Master's Thesis in Mathematics

Radboud University Nijmegen

Modal Degrees, Canonical Approximations and Dynamic Topological Logic

Investigations into completeness for modal logics

Author:
Niels C. Vooijs

Supervisors:
Wim Veldman
Nick Bezhanishvili
David Fernández-Duque

18th July 2024

Modal Degrees, Canonical Approximations and Dynamic Topological Logic

Master's Thesis in Mathematics

Modal Degrees, Canonical Approximations and Dynamic Topological Logic

Investigations into completeness for modal logics

Niels C. Vooijs

18th July 2024

Supervisors
Wim Veldman
Nick Bezhanishvili
David Fernández-Duque

Radboud University

Abstract

The overarching theme of this thesis is *completeness* in modal logic, to which our four main lines of research relate:

- degrees of completeness,
- quasi-canonicity,
- canonical approximations, and
- computable enumerability of dynamic topological logics.

We introduce degrees of completeness as a generalisation of Fine's degrees of incompleteness [19], and study degrees of pre-well-founded (WF) and converse pre-well-founded (CWF) frames. In particular, we show that there exist singleton and continuum sized CWF-frame degrees.

Our proof techniques for establishing the existence of these continuum sized degrees also turn out to have applications to the recently introduced notion of quasi-canonicity [42]. In particular we show that neither **GL** nor **Grz** is quasi-canonical, thus answering negatively a question posed by Takapui [41].

We also introduce a notion of approximations for logics, generalising the earlier notions of subframisations and stabilisations [4]. We study approximations in the complete lattice of canonical logics, and in particular compute the canonical approximations of **Grz.2** and **Grz.3**.

Finally we turn to dynamic topological logic. Using techniques developed by Konev et al. [28], we show that, under certain conditions on a class of CWF frames $\mathcal{F}$, the dynamic topological logic of dynamic frame structures over $\mathcal{F}$ is computably enumerable.

Preface

Before you lies the master's thesis 'Modal Degrees, Canonical Approximations and Dynamic Topological Logic', an investigation into four somewhat related topics in modal logic. It is the result of over a year of work, starting in Januar 2023 with subject orientation, and finishing now, mid July 2024. The thesis was supervised by Nick Bezhanishvili from the University of Amsterdam and co-supervised by David Fernández-Duque from the University of Barcelona and Wim Veldman from Radboud University.

During this period I have been asking more questions than answering ones, so many questions remain open. I was delighted when I had finally solved a problem, and sometimes desperate when everything I would try on a particular problem failed. Besides learning a lot about the many topics in modal logic we visited, more than are listed in this thesis in fact, I think I have got an impression of what it is like to do mathematical research, and I am grateful to be able to continue doing so.

None of this, however, would have been possible without the support of many people. First and foremost, this endeavor would not have been possible without my supervisors: Nick Bezhanishvili, David Fernández-Duque and Wim Veldman. In particular, I would like to express my deepest gratitude to Nick for our frequent discussions, his countless suggestions, comments and advice on this thesis, my TACL submission, and the slides for various of my presentations. In addition, I thank him for arousing my interest in modal logic in the first place.

I am extremely grateful to David, for the main supervision of what turned into Chapters 7 and 8, as well as his invaluable suggestions when we were investigating selection in the intuitionistic modal setting, a topic that did not make it into this final work. I would like to express my deepest appreciation to Wim, not only for the co-supervision of my thesis, but also for teaching, even after his retirement, so many highly interesting logic courses, which made specialising in mathematical logic possible for me at Radboud University.

Special thanks goes to Tommaso Moraschini for our discussions on degrees, and Guram Bezhanishvili for sharing with us his insights which led to Proposition 6.13. Many thanks go to Gilan Takapui, for discussing the notion of quasi-canonicity with me, and our collaboration on this topic, the results of which will be included in his master's thesis. I would like to thank Rodrigo Nicolau Almeida for his comments on a previous version of the proof which underlies Section 4.3, and Matías Menni for interesting suggestions regarding canonical approximations during the TACL conference, which I will surely investigate in the future.

I would like to acknowledge the organisers of TACL 2024 for organising this conference, and the programme committee for allowing me to present my work there. I would like to recognise my fellow students for their interesting contributions to the weekly master's thesis seminar I organised, and Radboud University for kindly providing us with a room to host this seminar series. Finally, I would like to mention my parents for their boundless support in demanding times.

Niels C. Vooijs
Heelsum, 18th July 2024

Contents

Chapter 1

Introduction

Modal logics are logics obtained by adding to classical proposition logic additional *modal* operators. With these modal operators, *modalities* such as necessity, obligation, ability and belief, as well as temporal aspects, can be expressed. As such, modal logics have applications ranging from linguistics and philosophy to mathematics and computer science. Applications within mathematics arise for example in provability logic, where modal operators are used to express provability and consistency, and intuitionistic propositional logic, which can be translated into a modal logic via the Gödel-McKinsey-Tarksi translation [34, Section 5].

In this thesis we study modal logics from a theoretical perspective. We investigate modal logics from several directions, all of which are related to *completeness*, mostly with respect to Kripke semantics. Four main lines of investigation can be distinguished in this thesis:

- degrees of completeness,
- quasi-canonicity,
- canonical approximations, and
- computable enumerability of dynamic topological logics.

We will give a short introduction to each of these topics in turn.

Degrees of completeness. It is well-known that there exist modal logics which are not Kripke complete [see e.g. 11, Section 6.4]. Fine [19] introduced *degrees of incompleteness* to measure this amount of incompleteness. Such a degree in essence groups together logics which cannot be distinguished between by the Kripke semantics.

Blok [8] characterised the cardinalities of these degrees of incompleteness. In fact only two cardinalities arise: 1 and continuum. This result is now known as Block's dichotomy theorem. However, what happens to the cardinalities when we restrict our attention to extensions of **K4**, the logic of transitive frames, is a major open problem [11, Problem 10.5].

Inspired by this open problem, G. Bezhanishvili, N. Bezhanishvili and Moraschini [5] introduced a variation on the notion of degrees, where they group logics together as soon they are not distinguished by *finite* Kripke frames. They show that for these degrees an analogue of the of the dichotomy theorem holds. However, when restricting to extensions of **K4**, they prove essentially the opposite result: any finite number as well as ω and continuum arise as the cardinality of such degree.

Inspired by this work, we introduce a general notion of degrees, which encompasses both Fine's original degrees of incompleteness as well as the finite frames variation. We study some general structural theory about degrees, and explore several new instances of the general notion of degrees. In particular we consider WF- and CWF-frame degrees; two instances of degrees which lie in-between the finite-frame degrees and Fine's degrees of incompleteness. Although we derive multiple results on cardinalities for these degrees, a full characterisation like in the finite frame case is still far off.

In addition to degrees for classes of frames, we also briefly consider degrees for classes of Kripke models, in particular WF-model degrees and CWF-model degrees.

Quasi-canonicity. Some of the techniques we develop while studying CWF-frame degrees surprisingly turn out to have applications in a different setting. Recall that a modal logic is called canonical when its canonical frames are frames of the logic.[1] Every canonical logic is Kripke complete, and as such canonicity is a major tool for proving Kripke completeness.

While studying topological d-semantics for modal logic, Takapui [42] introduced the notion of quasi-canonicity, a property *in-between* full canonicity and mere Kripke completeness, which turned out relevant for the applicability of one of his proof techniques. In particular, Takapui [41] posed the question of whether **GL** is quasi-canonical; a positive answer would improve his main result.

We show that quasi-canonicity lies *strictly* in-between canonicity and Kripke completeness, and answer the aforementioned question negatively. In particular we show that two of the most famous Kripke complete but non-canonical logics, namely **GL** and **Grz**, are not quasi-canonical either. Interestingly, the proofs for this follow readily from our work on CWF-frame degrees.

Canonical approximations. We also study canonicity from a different perspective. Recall that for a normal modal logic Λ, one can find a Kripke complete 'approximation' of it by taking the logic of the frames of Λ, or $\mathrm{Log}(\mathrm{Fr}(\Lambda))$ in symbols. This produces a

[1] A potentially different definition for canonicity that is sometimes used, is that only the canonical frame over countably many atomic propositions needs to be a frame of the logic. Whether these two definitions are equivalent is a major open problem [11, Problem 10.1].

least Kripke complete extension of Λ. A similar approach can produce least complete extensions for other semantics, e.g. topological semantics, as well.

However, the story does not end there. *Many* important classes of modal logics form complete lattices w.r.t. set-inclusion. Besides the class of logics that are complete for a certain semantics, like Kripke complete, these include the classes of subframe logics, stable logics, the logics axiomatisable by Sahlqvist formulas and the class of canonical logics. For any logic Λ and a complete lattice of modal logics, one can define an approximation of Λ from above or from below by taking the meet of its extensions in the lattice or the join of the logics of the lattice that it contains, respectively.

For the lattice of logics complete for a certain semantics, the approximation from above of a logic Λ is obtained by the procedure noted above, i.e. by taking the logic of the structures that validate Λ. In the setting of super-intuitionistic logics, G. Bezhanishvili, N. Bezhanishvili and Ilin [4] and Ilin [24] studied approximations for the lattices of subframe logics and stable logics. We explore approximations for the lattice of canonical logics. In particular we compute the canonical approximations of two extensions of **Grz**, namely **Grz.2** and **Grz.3**.

Computable enumerability of dynamic topological logics. As a final line of research we investigate computable enumerability for dynamic topological logics. Dynamic topological logic is a multi-modal logic that combines the usual unimodal logic with a linear temporal logic, giving it a total of three modal operators. This logic was introduced, in its current form, by Kremer and Mints [31], to study 'the confluence of three research areas: the topological semantics for S4, topological dynamics, and temporal logic.' [31]. While the name suggests a topological semantics, a Kripke semantics for it does exist.

Similar to unimodal logic, dynamic topological logics can be defined *semantically*, as the logic of a class of frames. However, contrary to unimodal logic, for many logics defined this way, no 'convenient' axiomatisations are known. As a result, it is not obvious whether these logics are computably enumerable.

Konev et al. [28] prove computable enumerability for a *fragment*, called DTL_1, of the dynamic topological logic of **S4**-frames. However, with some minor modifications, their techniques have a much wider applicability. We use these techniques to prove computable enumerability of the *full* dynamic topological logic of many classes of CWF-frames.

Although the final result is very computability theoretic, the proof is much more *semantic*. It mostly comprises a *completeness* result w.r.t. certain sequences of labelled trees. The computable enumerability of the logic immediately follows from this.

Outline. The thesis is organised as follows. Chapter 2 recalls the preliminaries required for the rest of the thesis, and introduces notations and conventions that are used thereafter. The reader familiar with modal logic can skip most of this chapter, and refer back to it as the need arises. The following chapters discuss the topics described above.

First, in Chapter 3 degrees of completeness are introduced, and their general theory is studied. The most important known results for degrees of incompleteness and finite-frame degrees, including the dichotomy theorem and the anti-dichotomy theorem, are stated. Towards the end of the chapter, straightforward results for our newly introduced WF-frame and -model degrees are derived. In Chapter 4 we continue our investigation into degrees, now focusing on CWF-frame, and, to a lesser extent, -model, degrees. Here we prove our main contributions regarding degrees of completeness, in particular the existence of infinitely many continuum sized CWF-model degrees.

In the short Chapter 5, we study quasi-canonicity. Our two main results in this chapter, namely that neither **GL** nor **Grz** is quasi-canonical, are proven using techniques developed in the previous chapter. Staying with canonicity, we study canonical approximations in Chapter 6. As our main results, we compute the canonical approximations of the logics **Grz.2** and **Grz.3**.

Chapter 7 introduces the syntax and semantics of dynamic topological logic, as well as the basics of computability theory and two famous theorems about trees, namely Kőnig's lemma and Kruskal's tree theorem. This chapter can be seen as setting the stage for the next chapter, and does not include novel material. With these preliminaries covered, in Chapter 8 we derive our computable enumerability result for certain dynamic topological logics, based on earlier work of Konev et al. [28].

Chapter 2

Preliminaries

In this chapter we give a brief overview of the basics of modal logic.

2.1 Introduction

While the reader is expected to be familiar with the basics of modal logic and Kripke semantics, for the sake of being self-contained, this chapter gives all the definitions and notational conventions that are used throughout this thesis. Basic knowledge about classical propositional logic, topology and ordinals is assumed. Kripke semantics, general frame semantics and algebraic semantics for modal logic are discussed. More specific topics, including depth, pre-well-foundedness, Fine-Rautenberg formulas and tree unravelling, are discussed towards the end of the chapter. For a properly motivated and more complete introduction to modal logic, we refer the reader to Blackburn, de Rijke and Venema [7]. For more information on any particular topic, we refer to Chagrov and Zakharyaschev [11].

2.2 Notations and Conventions

In this section we introduce basic mathematical notations and conventions.

Convention 2.1 (Ordinals)**.** We identify an ordinal number α, or ordinal for short, with the set of ordinals that are strictly smaller than α. The set of natural numbers, which is the smallest infinite ordinal, is denoted ω. We use the greek letters α, β, γ for variables over the ordinals, and λ for a variable over the limit ordinals.

Notation 2.2 (Tuple)**.** Let $n \in \omega$. We denote the n-ary tuple consisting of $a_0, \dots, a_{n-1}$ by $\langle a_0, \dots, a_{n-1} \rangle$.

Notation 2.3. Let X be a set. Then $\mathcal{P}(X)$ denotes the power set of X, i.e. the set of all subsets of X.

Regarding notation, we make a distinction between *functions* and *relations*.

Functions. We use the usual $f\colon A \to B$ notation for functions. When A and B are sets, this just indicates that f is a function (of sets). When A and B are structures, not just sets, the same notation indicates that f is a function on the underlying sets that respects this structure in some way. We always indicate the structure-preserving property that f satisfies.

