the logic of $\mathfrak{f}_1'$ as defined in [20, Lemma 5.4], which is a consistent NML extending GL. We take $\overline{\varphi_0}$ as in [20, page 55], and we will prove that $\neg\overline{\varphi_0}$ is true for every model over a scattered space for Λ. Note that $\neg\overline{\varphi_0} \notin \Lambda$, so that this indeed proves our theorem.

Let X be a scattered space and G be a general space over X such that $\Lambda \subseteq L_d(X, P)$. Assume that $\neg\overline{\varphi_0} \notin L_d(X, P)$, so that there exists a valuation v on (X, P) so that $v(\overline{\varphi_0})$ is nonempty. Now note as in the proof of [20, Lemma 5.16] that we have the following:

$$\{\overline{\varphi_n} \to \overline{\Diamond}\overline{\varphi_{n+1}}, \overline{\varphi_n} \to \bigwedge\{\neg\overline{\Diamond}\overline{\varphi_i} \mid i < n\} \mid n \in \omega\} \subseteq \Lambda$$

Here, $\overline{\Diamond}\psi$ is an abbreviation of $\Diamond\psi \vee \psi$. We may apply [20, Lemma 5.15], and we find:

$$\{\overline{\varphi_n} \to \Diamond\overline{\varphi_{n+1}}, \overline{\varphi_n} \to \bigwedge\{\neg\Diamond\overline{\varphi_i} \mid i < n\} \mid n \in \omega\} \subseteq \Lambda$$

From this, we conclude in particular that $\overline{\varphi_n} \to \Diamond\overline{\varphi_{n+1}} \in \Lambda$ for each n. As $v(\overline{\varphi_0})$ is nonempty we see that $v(\Diamond\overline{\varphi_1})$ is nonempty, so that $v(\overline{\varphi_1})$ is also nonempty. With induction, we find for each $n \in \omega$:

$$v(\overline{\varphi_n}) \neq \varnothing$$

Furthermore, we see that $v(\overline{\varphi_n}) \subseteq d(v(\overline{\varphi_{n+1}}))$ for each n. We take $A_n = v(\overline{\varphi_n})$ for each n, and now note that each A_n is nonempty and $A_n \subseteq d(A_{n+1})$ for all n. Now take $A = \bigcup_{n\in\omega} A_n$. Note that A is nonempty. We then see:

$$\begin{aligned} A &= \bigcup_{n\in\omega} A_n \\ &\subseteq \bigcup_{n\in\omega} d(A_{n+1}) \\ &\subseteq d(\bigcup_{n\in\omega} A_{n+1}) \\ &\subseteq d(\bigcup_{n\in\omega} A_n) \\ &= d(A) \end{aligned}$$

We deduce that A is a nonempty set with $A \subseteq d(A)$. But then $A \backslash d(A)$ is empty so that X is not a scattered space. We have found a contradiction and conclude that $v(\overline{\varphi_0}) = \varnothing$. We had made no assumptions on v, so that $(X, P) \vDash \neg\overline{\varphi_0}$. We conclude that there exists no class C of general scattered spaces such that $\Lambda \subseteq L_d(C)$ and $\neg\overline{\varphi_0} \notin L_d(C)$, so that indeed there is no class of general scattered spaces yielding the same logic as Λ. □

Even though scattered spaces are a natural class of topological spaces, we have shown that only restricting ourselves to general spaces over scattered spaces leaves us unable to obtain all possible GL-logics. It turns out that these problems fade when we consider all possible topological spaces, which we will discuss in the following chapter.

Chapter 7

Moving beyond scattered spaces

We have shown that there are limits to what is expressible with general d-semantics on scattered spaces. In this chapter, we show that by considering *all* topological spaces we can classify many more logics.

7.1 Capturing all GL-logics

We define a notion of topology that makes sense on all possible Kripke frames.

Definition 7.1. The *R-topology* or the *(generalized) upset topology* τ_R on a Kripke frame $F = (W, R)$ is the topology of the R-'upsets', so that a set $A \subseteq W$ is open if for each $x \in A$, if xRy then also $y \in A$.

One easily checks that indeed this defines a topology for any Kripke frame. We now slightly restrict our scope and focus on weakly transitive frames. As in theorem 2.9 we say that a frame K is weakly transitive if for all a, b and c:

$$aRb \wedge bRc \rightarrow a = c \vee aRc$$

As stated before, the weakly transitive frames are exactly the frames enjoying the wK4 axiom $(\Box p \wedge p) \rightarrow \Box\Box p$.