For example, when $\mathfrak{X}$ and $\mathfrak{Y}$ are topological spaces on sets X and Y respectively, $f\colon X \to Y$ indicates that f is a function from X to Y. We indicate that it is continuus by calling it a continuus function $f\colon \mathfrak{X} \to \mathfrak{Y}$.

Notation 2.4 (Function)**.** Let A, B be sets. We write $f\colon A \to B$ when f is a (total) function from A to B. In this case A is called the *domain* of f and B the *co-domain*. We write $f\colon A \hookrightarrow B$ when f is injective and $f\colon A \twoheadrightarrow B$ when it is surjective.

Notation 2.5 (Function application)**.** Let $n \in \omega$, $A_0, \dots A_{n-1}, B$ be sets and $f\colon A_0 \times \dots \times A_{n-1} \to B$ a function. When $a_0 \in A_0, \dots, a_{n-1} \in A_{n-1}$, we write $f(a_0, \dots, a_{n-1})$ for $f(\langle a_0, \dots, a_{n-1}\rangle)$.

Definition 2.6 (Sequence)**.** Let α be an ordinal. An α-sequence is a function with domain α.

Notation 2.7 (Partial function)**.** Let A, B be sets. We write $f\colon A \rightharpoonup B$ when f is a partial function from A to B, i.e. a $f\colon A' \to B$ for some subset $A' \subseteq A$. We call A' the *domain* of f, and write $\mathrm{dom}(f)$ for it.

Relations.

Notation 2.8 (Relation application)**.** Let R be an n-ary relation. Then for any $a_0, \dots, a_{n-1}$, we write

$$R(a_0, \dots, a_{n-1})$$

for $\langle a_0, \dots, a_{n-1}\rangle \in R$.

Notation 2.9 (Partial relation application)**.** Let R be an n-ary relation. Then for any $a_0, \dots, a_k$ with $k < n$, we write

$$R(a_0, \dots, a_k) \coloneqq \{\langle a_{k+1}, \dots, a_n\rangle \mid R(a_0, \dots, a_k)\}.$$

Topology. We use the letters $\mathfrak{X}$ and $\mathfrak{Y}$ for topological spaces. For a topological space $\mathfrak{X}$, we write $\mathfrak{X}_{\mathrm{w}}$ for the underlying set of points. For a set $Y \subseteq \mathfrak{X}_{\mathrm{w}}$, we denote the topological interior of Y, i.e. the largest open set contained in Y, by $\mathsf{Int}(Y)$ and the topological closure of Y, i.e. the least closed set extending Y, by $\mathsf{Cl}(Y)$.

2.3 Modal Logic Syntax

Modal logic does not refer to a single logic, but rather a family of logics over a variety of languages. What all have in common, is that it extends classical propositional logic with additional *modal operators*. In Section 7.3 we introduce a particular modal logic with more than one modal operator, but for now we restrict to the standard unimodal case.

In standard unimodal logic we extend the language of classical propositional logic with a single unary operator $\Box$, called 'box'. This box expresses necessity, and behaves similar to the $\forall$ in first-order logic. It can be interpreted in various ways, for example proof theoretically, epistemically, deontically or temporally. That is, $\Box\varphi$ can be read as 'φ is provable', '(someone) knows/believes φ', 'φ is obligatory' or 'from now on φ always holds'.

A second modal operator, $\Diamond$, is used as a shorthand for $\neg\Box\neg$. It expresses possibility, and behaves similar to $\exists$ in first-order logic. The proof theoretic, epistemic, deontic and temporal interpretations of $\Diamond\varphi$ would be 'φ is consistent', '(someone) holds it possible that φ', 'φ is permitted' or 'φ will/might hold in the future'.

Formally, we define the language of unimodal logic as follows:

Definition 2.10. Let P be some set of atomic propositions. Then the language of unimodal logic is defined by the following BNF:

$$\varphi ::= \top \mid \bot \mid p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \Box\varphi,$$

where p ranges over P. We use the following shorthands:

$$\begin{aligned} \varphi_1 \vee \varphi_2 &:= \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\ \varphi_1 \to \varphi_2 &:= \neg\varphi_1 \vee \varphi_2 \\ \Diamond\varphi &:= \neg\Box\neg\varphi. \end{aligned}$$

We introduce another shorthand notation, the relevance of which will become clear at the end of Section 2.5.

Notation 2.11. Let φ be a modal formula. We write $\overline{\Box}\varphi := \varphi \wedge \Box\varphi$ and $\overline{\Diamond}\varphi := \varphi \vee \Diamond\varphi$.

Using these definitions $\overline{\Diamond}$ is again dual to $\overline{\Box}$, in the sense that $\overline{\Diamond}\varphi$ is 'logically equivalent' to $\neg\overline{\Box}\neg\varphi$. More generally, we define a translation of formulas $\varphi \mapsto \overline{\varphi}$ replacing every $\Box$ by $\overline{\Box}$. More formally, this is defined as follows.

Definition 2.12. Define a translation of modal formulas φ into modal formulas φ_{refl} by induction on the formula φ, by $p := p$ for p an atomic proposition or $\top$ or

$\bot$, and for all ψ_1, ψ_2,

$$\begin{aligned} \overline{\psi_1 \wedge \psi_2} &:= \overline{\psi_1} \wedge \overline{\psi_2}, \\ \overline{\neg\psi_1} &:= \neg\overline{\psi_1}, \\ \overline{\Box\psi_1} &:= \overline{\Box}\overline{\psi_1}. \end{aligned}$$

We denote the set of subformulas of a formula φ by $\mathrm{Sub}(\varphi)$.

Just like classical propositional logic can be defined as the set of formulas that are true classically, we define modal logics as sets of formulas. To call a set of formulas a logic, it has to satisfy some properties. As we already said, it needs to extend classical logic, and obviously needs to be closed under substitution, and, similar to classical logic the modus ponens rule. There are two extra requirements about $\Box$, to make it actually behave like a necessity operator. In addition, if $\Diamond$ is introduced as a primitive symbol in the language, instead of a shorthand in the meta-language, the formula $\Diamond p \leftrightarrow \neg\Box\neg p$ needs to be in the logic, for an atomic proposition p.

Definition 2.13 (Normal modal logic)**.** Let Λ be set of modal formulas over some fixed countably infinite set of atomic propositions, say the natural numbers ω. It is called a *normal modal logic* iff Λ extends classical propositional logic, contains the K-axiom

$$\Box(p \to q) \to (\Box p \to \Box q),$$

and is closed under

substitution: if $\varphi \in \Lambda$, $p_0, \dots, p_{n-1}$ are atomic propositions and $\psi_0, \dots, \psi_{n-1}$ are modal formulas, then the substitution

$$\varphi[p_0 := \psi_0, \dots, p_{n-1} := \psi_{n-1}] \in \Lambda,$$

modus ponens: if $\varphi \to \psi \in \Lambda$ and $\varphi \in \Lambda$ then $\psi \in \Lambda$, and

necessitation: if $\varphi \in \Lambda$ then $\Box\varphi \in \Lambda$.

In this thesis we will only be concerned with normal modal logics, and therefore call them just modal logics, or even logics, for brevity.

Remark 2.14. It is sometimes useful to consider logics over a different set of atomic propositions, for example a set with larger cardinality than ω. Suppose P is some other set meant to be used as atomic formulas, and Λ a modal logic as defined above. Then we can induce a 'logic' in the modal language over atomic propositions P by taking all possible substitutions of formulas in Λ with atomic propositions in P.

This induced set then also has all the properties from the definition of normal modal logics, and if P is infinite we can get the original logic Λ back by repeating the previous process, now substituting with atomic propositions in ω. For the rest of this work, we will implicitly perform these substitutions whenever necessary, and identify a logic and its substitutions. •

There exists a least set of formulas which forms a normal modal logic, which we call the *basic modal logic* **K**. The set of all modal formulas also forms a normal modal logic, called the inconsistent logic **Fm**. In general we use bold names for concrete modal logics, and Λ for a variable logic.

Definition 2.15. Denote by $\mathcal{U}$ the set of all normal modal logics. When $\Lambda \in \mathcal{U}$, define

$$\mathrm{NExt}(\Lambda) := \{\Lambda' \in \mathcal{U} \mid \Lambda \subseteq \Lambda'\}$$

to be the set of all normal modal logics extending Λ.

2.4 Modal Algebras

In this section we define Boolean algebras and modal algebras. These are algebras with operators for the connectives of classical and modal logic respectively. The operators for conjunction and disjunction arise from the order theoretic structure called a lattice.

Definition 2.16 (Lattice)**.** A *lattice* is a partial order[1] $\langle X, \leq \rangle$ such that for every $x, y \in X$ there exist a supremum $x \vee y$ of x and y called their *join* and an infimum $x \wedge y$ called their *meet*. It is called *bounded* iff it has a least and greatest element.

Definition 2.17 (Complete lattice)**.** A *complete lattice* is a partial order $\langle X, \leq \rangle$ such that whenever $Y \subseteq X$ is a set of elements of X, then it has a supremum $\bigvee Y$ called the *join* of Y and an infimum $\bigwedge Y$ called the *meet* of Y.

Note that every complete lattice is a bounded lattice, since the join of the empty set is a least element and the meet of the empty set is a greatest element of the lattice.

Lattices can also be described algebraically.

Definition 2.18 (Algebraic lattice)**.** An *algebraic lattice* is an algebraic structure $\langle A, \wedge, \vee \rangle$ where $\wedge$ and $\vee$ are associative and commutative binary operators on A such that for all $a, b \in A$,

$$\begin{aligned} a \wedge a &= a & \qquad a \vee a &= a, \\ a \wedge (a \vee b) &= a & \qquad a \vee (a \wedge b) &= a. \end{aligned}$$

An *algebraic bounded lattice* is an algebraic structure $\langle A, \wedge, \vee, \bot, \top \rangle$ such that $\langle A, \wedge, \vee \rangle$ is an algebraic lattice and 0 and 1 are elements of A such that for all $a \in A$,

$$a \wedge \top = a \qquad\qquad a \vee \bot = a.$$

[1] Partial orders are formally defined in Definition 2.39 in the next section.

Algebraic (bounded) lattices induce (bounded) lattices and visa-versa. Therefore, we will identify between the two.

Proposition 2.19. *Let* $\langle X, \leq \rangle$ *be a lattice then* $\langle X, \wedge, \vee \rangle$ *is an algebraic lattice. If* $\langle X, \leq \rangle$ *is a bounded lattice with least element* $\bot$ *and greatest element* $\top$ *then* $\langle X, \wedge, \vee, \bot, \top \rangle$ *is an algebraic bounded lattice.*

Proposition 2.20. *Let* $\langle A, \wedge, \vee \rangle$ *be an algebraic lattice, and define for* $a, b \in A$, $a \leq b$ *iff* $a \wedge b = a$. *Then* $\langle A, \leq \rangle$ *is a lattice with* $\wedge$ *giving the binary meet and* $\vee$ *the binary join. If moreover* $\langle A, \wedge, \vee, \bot, \top \rangle$ *is an algebraic bounded lattice then* $\langle A, \leq \rangle$ *is a bounded lattice with least element* $\bot$ *and greatest element* $\top$.

Writing out $\langle A, \wedge, \vee, \bot, \top \rangle$ all the time is rather verbose. Therefore, from now on we will just write $\mathfrak{A}$ for such algebraic structure, and use the default notations for the meet, join etc. When there is a need to differentiate between different algebras for example, we will indicate the algebra in the subscript of the operator, as in $\wedge_{\mathfrak{A}}$, $\vee_{\mathfrak{A}}$ etc. We write $a \in \mathfrak{A}$ for $a \in A$.

The conjunction and disjunction in classical (and also intuitionistic) logic still satisfy one property that meet and join lattices do not: distributivity.

Definition 2.21 (Distributive lattice)**.** An (algebraic) (bounded) lattice $\mathfrak{A}$ is called distributive iff for all $a, b, c \in \mathfrak{A}$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

It can be noted that a second distributivity law follows:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

To interpret classical logic we still need an operator for the negation. The resulting algebraic structure is called a *Boolean algebra*, named after George Boole, who laid the foundations for Boolean algebra in Boole [9].

Definition 2.22 (Boolean algebra)**.** A *Boolean algebra* is a distributive (algebraic) bounded lattice $\mathfrak{A}$ enriched with a unary operator $\neg$, called the complement operator, such that for all $a \in \mathfrak{A}$,

$$a \wedge \neg a = \bot \quad \text{and} \quad a \vee \neg a = \top.$$

Example 2.23 (Powerset algebra)**.** Let X be some set. Then $\langle \mathcal{P}(X), \subseteq \rangle$ forms a bounded lattice with as meet the intersection $\cap$ and as join the union $\cup$. The least element is the empty set $\varnothing$ and the greatest one X itself. Clearly distributivity holds. In fact, set complements give a complement operator $\neg a := X \setminus a$ on this bounded lattice. Hence $\langle \mathcal{P}(X), \cap, \cup, \varnothing, X, X \setminus - \rangle$ forms a Boolean algebra, called the *powerset algebra* of X.

To interpret modal logic, in addition we need a unary operator to interpret $\Box$. This leads to *modal algebras*.

Definition 2.24 (Modal algebra). A *modal algebra* is a Boolean algebra $\mathfrak{A}$ enriched with a unary operator $\Box$ such that $\Box\top = \top$ and for all $a, b \in \mathfrak{A}$, $\Box(a \wedge b) = \Box a \wedge \Box b$.

We can interpret modal formulas as elements of a modal algebra. This requires the atomic propositions to be mapped to elements for the algebra. Such mapping is called a *valuation*,

Definition 2.25 (Modal algebra model). Let P be a set of atomic propositions. A *modal algebra model* with atomic propositions P, or just a modal algebra model for short, is a pair $\langle\mathfrak{A}, \mathfrak{V}\rangle$ where $\mathfrak{A}$ is a modal algebra and $\mathfrak{V} \colon P \to \mathfrak{A}$ a function, called the *valuation*.

Remark 2.26. Even when we always work over a modal language with infinitely many atomic propositions, it is often useful to be able to consider models with only a finite subset of these propositions. Hence we use this convention that every model *carries* its own set of atomic propositions around. ◆

Interpretation of modal formulas is then done in the obvious way.

Definition 2.27 (Modal algebra model interpretation). Let $\langle\mathfrak{A}, \mathfrak{V}\rangle$ be a modal algebra model and φ a modal formula such that every atomic proposition in φ is an atomic proposition of the valuation $\mathfrak{V}$. We define $[\![\varphi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} \in \mathfrak{A}$ by induction on the formula φ:

- If $\varphi = p$ is an atomic proposition then $[\![\varphi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} := \mathfrak{V}(p)$.
- If $\varphi = \psi_1 \wedge \psi_2$ then $[\![\varphi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} := [\![\psi_1]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} \wedge_{\mathfrak{A}} [\![\psi_2]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle}$.
- If $\varphi = \neg\psi$ then $[\![\varphi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} := \neg_{\mathfrak{A}} [\![\psi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle}$.
- If $\varphi = \Diamond\psi$ then $[\![\varphi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle} := \Diamond_{\mathfrak{A}} [\![\psi]\!]_{\langle\mathfrak{A},\mathfrak{V}\rangle}$.

2.5 Kripke Frames

The main semantics for modal logics we consider in this thesis is *Kripke semantics* or *relational semantics*. This is based on relational structures called *Kripke frames* or just *frames* for short. It is essentially a different word for a *directed graph*, without finiteness restrictions.

Definition 2.28 (Kripke frame). A *Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a set and R a binary relation on W. We write $\mathfrak{F}_{\mathrm{w}}$ for W, which we call the set of *worlds* or the *domain* of $\mathfrak{F}$. Elements of $\mathfrak{F}_{\mathrm{w}}$ are called *points*, *worlds* or *states*.

We write Fr for the class of all Kripke frames.

Terminology 2.29. Let $\mathfrak{F} = \langle W, R\rangle$ be a frame, and $x, y \in W$ points. We say

- x *sees* y or x is *below* y iff $R(x, y)$, and
- x is *strictly-below* y iff x sees y and y does not see x.

Many notions come in pairs, one looking upward an one downward, for example *ascending* and *descending*, or upsets and downsets. To save us having to duplicate every definition, we introduce the *converse* of a frame, which is the frame where the relation is turned around.

Definition 2.30 (Converse frame)**.** The *converse* R^{op} of R is the relation such that $R^{\mathrm{op}}(x, y)$ iff $R(y, x)$. The *converse frame* of $\mathfrak{F} = \langle W, R\rangle$ is $\mathfrak{F}^{\mathrm{op}} := \langle W, R^{\mathrm{op}}\rangle$.

Convention 2.31. We will use the letters $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ for Kripke frames, W, X, Y, Z for sets of points, R, S for relations and w, x, y, z as variables for points. We use $a, b, c, ...$ for named points in concrete frames.

For the rest of this section let $\mathfrak{F} = \langle W, R\rangle$ be a frame, unless noted otherwise.

Points. We introduce some standard terminology for points and sets of points in a frame.

Definition 2.32 (Reflexive, irreflexive)**.** A point $w \in W$ is called *reflexive* iff $R(x, x)$, and *irreflexive* otherwise.

Definition 2.33 (Chain)**.** A set $X \subseteq W$ is called a *chain* iff for all $x, y \in W$, $x = y$, $R(x, y)$ or $R(y, x)$.

Definition 2.34 (Anti-chain)**.** A set $X \subseteq W$ is called an *anti-chain* iff for all $x, y \in X$, $\neg R(x, y)$.

Definition 2.35 (Cluster)**.** A set $C \subseteq W$ is called a *cluster* iff for all $x, y \in C$, $R(x, y)$ and $R(y, x)$, and C is maximal with this property. It is called a *proper cluster* iff $|C| \geq 2$, and a *degenerate cluster* iff C consists of a single irreflexive point.

Note that every point is contained in a unique cluster.

Definition 2.36 (Final)**.** A point $x \in W$ is called *final* iff it sees no points other than potentially itself.

Definition 2.37 (Maximal)**.** Let $X \subseteq W$. A point $x \in X$ is said to be *maximal* for X iff for every $y \in R(x) \cap X$, we have $R(y, x)$. It is said to be *minimal* for X iff it is maximal for X in the converse frame $\mathfrak{F}^{\mathrm{op}}$.

Definition 2.38 (Upset)**.** A set $X \subseteq W$ is called an *upset* iff for all $x \in X$ and $R(x) \subseteq X$. It is called a *downset* iff it is an upset in the converse frame. The least upset or downset containing x is called the upset or downset *generated by* x respectively.

Frame properties. The relation of a frame can have all kinds of properties, for example be *transitive* or *reflexive*. When the relation of a frame has one of these properties, the frame is also said to have this property. For example, a frame $\langle W, R\rangle$ is called transitive iff R is transitive. For the sake of completeness, we include definitions for all the properties that we will use.

Definition 2.39. A frame $\mathfrak{F} = \langle W, R\rangle$ is called

- *weakly-transitive* iff for all $x, y, z \in W$, if $R(x, y)$, $R(y, z)$ and $x \neq z$ then $R(x, z)$,
- *transitive* iff for all $x, y, z \in W$, if $R(x, y)$ and $R(y, z)$ then $R(x, z)$,
- *reflexive* iff every point of $\mathfrak{F}$ is reflexive,
- *irreflexive* iff every point of $\mathfrak{F}$ is irreflexive,
- a *preorder* iff it is both reflexive and transitive,
- *anti-symmetric* iff for all $x, y \in W$, if $R(x, y)$ and $R(y, x)$ then $x = y$,
- a *partial order* iff it is both a preorder and anti-symmetric,
- *upward linear* iff for all $x \in W$, $R(x)$ is a chain in $\mathfrak{F}$,
- *linear* iff the entire domain W is a chain in $\mathfrak{F}$,
- a *linear order* iff it is both a partial order and linear,
- *symmetric* iff for all $x, y \in W$, if $R(x, y)$ then $R(y, x)$,
- an *equivalence relation* iff it is both a preorder and symmetric,
- *confluent* iff for all $x, y, z \in W$, if $R(x, y)$ and $R(x, z)$ then there exists $w \in W$ such that $R(y, w)$ and $R(z, w)$, and
- *rooted* iff there exists $r \in W$, called a *root*, such that $R^*(r) = W$, where R^* is the reflexive transitive closure of R, as defined in Definition 2.40.

Note that a rooted weakly-transitive frame is upward linear iff it is linear.

When $\mathcal{F}$ is a class of frames we write $\mathcal{F}_{\text{rooted}}$ for the class of rooted elements of $\mathcal{F}$.

Statements about cardinality are inherited from the set of worlds instead. For example, a frame $\mathfrak{F}$ is called finite iff $\mathfrak{F}_{\text{w}}$ is finite.

Operations on frames. There are several straightforward transformations that can be applied to frames. The most common ones are closure operations and disjoint unions of multiple frames.

Definition 2.40 (Closures)**.**

- The *reflexive closure* $\overline{R}$ of R is the least reflexive relation extending R, i.e. $\overline{R} = R \cup \{\langle x, x\rangle \mid x \in W\}$.
- The *irreflexivisation* of R is the largest irreflexive relation that R extends, i.e. $R \setminus \{\langle x, x\rangle \mid x \in W\}$.
- The *transitive closure* R^+ of R is the least transitive relation extending R.
- The *reflexive transitive closure* R^* of R is the least reflexive and transitive relation extending R.

The reflexive closure, transitive closure or reflexive transitive closure of $\mathfrak{F}$ is $\overline{\mathfrak{F}} := \langle W, \overline{R}\rangle$, $\mathfrak{F}^+ := \langle W, R^+\rangle$ or $\mathfrak{F}^* := \langle W, R^*\rangle$ respectively.

Definition 2.41 (Subframe)**.** Let $X \subseteq W$. Then $\mathfrak{F}{\restriction}X := \langle X, R \cap X^2\rangle$ is called the *restriction* of $\mathfrak{F}$ to X, and $\mathfrak{F}{\restriction}X$ is called a *subframe* of $\mathfrak{F}$.

Definition 2.42 (Generated subframe)**.** A subframe $\mathfrak{F}{\restriction}X$ of $\mathfrak{F}$ is called a *generated subframe* iff X is an upset in $\mathfrak{F}$. When $x \in W$, the subframe of $\mathfrak{F}$ *generated by* x is $\mathfrak{F}{\restriction}R^*(x)$.

Definition 2.43 (Disjoint union)**.** Let $\mathcal{F}$ be a set of frames. Then the *disjoint union* of $\mathcal{F}$ is the frame $\langle Y, S\rangle$ where

$$Y := \{\langle X, x\rangle \mid \langle X, R\rangle \in \mathcal{F}, x \in X\}$$

is the disjoint union of the domains of the elements of $\mathcal{F}$, and

$$S := \{\langle\langle X, x_1\rangle, \langle X, x_2\rangle\rangle \mid \langle X, R\rangle \in \mathcal{F}, \langle x_1, x_2\rangle \in R\}.$$

Morphisms. As in any category, we define notions of morphisms, functions that preserve some of the frame structure. Let for now $\mathfrak{F} = \langle X, R\rangle$ and $\mathfrak{G} = \langle Y, S\rangle$ be frames.

Definition 2.44 (Monotone, frame morphism)**.** A function $f\colon X \to Y$ is called *monotone* or a *frame morphism* from $\mathfrak{F}$ to $\mathfrak{G}$ iff for all $x, y \in X$, if $R(x, y)$ then $S(f(x), f(y))$. In this case we write $f\colon \mathfrak{F} \to \mathfrak{G}$.

Definition 2.45 (Embedding)**.** A frame morphism $f\colon \mathfrak{F} \to \mathfrak{G}$ is called an *embedding* iff it is injective and for all $x, y \in X$, if $S(f(x), f(y))$ then $R(x, y)$. In this case we write $f\colon \mathfrak{F} \hookrightarrow \mathfrak{G}$.

Definition 2.46 (Isomorphism)**.** A frame morphism $f\colon \mathfrak{F} \to \mathfrak{G}$ is called a *frame isomorphism* iff there exists a frame morphism $g\colon \mathfrak{G} \to \mathfrak{F}$ that is an inverse of f, i.e. $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case $\mathfrak{F}$ and $\mathfrak{G}$ are called *isomorphic.*

Equivalently, f is a frame isomorphism iff it is a surjective embedding.

For sequences we introduce special terminology.

Definition 2.47. Let α be an ordinal and $f\colon \alpha \to X$ an α-sequence of points of $\mathfrak{F}$. Then f is called *ascending* (w.r.t. $\mathfrak{F}$) iff it is monotone from $\langle \alpha, < \rangle$ to $\mathfrak{F}$ and *strictly-ascending* iff in addition for every $\beta, \gamma \in \alpha$ with $\beta < \gamma$, $\neg R^*(f(\gamma), f(\beta))$. It is called *descending* (w.r.t. $\mathfrak{F}$) iff it is ascending w.r.t. $\mathfrak{F}^{\mathrm{op}}$, and *strictly-descending* iff it is strictly-ascending w.r.t. $\mathfrak{F}^{\mathrm{op}}$.

Kripke models and modal interpretation. Modal formulas can be interpreted on Kripke frames, but like in the algebra case we need a valuation for assigning meaning to atomic propositions. A frame together with a valuation is called a Kripke model. Interpretation of modal formulas is defined via an induced modal algebra.

Definition 2.48. Let $\mathfrak{F} = \langle X, R \rangle$ be a Kripke frame, and $Y \subseteq X$ a subset of its points. Define

$$\Box_R Y \coloneqq \{x \in X \mid R(x) \subseteq Y\} \quad \text{and} \quad \Diamond_R Y \coloneqq R^{\mathrm{op}}(Y).$$

Instead of subscript R we will also write subscript $\mathfrak{F}$, as in $\Box_{\mathfrak{F}}$.

Proposition 2.49. *Let $\mathfrak{F} = \langle X, R \rangle$ be a Kripke frame. Then the powerset algebra of X enriched with $\Box_R$ as the modal operator forms a modal algebra.*

This algebra is called the modal algebra *induced by* $\mathfrak{F}$, and denoted $\mathfrak{F}^{\#*}$. This notation is due to the fact that taking the induced algebra of a Kripke frame can be factored into two steps, as we will see in the next section.

Definition 2.50 (Kripke model)**.** Let $\mathfrak{F}$ be a Kripke frame and P be a set of atomic propositions. Let *Kripke model* on $\mathfrak{F}$ with atomic propositions P is a pair $\langle \mathfrak{F}, \mathfrak{V} \rangle$ where $\mathfrak{V}\colon P \to \mathcal{P}(\mathfrak{F}_{\mathrm{w}})$ is a function called the *valuation.* We write $\mathfrak{M}_{\mathrm{fr}} \coloneqq \mathfrak{F}$.

Note that a valuation $\mathfrak{V}$ on a Kripke frame is also a valuation on a modal algebra. Hence Kripke model $\langle \mathfrak{F}, \mathfrak{V} \rangle$ induces a modal algebra model $\langle \mathfrak{F}^{\#*}, \mathfrak{V} \rangle$. This is used to interpret modal formulas on a Kripke model.

Definition 2.51 (Kripke model interpretation)**.** Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a Kripke model and φ a modal formula such that every atomic proposition in φ is an atomic proposition of $\mathfrak{M}$. Define $[\![\varphi]\!]_{\mathfrak{M}} \coloneqq [\![\varphi]\!]_{\langle \mathfrak{F}^{\#*}, \mathfrak{V} \rangle}$.