Lemma 7.2. *Let $F = (W, R)$ be a* wK4 *Kripke frame and let A be a subset of W. Then $d_{\tau_R}(A)$ consists of those points x such that there exists $y \in A$ with xRy and $x \neq y$. In particular, R is irreflexive if and only if $d_{\tau_R}(A) = \Diamond_R(A)$ for all $A \subseteq W$.*

Proof. Let F be any wK4-frame. Consider any $A \subseteq W$ and any $x \in W$. We will prove that $x \in d_{\tau_R}(A)$ if and only if there exists some $y \in A$ with xRy and $y \neq x$.

From 'left to right': Assume $x \in d_{\tau_R}(A)$. Let B be the set $\{z \mid xRz\} \cup \{x\}$. As F is a wK4-frame, this is indeed an R-upset so that B is an open set containing x. As $x \in d_{\tau_R}(A)$ there exists some $y \in A \cap B$ with $x \neq y$. Note that in that case xRy, so that we indeed obtain that there exists some $y \in A$ with xRy and $y \neq x$.

From 'right to left': Assume there exists some $y \in A$ with xRy and $x \neq y$. Now consider any $U \in \tau_R$ with $x \in U$. As U is an R-upset, we know that $y \in U$. As $y \in A$ and $y \neq x$, we indeed see that $x \in d_{\tau_R}(A)$.

We have the following:

$$x \in d_{\tau_R}(A) \iff \exists_{y \in A} xRy \wedge x \neq y$$
$$x \in \Diamond_R(A) \iff \exists_{y \in A} xRy$$

The first line is exactly the first part of the theorem. Furthermore, we see that $\Diamond_R(A) = d_{\tau_R}(A)$ for all A if R is irreflexive, and we see that R is irreflexive if $\Diamond_R(\{x\}) = d_{\tau_R}(\{x\})$ for all $x \in W$. □

Theorem 7.3. *Let $F = (W, R)$ be a* wK4 *Kripke frame, and let $F' = (W, R')$ be the irreflexive interior of F, that is:*

$$xR'y \iff xRy \wedge x \neq y$$

Then $L_d(F, \tau_R) = L(F')$.

Proof. Note that, by the previous lemma: $L_d(F, \tau_R) = L_d(F, \tau_{R'})$ and $L_d(F, \tau_{R'}) = L_d(F')$. □

The following is a well-known property of wK4. The standard argument uses the fact that wK4 is a Sahlqvist logic, see [10, Definition 3.51 and Theorem 5.91]. However, we give a direct argument.

Theorem 7.4. wK4 *is canonical.*

Proof. We will prove by contradiction. Let (F, P) be a descriptive wK4 frame. In particular, (F, P) is refined. Assume F is not a wK4-frame, so that there exist x, y, z with xRy, yRz, $x \neq z$ and $\neg(xRz)$.

As $x \neq z$ and as P is differentiated, we know that there exists some $A \in P$ with $x \notin A$ and $z \in A$. Furthermore, as $\neg(xRz)$ and as P is tight, there exists some $B \in P$ with $z \in B$ and $x \notin \Diamond_R B$. It follows immediately that $x \notin \Diamond_R(A \cap B)$. Furthermore, as $x \notin A$, we see that $x \notin A \cap B$. We deduce that

$$x \notin \Diamond_R(A \cap B) \cup (A \cap B).$$

Now note that $z \in A \cap B$. As xRy and yRz, we see that $x \in \Diamond_R\Diamond_R(A \cap B)$. We see that $x \in \Diamond_R\Diamond_R(A \cap B)$ but $x \notin \Diamond_R(A \cap B) \cup (A \cap B)$. We deduce that

$$\Diamond_R\Diamond_R(A \cap B) \not\subseteq \Diamond_R(A \cap B) \cup (A \cap B),$$

so that wK4 does not hold over (P, F). Via contradiction, we see that F is a wK4-frame. We conclude that wK4 is canonical. □

Recall from theorem 2.15 that every NML is sound and complete with respect to its descriptive frames.