Notation 2.52. We write

- $\mathfrak{M}, x \vDash \varphi$ iff $x \in [\![\varphi]\!]_{\mathfrak{M}}$,
- $\mathfrak{M} \vDash \varphi$ iff for all $x \in \mathfrak{F}_{\mathrm{w}}$, $\mathfrak{M}, x \vDash \varphi$,
- for a set of formulas Γ, $\mathfrak{M} \vDash \Gamma$ iff $\forall \varphi \in \Gamma.\ \mathfrak{M} \vDash \varphi$.
- $\mathfrak{F} \vDash \varphi$ iff for every Kripke model $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M} \vDash \varphi$, and
- for a set of formulas Γ, $\mathfrak{F} \vDash \Gamma$ iff for every $\forall \varphi \in \Gamma.\ \mathfrak{F} \vDash \varphi$.

We write $\nvDash$ instead of $\vDash$ for the negated statements.

Terminology 2.53. We say

- φ is *satisfied* in the point x of a model $\mathfrak{M}$ iff $\mathfrak{M}, x \vDash \varphi$,
- for a logic Λ, $\mathfrak{M}$ is a *model of* Λ, or a Λ-model for short, iff $\mathfrak{M} \vDash \Lambda$,
- φ is *satisfiable* on $\mathfrak{F}$ iff there exists a model $\mathfrak{M}$ on $\mathfrak{F}$ such that φ is satisfied in a point of $\mathfrak{M}$
- $\mathfrak{F}$ *validates* φ iff $\mathfrak{F} \vDash \varphi$, and
- for a logic Λ, $\mathfrak{F}$ is a *frame of* Λ, or a Λ-frame for short, iff $\mathfrak{F} \vDash \Lambda$.

Now the $\varphi \mapsto \overline{\varphi}$ translation from Definition 2.12 has the following meaning in Kripke semantics:

Proposition 2.54. *Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a Kripke model, and define $\overline{\mathfrak{M}} := \langle \overline{\mathfrak{F}}, \mathfrak{V} \rangle$. Then for any formula φ containing only atomic propositions of $\mathfrak{M}$, $[\![\overline{\varphi}]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\overline{\mathfrak{M}}}$. In particular $\mathfrak{F} \vDash \overline{\varphi}$ iff $\overline{\mathfrak{F}} \vDash \varphi$.*

Proof. The latter claim trivially follows from the former. We prove the former claim by induction on φ. The cases for atomic propositions, $\top$, $\bot$, conjunctions and negations are trivial.

For $\Box$, assume as induction hypothesis $[\![\overline{\varphi}]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\overline{\mathfrak{M}}}$. Then

$$[\![\overline{\Box\varphi}]\!]_{\mathfrak{M}} = [\![\overline{\Box}\overline{\varphi}]\!]_{\mathfrak{M}} = [\![\overline{\varphi}]\!]_{\mathfrak{M}} \cap [\![\Box\overline{\varphi}]\!]_{\mathfrak{M}} = [\![\overline{\varphi}]\!]_{\mathfrak{M}} \cap \Box_{\mathfrak{F}}[\![\overline{\varphi}]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\overline{\mathfrak{M}}} \cap \Box_{\mathfrak{F}}[\![\varphi]\!]_{\overline{\mathfrak{M}}},$$

where we use the induction hypothesis in the final step.

Now we need to use the definition of $\Box_{\overline{\mathfrak{F}}}$. Note that $\Box_{\overline{\mathfrak{F}}}Y$ for some set Y is the set of points x such that all successors in $\overline{\mathfrak{F}}$ are in Y. Note that the successors of x in $\overline{\mathfrak{F}}$ are precisely the successor of it in $\mathfrak{F}$ and x itself. Hence $\Box_{\overline{\mathfrak{F}}}Y = Y \cap \Box_{\mathfrak{F}}Y$. Using this we see

$$[\![\varphi]\!]_{\overline{\mathfrak{M}}} \cap \Box_{\mathfrak{F}}[\![\varphi]\!]_{\overline{\mathfrak{M}}} = \Box_{\overline{\mathfrak{F}}}[\![\varphi]\!]_{\overline{\mathfrak{M}}} = [\![\Box\varphi]\!]_{\overline{\mathfrak{M}}}.$$

$\square$

2.6 General Frames and Topology

In this section we introduce general frames which in a sense generalise Kripke frames and make some remarks on the relation between general frames and topology. General frames are essentially just Kripke frames where we restrict which valuations we allow on the frame. In particular, only valuations where each atomic proposition is valuated to a so called *admissible* subset of the frame are allowed.

Definition 2.55 (Family of admissibles)**.** Let $\mathfrak{F} = \langle W, R \rangle$ be a frame. Then $A \subseteq \mathcal{P}(W)$ is said to be a *family of admissibles* iff for all $a, a' \in A$,

- $\varnothing \in A$ and $W \in A$,
- $W \setminus a \in A$,
- $a \cap a' \in A$, and
- $\Diamond_R a \in A$.

Definition 2.56 (General frame)**.** Let $\mathfrak{F}$ be a frame and A a family of admissibles for it. Then $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ is called a *general frame.* A set $X \subseteq \mathfrak{F}_{\mathrm{w}}$ is called *admissible* in iff $X \in A$. We write $\mathfrak{f}_{\#} := \mathfrak{F}$.

We call a Kripke model $\langle \mathfrak{F}, \mathfrak{V} \rangle$ a *model of* a general frame $\langle \mathfrak{F}, A \rangle$ iff for every atomic proposition p of $\mathfrak{V}$, $\mathfrak{V}(p) \in A$. Similar to Kripke frames, we write $\mathfrak{f} \vDash \varphi$ for a general frame $\mathfrak{f}$ iff every model $\mathfrak{M}$ of $\mathfrak{f}$, $\mathfrak{M} \vDash \varphi$. Every Kripke frame and Kripke model induces a general frame, and similar to Kripke frames, every general frame induces a modal algebra.

Example 2.57 (Frame induced general frame)**.** Let $\mathfrak{F}$ be a Kripke frame. Then $\langle \mathfrak{F}, \mathcal{P}(\mathfrak{F}_{\mathrm{w}}) \rangle$ forms a general frame, denoted $\mathfrak{F}^{\#}$. Clearly, every Kripke model on $\mathfrak{F}$ is a model of $\mathfrak{F}^{\#}$.

This already explains half of the notion $\mathfrak{F}^{\#*}$ for the induced modal algebra of a Kripke frame: we first induce a general frame from the Kripke frame. The other half of the notion denotes inducing a modal algebra from this general frame.

Proposition 2.58. *Let $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ be a general frame. Then*

$$\langle A, \cap, \cup, \varnothing, \mathfrak{F}_{\mathrm{w}}, \mathfrak{F}_{\mathrm{w}} \setminus -, \Box_{\mathfrak{F}} \rangle$$

forms a modal algebra.

This algebra is called the algebra induced by $\mathfrak{f}$ and is denoted by $\mathfrak{f}^*$. Whenever $\mathfrak{f}$ is a general frame on $\mathfrak{F}$ and $\langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model of $\mathfrak{f}$, then $[\![-]\!]_{\langle \mathfrak{f}^*, \mathfrak{V} \rangle}$ and $[\![-]\!]_{\langle \mathfrak{F}^{\#*}, \mathfrak{V} \rangle}$ are identical.

For the model induced general frame we first need the notion of *generation* of general frames.

Definition 2.59 (Generation of general frames)**.** Let $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ be a general frame and $G \subseteq A$. Then $\mathfrak{f}$ is said to be *generated* by G iff A is the least set extending G such that A is a family of admissible sets on $\mathfrak{F}$.

Definition 2.60 (Model induced general frame)**.** Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a Kripke model. Define $\mathfrak{M}_{\mathrm{g}}$ to be the general frame on $\mathfrak{F}$ generated by the co-domain of $\mathfrak{V}$.

Proposition 2.61. *For a Kripke model $\mathfrak{M}$, $\mathfrak{M}$ is a model of $\mathfrak{M}_{\mathrm{g}}$ and $\mathfrak{M}_{\mathrm{g}}$ has as admissible sets precisely all sets $[\![\varphi]\!]_{\mathfrak{M}}$ for modal formulas φ. Moreover $\mathfrak{M}_{\mathrm{g}} \vDash \varphi$ iff $\mathfrak{M} \vDash \varphi'$ for some substitution φ' of φ.*

Important in the study of general frames is the connection with topology.

Definition 2.62 (Topology of a general frame)**.** Let $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ be a general frame. We define the *topology of* $\mathfrak{f}$ to be the topology generated by A.

Definition 2.63. Let $\mathfrak{f} = \langle W, R, A \rangle$ be a general frame. It is called

- *differentiated* iff the topology is totally separated, i.e. iff any distinct pair of points is separated by a clopen,
- *tight* iff for all $x, y \in W$ with $\neg R(x, y)$, there exists $a \in A$ with $y \in a$ and $x \notin \Diamond_R a$,
- *compact* iff the topology is compact,
- *discrete* iff the topology is the discrete topology,
- *refined* iff it is differentiated and tight, and
- *descriptive* iff it is refined and compact.

Proposition 2.64. *Let $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ be a general frame. Then A is a basis (in the topological sense).*

Proposition 2.65. *Let $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ be a compact frame. Then A is precisely the set of clopens of $\mathfrak{f}$.*

Proof. Let C denote the set of clopens in the topology generated by A. Since A generates the topology and is closed under complements, $A \subseteq C$.

Let $a \in C$. Since a is closed, the subspace of a is compact again. By the previous proposition A is a basis for the topology. Hence, as a is open, there exists an open cover $B \subseteq A$ for a. By compactness there exists a finite subcover $B' \subseteq B$ for a. Therefore $a = \bigcup B'$, which is a finite union of elements of A, hence in A. □

2.7 Bisimulation and p-Morphisms

In this section we discuss an important tool for working with Kripke frames: bisimulation. Arguably the most important special case of a bisimulation is a p-morphism, which is what the majority of this section is about.

Definition 2.66 (Bisimulation of frames)**.** Let $\mathfrak{F} = \langle X, R\rangle$ and $\mathfrak{G} = \langle Y, S\rangle$ be frames and $Z \subseteq X \times Y$ a relation. Then Z is called a *bisimulation* between $\mathfrak{F}$ and $\mathfrak{G}$ iff the following two conditions hold:

back condition: if $Z(x, y)$ and $S(y, y')$ then there exists x' such that $R(x, x')$ and $Z(x', y')$, and

forth condition: if $Z(x, y)$ and $R(x, x')$ then there exists y' such that $S(y, y')$ and $Z(x', y')$.

Definition 2.67 (Bisimulation of models)**.** Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kripke models and $Z \subseteq \mathfrak{M}_{\mathrm{w}} \times \mathfrak{N}_{\mathrm{w}}$ a relation. Then Z is called a *bisimulation* iff Z is a bisimulation between the underlying frames $\mathfrak{M}_{\mathrm{fr}}$ and $\mathfrak{N}_{\mathrm{fr}}$ and for all $\langle x, y\rangle \in Z$, x and y satisfy precisely the same atomic propositions.

We have already seen a special case of a bisimulation, namely the generated subframe.

Example 2.68 (Generated subframe)**.** Let $\mathfrak{F}$ be a frame and Y an upset in it. Then $\{\langle y, y\rangle \mid y \in Y\}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{F}{\restriction}Y$.

The importance of bisimulations is that they preserve the truth of modal formulas.

Proposition 2.69. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kripke models and $Z \subseteq \mathfrak{M}_{\mathrm{w}} \times \mathfrak{N}_{\mathrm{w}}$ a bisimulation between them. Let $\langle x, y\rangle \in Z$ and φ be a modal formula. Then $\mathfrak{M}, x \vDash \varphi$ iff $\mathfrak{N}, y \vDash \varphi$.*

Proof. A simple induction on the formula φ. The back and forth conditions precisely make the inductive step for $\Box$ go through. □

The rather obvious fact that whenever $\mathfrak{F}$ is a frame of a logic Λ, then so is every generated subframe of $\mathfrak{F}$, now easily follows from this proposition and the previous example. A similar fact holds for the other important special case of bisimulations: the p-morphisms.

Definition 2.70 (p-Morphism)**.** Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames and $f\colon \mathfrak{F}_{\mathrm{w}} \to \mathfrak{G}_{\mathrm{w}}$ a function. Then f is called a *p-morphism* from $\mathfrak{F}$ to $\mathfrak{G}$ iff (the graph of) f is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. If f is surjective then $\mathfrak{G}$ is called a *p-morphic image* of $\mathfrak{F}$. Similarly, when $\mathfrak{M}, \mathfrak{N}$ are Kripke models, a function $f\colon \mathfrak{M}_{\mathrm{w}} \to \mathfrak{N}_{\mathrm{w}}$ is called a *p-morphism* from $\mathfrak{M}$ to $\mathfrak{N}$ iff (the graph of) f is a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. If f is surjective then $\mathfrak{N}$ is called a *p-morphic image* of $\mathfrak{M}$.

For (graphs of) functions the forth condition simplifies precisely to monotonicity of the function. p-Morphisms of frames have the nice property that valuations can be pulled back along the p-morphism, making it reflect satisfiability.

Proposition 2.71. *Let $\mathfrak{F}, \mathfrak{G}$ be frames, $\mathfrak{N}$ a model on $\mathfrak{G}$ and $f\colon \mathfrak{F} \to \mathfrak{G}$ a surjective p-morphism. Then there exists a model $\mathfrak{M}$ on $\mathfrak{F}$ such that f is a p-morphism from $\mathfrak{M}$ to $\mathfrak{N}$. A formula φ is satisfied in a point of $\mathfrak{N}$ iff it is satisfied in a point of $\mathfrak{M}$. If Λ is a logic and $\mathfrak{F}$ a frame of Λ, then so is $\mathfrak{G}$.*

Proof. Say $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{V} \rangle$. Define $\mathfrak{M} := \langle \mathfrak{F}, f^{-1} \circ \mathfrak{V} \rangle$. Then a point x in $\mathfrak{M}$ satisfies, by definition, the same atomic formulas as $f(x)$ in $\mathfrak{N}$. Suppose φ is satisfied in a point x of $\mathfrak{M}$ or y of $\mathfrak{N}$. Then by Proposition 2.69 it is satisfied in $f(x)$ in $\mathfrak{N}$ or any f-preimage of y in $\mathfrak{M}$ respectively. The preservation of being a Λ-frame easily follows. □

The p-morphic images of a frame $\mathfrak{F}$, up to isomorphism, are induced by certain bisimulations between $\mathfrak{F}$ and $\mathfrak{F}$ itself.

Definition 2.72 (Bisimulation equivalence)**.** A *bisimulation equivalence* on $\mathfrak{F}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{F}$ which is also an equivalence relation.

Definition 2.73 (Quotient frame)**.** Let $\mathfrak{F} = \langle W, R \rangle$ be a frame and $\sim$ a bisimulation equivalence on it. Define $\mathfrak{F}/\!\sim \,:= \langle W/\!\sim, R' \rangle$ where $R'(X, Y)$ for equivalence classes X and Y of $\sim$ iff there exists $x \in X$ and $y \in Y$ such that $R(x, y)$. This is called the *quotient* of $\mathfrak{F}$ by $\sim$.

Proposition 2.74. *Let $\mathfrak{F}$ be a frame and $\sim$ be a bisimulation equivalence on $\mathfrak{F}$. Then the quotient map from $\mathfrak{F}$ to $\mathfrak{F}/\!\sim$, i.e. the map sending $x \in \mathfrak{F}_{\mathrm{w}}$ to its $\sim$ equivalence class, is a (surjective) p-morphism. In particular $\mathfrak{F}/\!\sim$ is a p-morphic image of $\mathfrak{F}$.*

In fact, any p-morphic image is induced, up to isomorphism, by a bisimulation equivalence.

A trivial but important example of a bisimulation equivalence is the following.

Example 2.75. Let $\mathfrak{F} = \langle X, R \rangle$ be a weakly-transitive frame, and define a binary relation $\sim$ on X by setting $x \sim y$ iff either $x = y$ or both $R(x, y)$ and $R(y, x)$. Then $\sim$ is a bisimulation equivalence, and the quotient $\mathfrak{F}/\!\sim$ is called the *skeleton*. Clearly $\mathfrak{F}/\!\sim$ is anti-symmetric.

p-Morphic images play an important role in a series of results that can be unified under the name *frame-based formulas* [6]. We only need the first result for modal logic in this space.

Theorem 2.76 ([18, Section 2])**.** *Let $\mathfrak{F}$ be a finite preorder. Then there exists a formula $\chi(\mathfrak{F})$ such that for any frame $\mathfrak{G}$, we have $\mathfrak{G} \not\models \chi(\mathfrak{F})$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.*

An intuitionistic analog of this theorem was first proven by Jankov [25], using algebraic methods. Later, Fine [18, Section 2] independently derived the result for modal logic, using frame theoretic techniques. Finally, Rautenberg [38, Proof of the Splitting Theorem] derived the same result for modal logic, using algebraic methods, while studying splittings of the lattice of normal modal logics. The formula $\chi(\mathfrak{F})$ is therefore called the *Fine-Rautenberg formula* or *Fine-Jankov formula* of $\mathfrak{F}$. Another proof of the theorem can be found in Chagrov and Zakharyaschev [11, Proposition 9.41].

2.8 Soundness and Completeness

In this section we recall the notions of soundness and completeness, and define Kripke completeness. We first introduce the following notation.

Notation 2.77. Let $\mathcal{S}$ be a class of frames, general frames or Kripke models. We write $\mathcal{S} \vDash \varphi$ iff for all $\mathfrak{X} \in \mathcal{S}$, we have $\mathfrak{X} \vDash \varphi$.

Definition 2.78 (Soundness)**.** A logic Λ is called *sound* w.r.t. a class of frames, general frames or Kripke models $\mathcal{S}$ iff for all formulas φ, $\varphi \in \Lambda$ implies $\mathcal{S} \vDash \varphi$.

There are two versions of *completeness*, a *weak* and a *strong* one. When left implicit, weak completeness is meant. We start with defining the weak completeness.

Definition 2.79 (Weak completeness)**.** A logic Λ is called *weakly complete*, or just complete for short, w.r.t. a class of frames, general frames or Kripke models $\mathcal{S}$ iff for all formulas φ, $\mathcal{S} \vDash \varphi$ implies $\varphi \in \Lambda$.

Definition 2.80 (Kripke completeness)**.** A logic Λ is called *Kripke complete* iff it is sound and complete w.r.t. some class of Kripke frames. We write Kripke for the set of all Kripke complete modal logics.

It should be noted that for a logic Λ, there exists a unique maximal class of Kripke frames w.r.t. which Λ is sound, namely the set

$$\mathrm{Fr}(\Lambda) := \{\mathfrak{F} \mid \mathfrak{F} \vDash \Lambda\} \tag{2.1}$$

of frames of Λ. Hence Λ is Kripke complete iff it is complete w.r.t. $\mathrm{Fr}(\Lambda)$. In Section 3.3 we will see a third equivalent formulation of Kripke completeness.

Before we turn to strong completeness, we introduce a stronger variant of Kripke completeness.

Definition 2.81 (Fmp, finite frame property)**.** A logic Λ is said to have the *finite model property*, or *fmp* for short, iff it is sound and complete w.r.t. some class of *finite* Kripke models. It is said to have the *finite frame property* iff it is sound and complete w.r.t. some class of finite Kripke frames.

Clearly, the finite frame property implies the finite frame property. However, perhaps surprisingly, the converse implication also holds, and both notions are equivalent [7, Theorem 3.28, 11, Theorem 8.47]. It has become standard to use the fmp terminology, and use the equivalence implicitly whenever necessary.

Proposition 2.82. *A modal logic Λ has the fmp iff it has the finite frame property.*

For strong completeness we need some additional definitions. Let us first note that a logic Λ is weakly complete w.r.t. a class of structures $\mathcal{S}$ iff for any formula φ, if $\neg\varphi \notin \Lambda$, then φ is satisfiable on some structure $\mathfrak{X} \in \mathcal{S}$. Now, $\neg\varphi \notin \Lambda$ means that Λ does not prove φ to be false, i.e. it means φ is consistent with Λ. We generalise this notion to sets of formulas.

Definition 2.83. Let Γ be a set of formulas and Λ a modal logic. Then Γ is called Λ-*consistent* for any $n \in \omega$ and any $\varphi_0, \dots, \varphi_{n-1} \in \Gamma$, their conjunction $\varphi_0 \wedge \dots \wedge \varphi_{n-1}$ is consistent, i.e.

$$\neg(\varphi_0 \wedge \dots \wedge \varphi_{n-1}) \notin \Lambda.$$

Definition 2.84 (Strong completeness)**.** Let P be a set meant to be used as atomic propositions. A logic Λ is called *P-strongly complete* w.r.t. a class of frames, general frames or Kripke models $\mathcal{S}$ iff whenever Γ is a Λ-consistent set of formulas over the atomic propositions P, then all formulas of Γ are satisfied in a single point of some model (with atomic proposition P) of $\mathcal{S}$. It is called strongly complete w.r.t. $\mathcal{S}$ it is κ-strongly complete w.r.t. $\mathcal{S}$ for all cardinals κ.

A logic is called *P-strongly Kripke complete* iff it is sound and P-strongly complete w.r.t. some class of Kripke frames, and strongly Kripke complete iff it is κ-strongly Kripke complete for all cardinals κ.

2.9 Canonicity

In this section we introduce the notion of canonicity of modal logics. Canonicity is a major tool for establishing Kripke completeness. It can be characterised in various ways, including algebraically and using canonical frames.

In order to define canonical frames, we need to introduce maximal consistent sets. Recall the definition of Λ-consistency from Definition 2.83. Then a set of formulas Γ is called *maximally Λ-consistent* iff it is Λ-consistent and maximal with this property. This is equivalent to the following definition, which we prefer since it decouples the *maximal consistency* from the logic Λ.

Definition 2.85 (Maximal consistent set)**.** Let P be a set of atomic propositions and Γ a set of formulas. Then Γ is called *maximally consistent* over atomic propositions P, or an MCS over P for short, iff for all formulas φ, ψ,

- $\top \in \Lambda$ and $\bot \notin \Lambda$,
- $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,
- $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$, and
- $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$.

If Λ is a modal logic and Γ contains (the substitution of) Λ (to atomic proposition P), then Γ is called a Λ-MCS.

Definition 2.86 (Canonical frame)**.** Let P be a set of atomic propositions and Λ a logic over these atomic propositions. Then the P*-canonical frame* of Λ is the frame $\mathbb{F}_P^\Lambda := \langle W, R\rangle$ where W is the set of Λ-MCSs and R is defined by

$$R(\Gamma, \Delta) \quad \text{iff} \quad \forall \Box\varphi \in \Gamma.\ \varphi \in \Delta.$$

Definition 2.87 (κ-Canonicity)**.** Let κ be a cardinal and Λ a logic. Then Λ is called κ*-canonical* iff the κ-canonical frame $\mathbb{F}_\kappa^\Lambda$ is a frame of Λ.

It is a well-known theorem that any modal logic is P-strongly complete w.r.t. its P-canonical frame [7, Theorem 4.22]. Hence any κ-canonical logic is κ-strongly Kripke complete.

Proposition 2.88. *Let Λ be a logic. Then the following are equivalent:*

(i) Λ *is κ-canonical for each cardinal κ,*

(ii) for every descriptive frame $\mathfrak{f}$ of Λ, the underlying Kripke frame $\mathfrak{f}_\#$ is a frame of Λ.

A logic is called *canonical* iff one of these equivalent properties hold.

2.10 Depth and Pre-well-foundedness

In this section we will define *depth* of points in a frame and introduce related notations. We will define it via transfinite induction. Using these, we define the notions of pre-well-foundedness and converse pre-well-foundedness.

Definition 2.89 (Points at depth α)**.** Let $\mathfrak{F} = \langle W, R\rangle$ be a frame and α an ordinal. We define a subset $W^{=\alpha} \subseteq W$ of points at *depth* α by induction on α. Let us also define

$$W^{<\alpha} := \bigcup \{W^{=\beta} \mid \beta < \alpha\}, \text{ and}$$
$$W^{\leq\alpha} := W^{=\alpha} \cup W^{<\alpha}.$$

For the inductive definition, let α be an ordinal, and assume as induction hypothesis that for all $\beta < \alpha$, $W^{=\beta}$ is already defined. Note that then also $W^{<\alpha}$ is defined. Now define

$$\begin{aligned} W^{=\alpha} := \{w \in W \mid\ & w \notin W^{<\alpha}, \\ & \forall x \in R(w)\ [\neg R^*(x,w) \implies x \in W^{<\alpha}], \\ & \forall \beta < \alpha\ \exists x \in R^*(w).\ x \in W^{=\beta}\}. \end{aligned}$$

We write $\mathfrak{F}^{<\alpha}$ for the subframe of $\mathfrak{F}$ on the points $W^{<\alpha}$ and analogously $\mathfrak{F}^{\leq\alpha}$ for the on the points $W^{\leq\alpha}$.

Remark 2.90. Note that these subsets are invariant under taking a transitive or reflexive closure of the frame $\mathfrak{F}$. For example $(\mathfrak{F}^*)^{<\alpha} = (\mathfrak{F}^{<\alpha})^*$. ♦

As a direction consequence of the definition, we get the following criterion for depth in the transitive case.

Proposition 2.91. *Let $\mathfrak{F}$ be a weakly-transitive frame, α an ordinal and $x \in \mathfrak{F}_{\mathrm{w}}$. Then $x \in \mathfrak{F}_{\mathrm{w}}^{\leq\alpha}$ iff for all successors y of x, either $y \in \mathfrak{F}_{\mathrm{w}}^{<\alpha}$ or x and y see each other.*

Definition 2.92 (Depth of a point)**.** Let $\mathfrak{F} = \langle W, R\rangle$ be a frame. Define a partial function $\mathrm{depth}_{\mathfrak{F}} : W \rightharpoonup \mathrm{Od}$ by setting $\mathrm{depth}_{\mathfrak{F}}(w) := \alpha$ where α is the smallest α such that w is in $\mathfrak{F}^{\leq\alpha}$, and $\mathrm{depth}_{\mathfrak{F}}(w)$ undefined when no such α exists.

Note that not every point has a depth. For example, any point from which there exists an infinite strictly-ascending sequence, has no depth. In fact, when assuming the axiom of dependent choice, having such sequence is equivalent to not having a depth.

Definition 2.93 (Deep points)**.** Let $\mathfrak{F} = \langle W, R\rangle$ be a frame and $w \in W$. We call w *deep* iff $\mathrm{depth}_{\mathfrak{F}}(w)$ is undefined. We write W^{deep} for the set of all deep points in $\mathfrak{F}$ and $\mathfrak{F}^{\mathrm{deep}}$ for its subframe, and call this the *deep part* of $\mathfrak{F}$.

Definition 2.94 (Upper points)**.** Let $\mathfrak{F} = \langle W, R\rangle$ be a frame. We write $W^{\mathrm{upper}} := \bigcup \{W^{\leq\alpha} \mid \mathrm{Od}(\alpha)\} = W \setminus W^{\mathrm{deep}}$ for the set of all non-deep points in $\mathfrak{F}$ and $\mathfrak{F}^{\mathrm{upper}}$ for its subframe, and call this the *upper part* of $\mathfrak{F}$.

With this theory of depth set up, we can introduce pre-well-foundedness and converse pre-well-foundedness in the following way.

Definition 2.95 (Cwf frame, wf frame)**.** A frame $\mathfrak{F}$ is called *conversely pre-well-founded*, or *cwf* for short, iff $\mathfrak{F} = \mathfrak{F}^{\mathrm{upper}}$, or equivalently $\mathfrak{F}^{\mathrm{deep}}$ is the empty frame. Analogously, a frame $\mathfrak{F}$ is called *pre-well-founded*, or *wf* for short, iff its converse frame $\mathfrak{F}^{\mathrm{op}}$ is cwf.