Lemma 7.5. *Let (F,P) be a general* GL*-frame. Then $d_{\tau_R}(A) = \Diamond_R A$ for all $A \in P$.*

Proof. Take any such frame (F,P). Note that GL is a K4 logic, so that it is certainly a wK4 logic. As (F,P) is descriptive, we know by the previous theorem that F is wK4. By lemma 7.2, we see that

$$d_{\tau_R}(A) = \{x \mid \exists_{y\in A} xRy \wedge x \neq y\}$$

and

$$\Diamond_R A = \{x \mid \exists_{y\in A} xRy\}$$

We see that $d_{\tau_R}(A) \subseteq \Diamond_R A$ for all $A \in P$. We will prove that the converse also holds.

Consider any $x \in \Diamond_R A$. Thus, there exists some $y \in A$ with xRy. Assume $x \notin d_{\tau_R}(A)$. Then, we see that $\{y \in A \mid xRy\} = \{x\}$. We note that $x \in \Diamond_R A$ so that there is no point $z \in A\backslash\Diamond_R A$ with xRz. But then $x \notin \Diamond_R(A\backslash\Diamond_R A)$, while $x \in \Diamond_R A$. We see that $\Diamond_R A \not\subseteq \Diamond_R(A\backslash\Diamond_R A)$. But this is impossible as (F,P) is a GL-frame and as $A \in P$. We conclude that $x \in d_{\tau_R}(A)$, so that indeed $\Diamond_R A = d_{\tau_R}(A)$. □

Do note that F need not be a GL-frame, as GL is not canonical. We see that these topological spaces are not necessarily scattered.

Theorem 7.6. *Let L be any normal modal logic extending* GL*. Then there exists a class C of general topological spaces such that $L = L_d(C)$.*

Proof. Consider any such logic L. As stated before, L is sound and complete with respect to its descriptive frames. We denote the class of descriptive L-frames by C. Given frame $(F,P) \in C$ with $F = (W,R)$, we can take the corresponding topological space $F' = (W,\tau_R)$. By the previous lemma, $d_{\tau_R}(A) = \Diamond_R A$ for any $A \in P$. We see that P is also a topological general space over F', and we deduce that $L(F,P) = L_d(F',P)$. By now taking $C' = \{(F',P) \mid (F,P) \in C\}$, we conclude that $L_d(C') = L(C) = L$. □

Indeed, we see that all GL-logics may be obtained via general d-semantics. It turns out that this result generalizes for all logics above wK4.

7.2 Capturing all wK4-logics.

Definition 7.7. Given Kripke frame $F = (W,R)$, we define the *irreflexified copy $F' = (W',R')$ of F* together with the *identification function* $\pi : W' \to W$ as follows:

1. Split W up into two disjoint sets of points W_r and W_i, with W_r containing all reflexive points according to R and W_i all irreflexive points.

2. Define $W' = W_i \sqcup (W_r \times \{0,1\})$, with $\sqcup$ denoting the disjoint union.

3. We define $\pi : W' \to W$ as $\pi(x) = x$ if $x \in W_i$ and $\pi(x, n) = x$ if $x \in W_r$ and $n \in \{0, 1\}$.
4. We define R' on W' as $xR'y$ if $\pi(x) \neq \pi(y)$ and $\pi(x)R\pi(y)$. For each $x \in W_r$, we in addition take $(x, 0)R'(x, 1)$ and $(x, 1)R'(x, 0)$.

Intuitively, F' is the frame in which each reflexive point of F is replaced with two irreflexive points that see each other, and π identifies each point in F' with its corresponding point in F.

Note that an equivalent construction has been considered in [15] (See also: [7] for a more accessible paper in English) to prove that wK4 is sound and complete with respect to d-semantics. We show some properties of these irreflexified copies.

Theorem 7.8. *Let $F = (W, R)$ be a Kripke frame, let $F' = (W', R')$ be its irreflexified copy and let π be its identification function. Then*

1. *F' is an irreflexive Kripke frame.*
2. *π is surjective, and $\pi(x)R\pi(y)$ for all $x, y \in W'$ with $xR'y$.*

Proof. For the first: Assume that F' contains a reflexive point. We split this in the case where this point is in W_r and the case where this point is in W_i. First, assume that there exists some $x \in W_r$ such that $(x, 0)R'(x, 0)$ or $(x, 1)R'(x, 1)$. But this is of course impossible: $\pi(x, 0) = \pi(x, 1) = x$, so we must have that $x \neq x$. Likewise, cannot have that $x \in W_i$, as then $xR'x$ only if $x \neq x$. Indeed, F' is irreflexive.

For the second: Surjectivity is trivial. Now, assume $xR'y$. If $\pi(x) \neq \pi(y)$, we indeed automatically obtain that $\pi(x)R\pi(y)$. Now, assume $\pi(x) = \pi(y)$. Then there exists some z such that $x = (z, 0)$ and $y = (z, 1)$, or $x = (z, 1)$ and $y = (z, 0)$. But then z must be a reflexive point in F. As $z = \pi(x) = \pi(y)$ we conclude that indeed $\pi(x)R\pi(y)$. □

Note that the following is already known and can be found in [7] and [15].

Lemma 7.9. *Let F be a* wK4 *frame. Then its irreflexified copy is also a* wK4 *frame.*

Proof. Let $F = (W, R)$ be a wK4 frame and let $F' = (W', R')$ be its irreflexified copy. Assume $xR'yR'z$. By theorem 7.8, we see that $\pi(x)R\pi(y)R\pi(z)$. As F is weakly transitive, we deduce that $\pi(x)R\pi(z)$ or $\pi(x) = \pi(z)$.

If $\pi(x) = \pi(z)$, we see that there exists some $a \in W$ such that $x = (a, i)$ and $z = (a, j)$ with $i \in \{0, 1\}$ and $j \in \{0, 1\}$. In any case, immediately it follows that $x = z$ or $xR'z$.

Now assume $\pi(x) \neq \pi(z)$. We must have that $\pi(x)R\pi(z)$ and $\pi(x) \neq \pi(z)$, so that $xR'z$. Indeed, F' is wK4.

Theorem 7.10. *Let F be a* wK4 *frame and let P be a general frame over F. Then there exists a general frame P' over the irreflexified copy F' of F such that $L(F, P) = L(F', P')$.*

Proof. Consider any such general frame (F, P). We define P' as follows:

$$P' = \{\pi^{-1}(A) \mid A \in P\}$$

We will first prove that $\pi^{-1}(\Diamond_R A) = \Diamond_{R'} \pi^{-1}(A)$ for each $A \in P$.

Consider any $x \in \pi^{-1}(\Diamond_R A)$, so that $\pi(x) \in \Diamond_R A$. We see that there exists some $y \in A$ with $\pi(x) R y$. Note that, as π is surjective, there are either one or two elements of $\pi^{-1}(A)$ with y as their image.

First, assume $\pi(x) \neq y$. Now take any element z such that $\pi(z) = y$. We see that $xR'z$ and $z \in \pi^{-1}(A)$, so that indeed $x \in \Diamond_{R'} \pi^{-1}(A)$.

Now assume $\pi(x) = y$. We see that yRy, so that $x = (y, i)$ with $i \in \{0, 1\}$. Now note that $\pi(y, 1 - i) = y \in A$, and that $xR'(y, 1 - i)$. We deduce that $x \in \Diamond_{R'} \pi^{-1}(A)$. We conclude $\pi^{-1}(\Diamond_R A) \subseteq \Diamond_{R'} \pi^{-1}(A)$.

Conversely, assume $x \in \Diamond_{R'} \pi^{-1}(A)$. Thus, there exists some $y \in \pi^{-1}(A)$ such that $xR'y$. Now note that $\pi(y) \in A$. By theorem 7.8 we also see that $\pi(x) R \pi(y)$, so that indeed $\pi(x) \in \Diamond_R A$. We conclude that $x \in \pi^{-1}(\Diamond_R A)$, and conclude that $\pi^{-1}(\Diamond_R A) = \Diamond_{R'} \pi^{-1}(A)$.

As π is surjective, we also see that

$$\pi^{-1}(W \backslash A) = W' \backslash \pi^{-1}(A)$$

and

$$\pi^{-1}(A \cup B) = \pi^{-1}(A) \cup \pi^{-1}(B).$$

Furthermore, π is a surjection from P' to P. This, together with the fact that $\pi^{-1}(\Diamond_R A) = \Diamond_{R'} \pi^{-1}(A)$ instantly gives us that P' is a general space over F', isomorphic to P as a modal algebra. As a result, we may immediately conclude that $L(F, P) = L(F', P')$ □

Remark. In the theorem above, it is essential that (F', P') is general frame, i.e., there is no guarantee that the logics of F and F' coincide.