Remark 2.96. The terminology around *well-foundedness* (alternatively spelled *well-foundedness*) in the literature is a bit fuzzy. Arguably the most common definition requires the absence of any loops, so in particular implies irreflexivity. A version which allows reflexive points is also sometimes used. However, our pre-well-foundedness differs from both of these, in allowing even proper clusters to exist. Our pre-well-foundedness is in this sense like the well-foundedness condition of *prewellorders*. Formally, a transitive frame $\mathfrak{F}$ is pre-well-founded iff the irreflexivisation of the skeleton of $\mathfrak{F}$ is well-founded in the classic sense.

Our pre-well-foundedness differs in another way from the usual well-foundedness: it is technically also defined for frames that are not transitive (or weakly-transitive). By Remark 2.90 it is then equivalent to the pre-well-foundedness of the transitive closure. Combining these two features of pre-well-foundedness, it follows that every finite frame is pre-well-founded, which is one of the main motivations for this variant on well-foundedness. •

Given the axiom of dependent choice, pre-well-foundedness can also be expressed using sequences. This is a very useful characterisation in practise.

Proposition 2.97. *Let $\mathfrak{F}$ be a transitive frame. Then the following are equivalent:*

(i) $\mathfrak{F}$ *is* WF,

(ii) there exists no infinite strictly-descending sequence in $\mathfrak{F}$*, and*

(iii) there exists no infinite descending sequence in the irreflexivisation of the skeleton of $\mathfrak{F}$*.*

We end this section with a depth preservation result for p-morphisms.

Proposition 2.98. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames and $f\colon \mathfrak{F} \to \mathfrak{G}$ a p-morphism. Then for all α,*

$$f(\mathfrak{F}^{\leq\alpha}_{\mathrm{w}}) \subseteq \mathfrak{G}^{\leq\alpha}_{\mathrm{w}}.$$

Proof. Write $\mathfrak{F} = \langle X, R\rangle$ and $\mathfrak{G} = \langle Y, S\rangle$. We prove this by induction on α. Suppose, as induction hypothesis, that for every $\beta < \alpha$, the claim holds, and let $x \in \mathfrak{F}^{\leq\alpha}_{\mathrm{w}}$. We show that $f(x) \in \mathfrak{G}^{\leq\alpha}_{\mathrm{w}}$.

For let $y' \in S(f(x))$ such that $\neg S^*(y', f(x))$. Since f a p-morphism, there exists a preimage y of y', such that $R(x, y)$. As f is a frame morphism, we also see $\neg R^*(y, x)$. Then $y \in \mathfrak{F}^{<\alpha}_{\mathrm{w}}$, so by the induction hypothesis $y' = f(y) \in \mathfrak{G}^{<\alpha}_{\mathrm{w}}$. Hence $f(x) \in \mathfrak{G}^{\leq\alpha}_{\mathrm{w}}$. □

In particular, taking p-morphic images preserves converse pre-well-foundedness.

Table 2.1: *List of modal axioms, with their axiom name (a.n.), formula name (f.n.) and the corresponding frame condition.*

a.n.	f.n.	formula	frame condition
t	*refl*	$\Box p \to p$	reflexivity
4	*trans*	$\Box p \to \Box^2 p$	transitivity
.3	*lin*	$\Diamond p \wedge \Diamond q \to (\Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(\Diamond p \wedge q))$	upward linearity
.2	*confl*	$\Diamond \Box p \to \Box \Diamond p$	confluence (Church-Rosser prop.)
.1	*ma*	$\Box \Diamond p \to \Diamond \Box p$	McKinsey property
gl	*la*	$\Box(\Box p \to p) \to \Box p$	irreflexivity + transitivity + converse pre-well-foundedness
grz	*grz*	$\Box(\Box(p \to \Box p) \to p) \to p$	reflexivity + transitivity + converse pre-well-foundedness + absence of proper clusters

2.11 Modal Logics

Throughout this thesis we come across modal logics. This section gives an overview of important axioms and modal logics.

For simplicity, the names of many modal logics are derived by composing the names of their axioms. However, due to historic reasons, rather peculiar axiom names used for this, for example *4* to designate the transitivity axiom and *.2* for the confluence axiom. Sometimes this name that is used for naming the logics is also used as the name of the modal formula [7], and sometimes a different but more descriptive name is used for the modal formula [11]. We will refer to the former as the *axiom name* and the latter as the *formula name*.

A *frame condition* for a formula φ is a property such that for any frame $\mathfrak{F}$, this $\mathfrak{F}$ has the property iff $\mathfrak{F} \vDash \varphi$. Table 2.1 lists the most important modal axioms, with their axiom name, formula name, formula and frame condition.

The McKinsey property mentioned as the frame condition of *ma*, is not a first-order condition [7, Example 3.11]. However, when restricted to transitive frames, it becomes the property that every point sees a final point [11, Proposition 3.46]. Hence the conjunction $trans \wedge ma$ has as frame condition the transitive frames in which every point sees a final point.

In addition to the formulas in the table, we define a family of formulas bw_n for $n \in \omega \setminus \{0\}$ over atomic propositions $p_0, \dots, p_n$ by

$$bw_n := \bigwedge\{\Diamond p_i \mid i \leq n\} \to \bigvee\{\Diamond(p_i \wedge (p_j \vee \Diamond p_j)) \mid i, j \leq n, i \neq j\}.$$

The frame condition of bw_n is that for every point x of the frame, the set of successors of x contains *no* anti-chain of size at least $n + 1$ [11, Proposition 3.42].

Logic **K** is the least normal modal logic, i.e. the logic of all Kripke frames. Names for other logics are then derived by appending the (capitalised) axiom names to a base logic like **K**. For example **K4** := **K** ⊕ *trans* and **KT** := **K** ⊕ *refl*. There are, due to historic reasons again, several exceptions to this rule. The logic **K4** ⊕ *refl* is called **S4**, not **KT4**. The axioms *gl* and *grz* serve as base logic names, i.e. they replace the **K**: **GL** := **K** ⊕ *gl* and **Grz** := **K** ⊕ *grz*.

2.12 Trees and Tree Unravelling

Trees are a special case of frames that are particularly easy to reason about. As such, we will encounter them multiple times in this thesis. However, while intuitively simple, there are various ways to formalise trees as frames. Therefore, we define multiple notions of trees in this section. In addition, we consider a version of tree unravelling, a common technique for turning frames into trees.

We start by defining trees in the non-transitive setting, and then work our way up from there to define increasingly general transitive notions of trees. In order to differentiate the non-transitive notion of a tree from transitive versions of trees we introduce after it, we call it a strict tree.

Definition 2.99 (Strict tree)**.** A frame $\mathfrak{F} = \langle W, R \rangle$ is called a *strict tree* iff

- $\mathfrak{F}$ is rooted, say with root r,
- $\mathfrak{F}$ is acyclic, i.e. the transitive closure of $\mathfrak{F}$ is irreflexive,[2] and
- every non-root has a unique predecessor, i.e. for every $x \in W \setminus \{r\}$ there exists a unique $y \in W$ such that $R(y, x)$.

Taking the transitive closure of a strict tree gives a transitive notion of trees.

Definition 2.100 (Irreflexive transitive tree, reflexive transitive tree)**.** A frame $\mathfrak{F}$ is called an *irreflexive transitive tree* if it is the transitive closure of a strict tree. It is called a *reflexive transitive tree* if it is the reflexive transitive closure of a strict tree.

A final version of trees we introduce allows mixing reflexive and irreflexive points and having clusters in the tree. We call these tree-like frames. More precisely, a tree-like frame is the result of replacing points in an irreflexive transitive tree by mixed reflexive-irreflexive clusters. Note that such frames need no longer be transitive, but are still weakly-transitive.

Definition 2.101 (Tree-like frame)**.** A frame $\mathfrak{F}$ is called *tree-like* if the irreflexivisation of the skeleton of $\mathfrak{F}$ is an irreflexive transitive tree.

[2] Hence cannot contain any clusters.

Each of these notions of trees gives a notion of *forests*, by taking disjoint unions of the respective trees. This yields definitions for *strict forests*, *irreflexive transitive forests* and *forest-like* frames respectively.

Unravelling. A rather obvious question is now whether we can transform frames into trees or forests somehow, while preserving some properties of the frame. Such transformation is called *tree unravelling* or *unwinding*. The main property that we want from any unravelling is that the original frame is a p-morphic image of its unravelling. Even more than trees, there are various different tree unravellings. We will discuss two of them.

The most common unravelling is to turn a rooted frame $\mathfrak{F}$ into a strict tree by taking as points all the paths in $\mathfrak{F}$ starting from the root, and let a path x see another path y iff y is the path of x with a single step in $\mathfrak{F}$ added at the end [7, Proposition 2.15, 11, Theorem 3.18]. We call this the *path unravelling*, to distinguish it from a second unravelling that we discuss. Irreflexive transitive trees can be obtained from transitive frames by taking the transitive closure of this path unravelling.

Definition 2.102 (Path unravelling)**.** Let $\mathfrak{F} = \langle X, R \rangle$ be a frame, and $x_0 \in X$. Define the *path unravelling* of $\mathfrak{F}$ around x_0 to be the frame with as its worlds the finite non-empty ascending sequences in $\mathfrak{F}$ starting in x_0 and as its relation $\vec{R}$ where $\vec{R}(\vec{x}, \vec{y})$ iff $\vec{x}$ is an initial segment of $\vec{y}$ and $\vec{y}$ has length precisely one more than $\vec{x}$.

Proposition 2.103. *Let $\mathfrak{F}$ be a frame and $x_0 \in \mathfrak{F}_{\mathrm{w}}$. Then its path unravelling $\vec{\mathfrak{F}}$ around x_0 is a strict tree. Moreover the function $f\colon \vec{\mathfrak{F}}_{\mathrm{w}} \to \mathfrak{F}_{\mathrm{w}}$ mapping a finite sequence $\vec{x}$ to its last element is a surjective p-morphism from $\vec{\mathfrak{F}}$ to $\mathfrak{F}$. If $\mathfrak{F}$ is transitive then f is a p-morphism from transitive closure $\vec{\mathfrak{F}}^+$ of $\vec{\mathfrak{F}}$ to $\mathfrak{F}$. If $\mathfrak{F}$ is a preorder then f is a p-morphism from the reflexive transitive closure $\vec{\mathfrak{F}}^*$ of $\vec{\mathfrak{F}}$ to $\mathfrak{F}$.*

One drawback of the previous approach is that it does not preserve finiteness; a property we crucially rely on in Chapter 8. Hence we need a second unravelling. Still, in the transitive setting, there are several possibilities for unravellings that preserve finiteness. We here choose an unravelling, which we call the *finite tree-like unravelling*, where a point of the unravelling is a point in the original frame together with a *maximal* path to it this point.

Definition 2.104 (Finite tree-like unravelling)**.** Let $\mathfrak{F} = \langle X, R \rangle$ be a weakly-transitive Kripke frame. Its *finite tree-like unravelling* is the frame $\langle Y, S \rangle$ where

- $Y \subseteq \mathcal{P}(X) \times X$ is the set of pairs $\langle Z, x \rangle$ such that Z is maximal with the following two properties: $Z \subseteq R^{*\mathrm{op}}(x)$ and Z is *free of anti-chains*, i.e. for any $z_1, z_2 \in Z$, $z_1 = z_2$, $R(z_1, z_2)$ or $R(z_2, z_1)$,
- $S(\langle Z_1, x_1 \rangle, \langle Z_2, x_2 \rangle)$ iff $Z_1 \subseteq Z_2$ and $R(x_1, x_2)$.

Proposition 2.105. *Let $\mathfrak{F}$ be a finite weakly-transitive Kripke frame. Then its finite tree-like unravelling $\mathfrak{G}$ is a finite forest-like frame. If $\mathfrak{F}$ is rooted then so is $\mathfrak{G}$.*

Lemma 2.106. *Let $\mathfrak{F}$ be a finite weakly-transitive Kripke frame and $\mathfrak{G}$ its finite tree-like unravelling. Then the function $h\colon \mathfrak{G} \to \mathfrak{F}$ that maps the pair $\langle Z, x\rangle$ to x is a surjective p-morphism.*

Proof. Let us write $\mathfrak{F} = \langle X, R\rangle$. Monotonicity is trivial by de definition relation on $\mathfrak{G}$. So suppose $x_1, x_2 \in \mathfrak{F}_{\mathrm{w}}$ such that $R(x_1, x_2)$, and let $\langle Z_1, x_1\rangle$ be some h-preimage of x_1. Then $Z_1 \subseteq R^{*\mathrm{op}}(x_1) \subseteq R^{*\mathrm{op}}(x_2)$ and Z_1 is free of anti-chains. Clearly, we can extend Z_1 into a maximal such set Z_2. Then $\langle Z_2, x_2\rangle \in \mathfrak{G}_{\mathrm{w}}$, obviously maps to x_2 under h and is a successor of $\langle Z_1, x_1\rangle$ in $\mathfrak{G}$. □

The finite tree-like unravelling is functorial, in the following sense.

Lemma 2.107. *Let, for $i \in \{1, 2\}$, $\mathfrak{F}_i$ be a finite weakly-transitive Kripke frame, $\mathfrak{G}_i$ its finite tree-like unravelling and $h_i\colon \mathfrak{G}_i \to \mathfrak{F}_i$ the surjective p-morphism from the previous lemma. Let $f\colon \mathfrak{F}_1 \to \mathfrak{F}_2$ be a monotone function. Then there exists a monotone function $g\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ such that $h_2 \circ g = f \circ h_1$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{G}_1 & \overset{g}{\dashrightarrow} & \mathfrak{G}_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ \mathfrak{F}_1 & \underset{f}{\longrightarrow} & \mathfrak{F}_2 \end{array}$$

Proof. Write $\mathfrak{F}_i = \langle X_i, R_i\rangle$. Let $\langle Z_1, x\rangle \in \mathfrak{G}_{1,\mathrm{w}}$. Then $Z_1 \subseteq R_1^{*\mathrm{op}}(x)$ and Z_1 is free of anti-chains. Since f is monotone, $f(Z_1) \subseteq f(R_1^{*\mathrm{op}}(x)) \subseteq R_2^{*\mathrm{op}}(f(x))$ and $f(Z_1)$ is free of anti-chains. Then there exists a maximal set Z_2 with these properties extending $f(Z_1)$.