Lemma 7.11. *Let L be a* wK4 *logic. Then there exists some class C of general Kripke frames such that $L = L(C)$ and F is irreflexive for each $(F, P) \in C$.*

Proof. Consider any wK4 logic L. As wK4 is canonical there exists some class C of general frames over wK4 frames such that $L = L(C)$. Now, take C' to be the class of general frames (F', P') for each $(F, P) \in C$, with F' the irreflexive copy of F and P' as defined in theorem 7.10. By the same theorem, we conclude that $L(C') = L(C) = L$. □

Theorem 7.12. *Let L be a* wK4*-logic. Then there exists some class C of general topological spaces so that $L_d(C) = L$.*

Proof. Consider any wK4 logic L. As wK4 is canonical there exists a class C of general frames over wK4-frames such that $L = L(C)$. By the previous lemma there exists a class C' of general frames so that F is irreflexive for each $(F, P) \in C$ and $L(C') = L(C) = L$.

Note that we can view F as a topological space with the upset topology for any $(F, P) \in C'$. Note that each such F is an irreflexive wK4 frame, so that (F, P)

is also a general topological frame by lemma 7.2. Futhermore, that theorem gives us that $L(F,P) = L_d(F,P)$. We conclude that $L_d(C') = L(C') = L$. □

We have shown that general d-semantics allows us to describe not only all GL-logics, but all wK4-logics as long as we do not restrict ourselves to scattered spaces. Note that this is, in a certain sense, a maximal result: We have for all topological spaces X and all $A \subseteq X$ that $d(d(A)) \subseteq d(A) \cup A$, so that we can only obtain extensions of wK4 by considering (general) d-semantics.

7.3 Characterizing topological Kripke frames

All tools we have discussed so far mainly deal with the fact that we may view Kripke frames as topological spaces. It is now natural to wonder if the reverse is possible. It may be that any logic complete with respect to topological spaces is complete with respect to Kripke frames. This section aims to provide a necessary and sufficient condition for when we can view a topological space as a Kripke frame, to understand better the structures topological spaces enjoy that Kripke frames lack.

Definition 7.13. We say that a topological space (X, τ) satisfies the *infinite distribution law* if, for all $\{A_i\}_{i \in I} \subseteq X$:

$$d(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} d(A_i)$$

We say that a topological space (X, τ) is *inherited from a Kripke frame* if there exists a relation R on X so that $\tau = \tau_R$.

Note that $d(A \cup B) = d(A) \cup d(B)$ for all topological spaces, so that d already distributes over all finite unions.

Theorem 7.14. *Let (X, τ) be a topological space that is inherited from a Kripke frame. Then X satisfies the infinite distribution law.*

Proof. Consider a topological space (X, τ) that is inherited from a Kripke frame, and let R be the relation on X such that $\tau = \tau_R$. Take R' to be the irreflexive interior of the transitive closure of R. Note that $\tau_{R'} = \tau_R = \tau$. By lemma 7.2, we deduce that $d = d_{\tau_{R'}} = \Diamond_{R'}$. Consider any $\{A_i\}_{i \in I} \subseteq X$. We now see:

$$\begin{aligned}
d(\bigcup_{i \in I} A_i) &= \Diamond_{R'} \bigcup_{i \in I} A_i \\
&= \{x \in X \mid \exists_i \exists_{y \in A_i} x R' y\} \\
&= \bigcup_{i \in I} \{x \in X \mid \exists_{y \in A_i} x R' y\} \\
&= \bigcup_{i \in I} \Diamond_{R'} A_i \\
&= \bigcup_{i \in I} d(A_i)
\end{aligned}$$

Indeed, (X, τ) satisfies the infinite distribution law. $\square$

It turns out that this 'infinite distribution law' exactly characterizes the spaces that are inherited from a Kripke frame.

Theorem 7.15. *Let (X, τ) be a topological space with the infinite distribution law. Then (X, τ) is inherited from a Kripke frame.*

Proof. Consider a topological space (X, τ) with the infinite distribution law. We define a relation R on X as follows:

$$xRy \iff x \in d(\{y\})$$

We claim that $\tau_R = \tau$. We will first prove that $\Diamond_R = d_\tau$. Consider any $A \subseteq X$. We find:

$$\begin{aligned} x \in d_\tau(A) &\iff x \in d_\tau(\bigcup_{y \in A} \{y\}) \\ &\iff x \in \bigcup_{y \in A} d_\tau(\{y\}) \\ &\iff \exists_{y \in A} x \in d_\tau(\{y\}) \\ &\iff \exists_{y \in A} xRy \\ &\iff x \in \Diamond_R A \end{aligned}$$

Now, we claim that R is irreflexive and weakly transitive. Consider any $x, y \in X$. We note the following:

$$\begin{aligned} x \in d(\{y\}) &\iff \forall_{U \in \tau} [x \in U \to \exists_{z \in U \cap \{y\}} [z \neq x]] \\ &\iff x \neq y \wedge \forall_{U \in \tau} x \in U \to y \in U \end{aligned}$$

In particular, we cannot have that $x \in d(\{x\})$ for any $x \in X$, so that R is irreflexive. We prove that it is weakly transitive:

Consider any $x, y, z \in X$ with xRy and yRz. We will prove that $x = z$ or xRz. Assume that $x \neq z$.Take any open U with $x \in U$. As xRy, we see that $y \in U$, and as yRz, we find that $z \in U$. Thus, for each open U, we deduce that $x \in U$ implies $z \in U$. Of course, it follows that either $x = z$ or xRz. Indeed, R is weakly transitive.

By lemma 7.2 we now see that $d_{\tau_R} = \Diamond_R = d_\tau$, so that $d_{\tau_R} = d_\tau$. Now note that $cl_{\tau'}(A) = A \cap d_{\tau'}(A)$ for any topology τ' on any topological space X' and for all $A \subseteq X'$, where $cl_{\tau'}$ denotes the closure operator. As $d_{\tau_R} = d_\tau$, we see that $cl_{\tau_R} = cl_\tau$. We deduce that (X, τ_R) has the same closed sets as (X, τ), so that they must have the same open sets. We conclude that $\tau_R = \tau$, so that indeed (X, τ) is inherited from a Kripke frame. $\square$

This shows us that many topological spaces cannot be inherited from Kripke frames, but it may still be possible that many topological spaces are the surjective image of Kripke frames. It turns out that we can also give a condition for when a space is the surjective image of some Kripke frame.

Theorem 7.16. *Let X be a topological space X with the infinite distribution law, let Y be any topological space and let $f : X \to Y$ be an onto d-morphism. Then Y satisfies the infinite distribution law.*

Proof. Consider any such X, Y and f and consider any $\{A_i\}_{i\in I} \subseteq Y$. Then by theorem 2.29:

$$\begin{aligned}
f^{-1}(d_Y(\bigcup_{i\in I} A_i)) &= d_X(f^{-1}(\bigcup_{i\in I} A_i)) \\
&= d_X(\bigcup_{i\in I} f^{-1}(A_i)) \\
&= \bigcup_{i\in I} d_X(f^{-1}(A_i)) \\
&= \bigcup_{i\in I} f^{-1}(d_Y(A_i)) \\
&= f^{-1}(\bigcup_{i\in I} d_Y(A_i))
\end{aligned}$$

We see that $f^{-1}(d_Y(\bigcup_{i\in I} A_i)) = f^{-1}(\bigcup_{i\in I} d_Y(A_i))$. As f is onto, we must have that $d_Y(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} d_Y(A_i)$. We conclude that Y satisfies the infinite distribution law. $\square$

Corollary 7.16.1. *Let $\alpha \geq \omega + 1$ be an ordinal equipped with the interval topology. Then there does not exist a Kripke frame $\mathbb{F} = (W, R)$ with an onto d-morphism $f : W \to \alpha$.*

Proof. All we have to do is note that α does not satisfy the infinite distribution law, as $d(\{n\}) = \varnothing$ for each $n \in \omega$, but $d(\bigcup_{n\in\omega}\{n\}) = d(\omega) = \{\omega\}$. $\square$

We have shown that there exist topological spaces that are not the surjective image of Kripke frames. We see that it may be possible for a logic to be sound and complete with respect to (general) scattered spaces but not with respect to (general frames over) GL-frames. We leave this as an open question.

Chapter 8

Conclusion and future work

In this thesis we defined the concept of a general topological frame for d-semantics, which is a topological frame with a restricted set of admissible sets akin to general Kripke semantics. We started by defining the least general space over a topological space. This allowed us to show that the logics of these least spaces are normal extensions of GL.3, and gave for each normal extension of GL.3 an ordinal whose least space is that logic. We then gave completeness results for general topological semantics for ordinal numbers, thereby extending the Abashidze-Blass theorem. We first showed that each logic enjoying the FMP is the logic of a general space over an ordinal $\alpha \leq \omega^\omega$. We then introduced the class of GL-semicomplete logic, a class extending the Kripke complete logics, and showed that each such logic is the logic of a general space over some countable ordinal. We finally showed that there exists a GL-logic that can not be obtained by only considering scattered spaces, and we showed that it is possible to obtain all wK4-logics if one considers general spaces over all topological spaces.