Now we would like to set $g(\langle Z_1, x\rangle)$ to be $\langle Z_2, f(x)\rangle$. However, to make g monotone, we need to chose the Z_2 in a consistent manner: whenever Z_1' extends Z_1 we want Z_2' to extend Z_2. This is easily achieved using tree induction since the set Z_1 that occur as the first element of a point in $\mathfrak{G}_1$ form a reflexive transitive tree under inclusion. □

Unravelling of frames easily extends to models.

Definition 2.108 (Finite tree-like unravelling)**.** Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a finite weakly-transitive Kripke model. Its *finite tree-like unravelling* is the model $\langle \mathfrak{G}, h^{-1} \circ \mathfrak{V}\rangle$ where $\mathfrak{G}$ is the finite tree-like unravelling of $\mathfrak{F}$ and $h\colon \mathfrak{G} \to \mathfrak{F}$ is the surjective p-morphism from Lemma 2.106.

Lemma 2.109. *Let $\mathfrak{M}$ be a finite weakly-transitive Kripke model and $\mathfrak{N}$ its finite tree-like unravelling. Then the surjective p-morphism of frames h of Lemma 2.106 is a p-morphism of Kripke models.*

Proof. Trivial, since by definition of the valuation of $\mathfrak{N}$, h preserves and reflects atomic propositions. $\square$

With these preliminaries settled, we can move to the first topic of this thesis.

Chapter 3

Degrees of Completeness

In this chapter we introduce degrees of completeness, generalising Fine's notion of degrees of incompleteness. We consider degrees in their full generality, as well as concrete examples, most notably WF-frame degrees. In the next chapter we continue this path by studying CWF-frame degrees in more detail.

3.1 Introduction

Fine [19] introduced the notion of degrees of incompleteness, which measures the number of logics that are indistinguishable from a given logic by Kripke frame semantics. Formally, for a normal modal logic Λ, the *degree of incompleteness* of Λ is the cardinally of the set

$$\{\Lambda' \in \mathcal{U} \mid \mathrm{Fr}(\Lambda') = \mathrm{Fr}(\Lambda)\},$$

i.e. set of normal modal logics with the same frame class [19]. Here the $\mathrm{Fr}(-)$ notion from eq. (2.1) is used.

This definition of degrees can be generalised in two directions. For one, it is possible to restrict the logics Λ' to a smaller set of logics, for example the extensions of some base logic like **K4** or **S4**. Second, instead of looking at the frame class of the logics, one can consider a subclass of all frames. For example, G. Bezhanishvili, N. Bezhanishvili and Moraschini [5] introduce what they call *degrees of fmp*, where they restrict the frame classes to only finite frames. One can generalise this a step further, by not necessarily using a class of frames, but an arbitrary class of structures on which modal logic can be interpreted.

This leads to a whole family of different notions of degrees of logics, of which degrees of incompleteness and degrees of fmp are just two examples. In this and the next chapter, we mostly investigate two other notions of degrees, namely degrees w.r.t. the class of WF frames and degrees w.r.t. the class of CWF frames. In both cases we always restrict the set of logics to the extensions of **K4** or some extension of it.

First, in Section 3.2 we introduce an abstract framework of semantics for modal logic. This allows us to introduce and study degrees in their full generality in Section 3.3. Next, in Section 3.4 we consider degrees of incompleteness, WF- and CWF-degrees, and degrees of fmp, and the relation between these different notions of degrees. Furthermore, we note that Block's celebrated dichotomy theorem trivially extends to WF- and CWF-degrees.

In the final two sections we focus on WF-frame and -model degrees. First, in Section 3.5, we show that **GL.3** has infinite WF-frame degree. In Section 3.6 we then consider WF-model completeness. In particular we prove, in contrast to the setting of finite frames, that WF-model completeness and WF-frame completeness are not equivalent.

3.2 Abstract Semantics

To be able to define degrees for any semantics of modal logic, we first need a definition of what a semantics is. In this section we define this in the most general sense possible and consider some notable examples and constructions.

Definition 3.1 (Semantics for the modal language)**.** A *semantics for the modal language*, or just semantics for short, is a pair $\langle \mathcal{S}, \vDash \rangle$ where $\mathcal{S}$ is a class and $\vDash \subseteq \mathcal{S} \times \mathbf{Fm}$ a relation. We will call the elements of $\mathcal{S}$ simply *objects*.

Note that for now we do not impose any requirements on the relation $\vDash$, although various axioms, e.g. modus ponens, would seem logical. We introduce notation for taking the logic of class of objects and the class of objects of a logic.

Notation 3.2. Let $\langle \mathcal{S}, \vDash \rangle$ be a semantics, $\mathcal{S}' \subseteq \mathcal{S}$, $X \in \mathcal{S}$, $\Gamma \subseteq \mathbf{Fm}$ and $\varphi \in \mathbf{Fm}$. We write

$$\begin{aligned} X \vDash \Gamma \quad &\text{iff} \quad \forall \varphi \in \Gamma.\ X \vDash \varphi, \\ \mathcal{S}' \vDash \varphi \quad &\text{iff} \quad \forall X \in \mathcal{S}'.\ X \vDash \varphi, \\ \mathcal{S}' \vDash \Gamma \quad &\text{iff} \quad \forall \varphi \in \Gamma, X \in \mathcal{S}'.\ X \vDash \varphi. \end{aligned}$$

Definition 3.3 (Logic of objects)**.** Let $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash \rangle$ be a semantics and $\mathcal{S}' \subseteq \mathcal{S}$. Define *the logic of* $\mathcal{S}'$ to be

$$\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}') := \{\varphi \in \mathbf{Fm} \mid \mathcal{S}' \vDash \varphi\}.$$

Definition 3.4 (Objects of logic)**.** Let $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash \rangle$ be a semantics and $\Gamma \subseteq \mathbf{Fm}$. Define *the* $\boldsymbol{\mathcal{S}}$*-semantics of* Γ to be

$$\boldsymbol{\mathcal{S}}(\Gamma) := \{X \in \mathcal{S} \mid X \vDash \Gamma\}.$$

The two primitive semantics for modal logic that we consider are the semantics of pointed Kripke models and that of modal algebra models. Other semantics, like those of Kripke frames or modal algebras can be derived by taking quotients.

Example 3.5 (Pointed Kripke model semantics)**.** Take $\mathcal{S}$ to be the class of pointed Kripke models $\langle \mathfrak{M}, x\rangle$, and define $\vDash$ as it is usually defined, i.e. as in Definition 2.51.

Example 3.6 (Modal algebra model semantics)**.** Take $\mathcal{S}$ to be the class of modal algebra models $\langle \mathfrak{A}, \mathfrak{V}\rangle$, and define $\vDash$ by $\langle \mathfrak{A}, \mathfrak{V}\rangle \vDash \varphi$ iff $[\![\varphi]\!]_{\langle \mathfrak{A}, \mathfrak{V}\rangle} = [\![\top]\!]_{\langle \mathfrak{A}, \mathfrak{V}\rangle}$.

Note that Kripke model semantics arises from pointed Kripke model semantics by grouping all pointed versions of a given Kripke model together, and requiring formulas to be valid in each point instead of a single point. Similarly, Kripke frame semantics arises from Kripke model semantics by grouping Kripke models together with identical underlying frame. This approach of grouping structures is a general method for creating coarser semantics from finer ones, which we call quotient semantics.

Definition 3.7 (Quotient semantics)**.** Let $\langle \mathcal{S}, \vDash\rangle$ be a semantics, $\mathcal{S}'$ a class and $q\colon \mathcal{S} \to \mathcal{S}'$ a surjection. Then the *quotient of* $\langle \mathcal{S}, \vDash\rangle$ *by* q is the pair $\langle \mathcal{S}', \vDash'\rangle$ where $\vDash'$ is defined by, for all $X' \in \mathcal{S}'$ and $\varphi \in \mathbf{Fm}$,

$$X' \vDash' \varphi \quad \text{iff} \quad \{X \in \mathcal{S} \mid q(X) = X'\} \vDash \varphi.$$

Such $\langle \mathcal{S}', \vDash'\rangle$ is called a *quotient of* $\langle \mathcal{S}, \vDash\rangle$, and q is called the *quotient map.*

Clearly, also modal algebra semantics arises as a quotient semantics of modal algebra model semantics, and general frame semantics as a quotient semantics of Kripke model semantics.

A second way to create new semantics, is by restricting the class of objects. For example, instead of the semantics of all Kripke frames, one can consider only the finite frames. We call this a subsemantics.

Definition 3.8 (Subsemantics)**.** Let $\langle \mathcal{S}, \vDash\rangle$ be a semantics and $\mathcal{S}' \subseteq \mathcal{S}$. Then the *subsemantics* of $\langle \mathcal{S}, \vDash\rangle$ induced by $\mathcal{S}'$ is the pair $\langle \mathcal{S}', \vDash'\rangle$ where $\vDash'$ is the restriction of $\vDash$ to $\mathcal{S}' \times \mathbf{Fm}$.

We will mostly be concerned with the subsemantics of Kripke frame semantics of all WF or CWF frames. In Section 3.6 we also consider the subsemantics of Kripke model semantics of all models on WF or CWF frames.

Many basic properties that are known for frame classes still hold in the general setting:

Proposition 3.9. *Let* $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash\rangle$ *be a semantics,* $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$*, and* $\Gamma_1, \Gamma_2 \subseteq \mathbf{Fm}$*. Then*

(i) if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ *then* $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_2) \subseteq \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)$,

(ii) if $\Gamma_1 \subseteq \Gamma_2$ *then* $\mathcal{S}(\Gamma_2) \subseteq \mathcal{S}(\Gamma_1)$,

(iii) $\mathcal{S}_1 \subseteq \mathcal{S}(\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1))$,

(iv) $\Gamma_1 \subseteq \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}(\Gamma_1))$,

(v) $\mathcal{S}(\Gamma_1) = \mathcal{S}(\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}(\Gamma_1)))$, *and*

(vi) $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1) \subseteq \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}(\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1)))$.

3.3 Completeness and Degrees

Now that we have defined semantics in a general way, we can define degrees of completeness in this general setting. The term degree of completeness is used to differentiate from Fine's degree of incompleteness, which is only a special case, and potential other definitions of degrees that take a more syntactic approach, see Section 6.2. The degrees of completeness relate strongly to completeness (and incompleteness), which we will define first.

Maybe the most natural definition for Λ being complete would be that the logic of the objects of Λ is again Λ. However, one inclusion is automatic, so we only request the other.

Definition 3.10 (Completeness)**.** Let Λ be a logic and $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash \rangle$ a semantics. Then Λ is said to to $\boldsymbol{\mathcal{S}}$-*complete* iff

$$\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda)) \subseteq \Lambda.$$

Proposition 3.11. *Let $\boldsymbol{\mathcal{S}}$ be a semantics and $\Lambda \in \mathcal{U}$. Then the following are equivalent:*

(i) Λ *is $\boldsymbol{\mathcal{S}}$-complete,*

(ii) $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda)) = \Lambda$,

(iii) there exists $\Lambda' \in \mathcal{U}$ *such that* $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda')) = \Lambda$.

Proof. (i) $\Rightarrow$ (ii): One inclusion by assumption, and the other by Proposition 3.9 (iv).

(ii) $\Rightarrow$ (iii): Take $\Lambda' := \Lambda$.

(iii) $\Rightarrow$ (i): Find $\Lambda' \in \mathcal{U}$ such that $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda')) = \Lambda$. Then

$$\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda)) = \mathrm{Log}_{\boldsymbol{\mathcal{S}}}\Big(\boldsymbol{\mathcal{S}}\Big(\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda'))\Big)\Big) = \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda')) = \Lambda$$

by Proposition 3.9 (v). □

For Kripke semantics, this notion of completeness coincides with Kripke completeness.

Example 3.12. When we take for $\boldsymbol{\mathcal{S}}$ the Kripke semantics, a logic is $\boldsymbol{\mathcal{S}}$-complete iff it is Kripke complete as in Definition 2.80. For Λ is $\boldsymbol{\mathcal{S}}$-complete iff it satisfies $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\mathrm{Fr}(\Lambda)) \subseteq \Lambda$, i.e. whenever $\mathrm{Fr}(\Lambda) \vDash \varphi$ then $\varphi \in \Lambda$. This means precisely Λ is complete w.r.t. $\mathrm{Fr}(\Lambda)$, and as already noted in Section 2.8, $\mathrm{Fr}(\Lambda)$ is the largest class of frames w.r.t. which Λ is sound.

Remark 3.13. Note that we already defined (weak) completeness in Section 2.8, with a different meaning from the $\boldsymbol{\mathcal{S}}$-completeness we define here. Like Kripke completeness, our generic notion of $\boldsymbol{\mathcal{S}}$-completeness requires completeness w.r.t. *only those structures of the semantics* $\boldsymbol{\mathcal{S}}$ *that validate the logic.* In a sense, it combines soundness and weak completeness.

To clearly distinguish between these two notions, we use the following convention. For (weak) completeness, we indicate the class of structures always using 'w.r.t.', and *after* the word *completeness*. For $\boldsymbol{\mathcal{S}}$-completeness, we indicate the semantics *in front of* the word *completeness*. As an example, compare 'Λ is complete w.r.t. WF frames' with 'Λ is WF-frame complete'. •

A degree of completeness is the set of logics that are indistinguishable from one-another for the semantics.