In the future, our aim is to better understand GL-semicomplete logics and the notion of quasicanonicity, with the main question being whether there exists a GL-logic complete with respect to general scattered spaces that is not GL-semicomplete. The existance of such a logic would show that scattered space semantics is strictly more general than Kripke semantics. Another aim is to consider these general semantics for polymodal logics and see which polymodal GL-logics can be encapsulated by general polytopological spaces, e.g., extensions of GLB or GLP.

Another direction for future research would be a deeper dive into least general spaces. We have seen that GL.3 is the logic of the least general spaces for GL. This mirrors the fact that GL.3 is the logic of the zero-generated universal Kripke frame of GL, see [13, page 275]. A natural question would be whether the logic of all least spaces coincides with the logic of the zero-generated universal frame for wK4. Will this logic be finitely axiomatizable? If so, what is its axiomatization? In general, one could study the spectrum of logics that are complete for least topological spaces. Can such logics be fully characterized? Note that these logics will all extend wK4. Of course, the notion of least spaces for polymodal logics would also be interesting to discuss, as these in general have much more structure and in turn have a much larger and more complicated set of admissible sets.

Bibliography

[1] M. A. Abashidze. Ordinal completeness of the Gödel-Löb modal system. *Intensional logics and the logical structure of theories*, pages 49–73, 1985. In Russian.

[2] J. Aguilera and G. Stepanov. Completeness of the GL.3 provability logic for the intersection of normal measures. 2024. Extended abstract for the conference of Topology, Algebra and Categories in Logic 2024.

[3] L.D. Beklemishev and D. Gabelaia. *Topological Interpretations of Provability Logic*, pages 257–290. Springer, 2014.

[4] L.D. Beklemishev and Y. Wang. Periodic frames, 2024. Preprint.

[5] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of Spatial Logics*, pages 217–298. Springer, 2007.

[6] G. Bezhanishvili, L. Esakia, and D. Gabelaia. Some results on modal axiomatization and definability for topological spaces. *Studia Logica*, 81:325–355, 2005.

[7] G. Bezhanishvili, L. Esakia, and D. Gabelaia. Spectral and T_0-spaces in d-semantics. In *Proceedings of the 8th International Tbilisi Conference on Logic, Language, and Computation*, pages 16—29, 2009.

[8] G Bezhanishvili, D Gabelaia, and J Lucero-Bryan. Topological completeness of logics above S4. *The Journal of Symbolic Logic*, 80(2):520–566, 2015.

[9] G. Bezhanishvili and P. J. Morandi. Scattered and hereditarily irresolvable spaces in modal logic. *Arch. Math. Log.*, 49(3):343–365, 2010.

[10] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.

[11] A. Blass. Infinitary combinatorics and modal logic. *The Journal of Symbolic Logic*, 55(2):761–778, 1990.

[12] G. S. Boolos. *The Logic of Provability.* Cambridge University Press, 1994.

[13] A. Chagrov. *Modal Logic.* Oxford University Press, 1997.

[14] L. Esakia. Diagonal constructions, Löb's formula and Cantor's scattered spaces. *Studies in logic and semantics*, pages 128–143, 1981. In Russian.

[15] L Esakia. Weak transitivity - a restitution. *Logical Investigations*, (8):244–255, 2001. In Russian.

[16] D. Fernández-Duque. The polytopologies of transfinite provability logic. *Archive for Mathematical Logic*, 53(3-4):385–431, 2014.

[17] T. Jech. *Set Theory: The Third Millennium Edition, revised and expanded.* Springer, 2007.

[18] J.C McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, pages 141–191, 1944.

[19] H. Simmons. Topological aspects of suitable theories. *Proceedings of the Edinburgh Mathematical Society*, 19:383–391, 1975.

[20] N.C. Vooijs. Modal degrees, canonical approximations and dynamic topological logic. Master's thesis, Radboud University Nijmegen, 2024.

[21] Y. Wang. General topological frames for polymodal provability logic. Master's thesis, University of Amsterdam, 2023.