Definition 3.14 (Degree of completeness)**.** Let $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash \rangle$ be a semantics, $\mathcal{X}$ a set of logics and $\Lambda \in \mathcal{X}$. The *degree of* $\boldsymbol{\mathcal{S}}$*-completeness of* Λ *over* $\mathcal{X}$ is the set

$$\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda) := \{\Lambda' \in \mathcal{X} \mid \boldsymbol{\mathcal{S}}(\Lambda') = \boldsymbol{\mathcal{S}}(\Lambda)\}.$$

We will often call it the $\boldsymbol{\mathcal{S}}$*-degree of* Λ *over* $\mathcal{X}$ for short.

Note that, in contrast to Fine, we say the degree is the set of logics, and not the cardinality of this set. This has several advantages. The main advantage is that it is now possible to talk about the *elements* of a degree. For example, later in this section, we will talk about complete elements of a degree.[1] This also makes it possible to naturally state many order-theoretic properties, such as closedness under intersections, meets, joins, etc. And when comparing degrees w.r.t. different semantics, one degree being included in the other is a much more informative statement then just an inequality on their cardinalities.

A secondary advantage is that without taking the cardinality, no choice axioms are involved in merely *defining* what degrees of completeness are. Finally, there is no real disadvantage, since statements about the cardinality of a degree can still be phrased conveniently. Compare for example 'degree 1' with 'singleton degree', and note that terms like *infinite* or *continuum* apply regardless.

[1]Indeed, there can be at most one $\boldsymbol{\mathcal{S}}$-complete logic in a $\boldsymbol{\mathcal{S}}$-degree.

Structural theory. Since we define a degree to be a set of logics rather than a cardinality, we can endow it with subset-inclusion $\subseteq$, and study the resulting poset, in order to get a better understanding of degrees. We will also relate completeness of logics to this degree structure.

First, degrees are *convex.*

Proposition 3.15. *Let* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$ *be a degree and* $\Lambda_1, \dots, \Lambda_3 \in \mathcal{X}$ *such that* $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3$*. If* Λ_1 *and* Λ_3 *are in* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$ *then so is* Λ_2*.*

Proof. Trivial. □

Second, a degree can contain at most one complete logic, and if it does then that logic forms a top element for the degree.

Proposition 3.16. *Let* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$ *be a degree and* $\Lambda_1 \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$*. If* Λ_1 *is* $\boldsymbol{\mathcal{S}}$*-complete then it is a top in the degree, i.e. for all* $\Lambda_2 \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$*,* $\Lambda_2 \subseteq \Lambda_1$*. In particular* Λ_1 *is the unique* $\boldsymbol{\mathcal{S}}$*-complete element of* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$*.*

Proof. Suppose $\Lambda_2 \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$, so $\boldsymbol{\mathcal{S}}(\Lambda_2) = \boldsymbol{\mathcal{S}}(\Lambda_1)$. Then

$$\Lambda_2 \subseteq \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda_2)) = \mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda_1)) \subseteq \Lambda_1,$$

where the first inclusion follows by Proposition 3.9 (iv). □

We always instantiate $\mathcal{X}$ with the set of normal extensions of some base logic, or the set of $\boldsymbol{\mathcal{S}}'$-complete such extensions, where the semantics $\boldsymbol{\mathcal{S}}$ used for the degrees is a subsemantics of $\boldsymbol{\mathcal{S}}'$. In these cases $\mathrm{Log}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{\mathcal{S}}(\Lambda))$ will always be an element of $\mathcal{X}$, hence every degree contains precisely one $\boldsymbol{\mathcal{S}}$-complete logic.

Let us look at an example of degrees that are not studied (for good reason): degrees of modal algebra semantics.

Example 3.17 (Modal algebra degrees)**.** Let $\boldsymbol{\mathcal{S}}$ be the modal algebra semantics. It is well-known [7, Section 5.2] that every normal modal logic is complete w.r.t. modal algebras. Since any $\boldsymbol{\mathcal{S}}$-degree contains at most one $\boldsymbol{\mathcal{S}}$-complete logic, every $\boldsymbol{\mathcal{S}}$-degree is singleton.

Multiple semantics. We show that taking subsemantics and quotient semantics makes degrees larger, i.e. more coarse grained.

Proposition 3.18. *Let* $\boldsymbol{\mathcal{S}}$ *be a semantics,* $\boldsymbol{\mathcal{S}}'$ *a subsemantics of it,* $\mathcal{X}$ *a set of modal logics and* $\Lambda \in \mathcal{X}$*. Then* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda) \subseteq \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}'}(\Lambda)$*.*

Proof. Write $\mathcal{S}'$ for the class underlying $\boldsymbol{\mathcal{S}}'$, and let $\Lambda' \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$. Then $\Lambda' \in \mathcal{X}$ and $\boldsymbol{\mathcal{S}}(\Lambda') = \boldsymbol{\mathcal{S}}(\Lambda)$. Hence

$$\boldsymbol{\mathcal{S}}'(\Lambda') = \boldsymbol{\mathcal{S}}(\Lambda') \cap \mathcal{S}' = \boldsymbol{\mathcal{S}}(\Lambda) \cap \mathcal{S}' = \boldsymbol{\mathcal{S}}'(\Lambda),$$

so $\Lambda' \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}'}(\Lambda)$. □

Proposition 3.19. *Let $\boldsymbol{\mathcal{S}}$ be a semantics, $\boldsymbol{\mathcal{S}}'$ a quotient of it, $\mathcal{X}$ a class of modal logics and $\Lambda \in \mathcal{X}$. Then* $\deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda) \subseteq \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}'}(\Lambda)$.

Proof. Say $\boldsymbol{\mathcal{S}} = \langle \mathcal{S}, \vDash \rangle$, $\boldsymbol{\mathcal{S}}' = \langle \mathcal{S}', \vDash' \rangle$, and $q\colon \mathcal{S} \to \mathcal{S}'$ is the quotient map. Let $\Lambda' \in \deg^{\mathcal{X}}_{\boldsymbol{\mathcal{S}}}(\Lambda)$. Then $\Lambda' \in \mathcal{X}$ and $\boldsymbol{\mathcal{S}}(\Lambda') = \boldsymbol{\mathcal{S}}(\Lambda)$. We show that $\boldsymbol{\mathcal{S}}'(\Lambda') \subseteq \boldsymbol{\mathcal{S}}'(\Lambda)$; the other inclusion is entirely analogous.

Let $X' \in \boldsymbol{\mathcal{S}}'(\Lambda')$. Then $X' \vDash' \Lambda'$, so

$$\{X \in \mathcal{S} \mid q(X) = X'\} \vDash \Lambda'.$$

But $\boldsymbol{\mathcal{S}}(\Lambda') = \boldsymbol{\mathcal{S}}(\Lambda)$, so in the semantics $\boldsymbol{\mathcal{S}}$ any object validates Λ' iff it validates Λ. Hence

$$\{X \in \mathcal{S} \mid q(X) = X'\} \vDash \Lambda,$$

from which it follows by the definition of a quotient that $X' \vDash' \Lambda$. Therefore $X' \in \boldsymbol{\mathcal{S}}'(\Lambda)$. □

3.4 Concrete Degrees of Completeness

In this section we have a first look at specific instantiations of this general notion of degrees. In particular, we consider degrees w.r.t. Kripke frames (like Fine's degrees of incompleteness), finite frames (like degrees of fmp), WF frames and CWF frames.

The dichotomy theorem. A natural question to ask, and one that was indeed already raised by Fine [19], is which cardinals arise as the cardinality of a degree. This question was answered by Blok [8], and the answer might be surprising.

Theorem 3.20 (Block's dichotomy theorem, [8])**.** *Let* $\Lambda \in \mathrm{NExt}(\mathbf{K})$. *Then* $\deg^{\mathrm{NExt}(\mathbf{K})}_{\mathrm{Fr}}(\Lambda)$ *is singleton or continuumly sized.*

Moreover, Blok [8] gives a criterion for when a logic has singleton degree. Given a complete lattice $\mathcal{X}$ of logics, a logic Λ_1 is called a *splitting* of $\mathcal{X}$ iff there exists a logic Λ_2 such that $\mathcal{X}$ is the disjoint union of the upset of Λ_1 and the downset of Λ_2. A logic is called a *join-splitting* of $\mathcal{X}$ iff it is the join of splittings of $\mathcal{X}$. Then $\deg^{\mathrm{NExt}(\mathbf{K})}_{\mathrm{Fr}}(\Lambda)$ is singleton iff Λ is a join-splitting of $\mathrm{NExt}(\mathbf{K})$. We refer to Chagrov and Zakharyaschev [11, Section 10.5] for more details and the proofs.

This is a particularly notable theorem, as it completely describes Fine's degrees of incompleteness. However, one is interested only in logics of transitive frames, i.e. extensions of **K4**. Unfortunately, it is an open problem whether Block's dichotomy theorem generalises to other lattices than $\mathrm{NExt}(\mathbf{K})$, such as $\mathrm{NExt}(\mathbf{K4})$ or $\mathrm{NExt}(\mathbf{S4})$ [11, Problem 10.5].

As a first step in the direction of solving this problem, G. Bezhanishvili, N. Bezhanishvili and Moraschini [5] introduce and analyse what they call degrees of fmp. In our framework, these would be finite frame degrees. Still over NExt($\mathbf{K}$), they extend Block's dichotomy theorem to this setting. However, over NExt($\mathbf{K4}$) and NExt($\mathbf{S4}$) they prove the opposite result: assuming the continuum hypothesis, every cardinal up to continuum is realised as the cardinality of a degree.

Proposition 3.21 ([5, Theorem 7.1])**.** *Let* $\Lambda \in \mathrm{NExt}(\mathbf{K})$*. Then* $\deg_{\mathrm{Fr}_{\mathrm{fin}}}^{\mathrm{NExt}(\mathbf{K})}(\Lambda)$ *is singleton or continuum. It is singleton iff* Λ *is a join-splitting of* NExt($\mathbf{K}$)*.*

Theorem 3.22 (Anti-dichotomy theorem, [5, Theorem 7.3])**.** *Let* $\Lambda_0 \subseteq \mathbf{Grz}$ *be a normal modal logic with the fmp such that* $\mathbf{Grz}$ *is a join-splitting of* NExt(Λ_0)*. Then for any cardinal* $\kappa \in \omega \cup \{\omega, 2^\omega\}$*, there exists a logic* $\Lambda \in \mathrm{NExt}(\Lambda_0)$ *such that*

$$\left|\deg_{\mathrm{Fr}_{\mathrm{fin}}}^{\mathrm{NExt}(\Lambda_0)}(\Lambda)\right| = \kappa.$$

In particular $\Lambda_0 \in \{\mathbf{K4}, \mathbf{S4}, \mathbf{Grz}\}$ have the required properties.

Pre-well-foundedness. With the situation for finite frame degrees completely understood, a natural next step is to look for a degree that is *in-between* Fine's degree of incompleteness and the finite frame degree. Maybe the most obvious generalisation of finite frames would be countable frames, but this does not provide a lot of structure on frames. Every finite frame also has the properties of being WF and CWF, and these are useful structural properties of the frame. Hence, we introduce WF-frame degrees and CWF-frame degrees, which we will study in the following sections and the next chapter.

We note that these new degrees are really in-between degrees of incompleteness and finite frame degrees. This immediately provides a dichotomy theorem over NExt($\mathbf{K}$).

Lemma 3.23. *Let* $\mathcal{X}$ *be a set of logics,* $\Lambda \in \mathcal{X}$ *and* $\mathcal{F}$ *be either* $\mathrm{Fr}_{\mathrm{wf}}$ *or* $\mathrm{Fr}_{\mathrm{cwf}}$*. Then*

$$\deg_{\mathrm{Fr}_{\mathrm{fin}}}^{\mathcal{X}}(\Lambda) \subseteq \deg_{\mathcal{F}}^{\mathcal{X}}(\Lambda) \subseteq \deg_{\mathrm{Fr}}^{\mathcal{X}}(\Lambda).$$

Proof. By Proposition 3.18. □

Theorem 3.24 (Dichotomy theorem)**.** *Let* $\mathcal{F} \in \{\mathrm{Fr}_{\mathrm{fin}}, \mathrm{Fr}_{\mathrm{wf}}, \mathrm{Fr}_{\mathrm{cwf}}, \mathrm{Fr}\}$*. If* Λ *is a join-splitting of* NExt($\mathbf{K}$) *then* $\deg_{\mathcal{F}}^{\mathrm{NExt}(\mathbf{K})}(\Lambda)$ *is singleton. Otherwise it is continuum.*

Proof. Suppose Λ is a join-splitting. By Proposition 3.21 the finite frame degree is singleton. By the previous lemma $\deg_{\mathcal{F}}^{\mathrm{NExt}(\mathbf{K})}(\Lambda)$ is at most singleton. But any degree is non-zero, so at least singleton.

Suppose Λ is not a join-splitting. By Theorem 3.20 $\deg_{\mathrm{Fr}}^{\mathcal{X}}(\Lambda)$ is continuumly sized. By the previous lemma $\deg_{\mathcal{F}}^{\mathrm{NExt}(\mathbf{K})}(\Lambda)$ is at least continuum. Since there are only continuumly many logics, any degree is always at most continuum. □

Note that this actually applies to any class of frames $\mathcal{F}$ that contains all finite ones.

Degrees as lattices. We make one more observation about these four notions of degrees. As was already noted in the previous section, one can endow degrees with the inclusion-order and study its order theoretic properties. We already saw that, for a nice enough class of logics $\mathcal{X}$, any degree over $\mathcal{X}$ has a top element and is convex. For these the four concrete notions of degrees, we add a property to this list: closedness under binary intersections. It will follow that, again for 'good' $\mathcal{X}$, a degree over $\mathcal{X}$ forms a lattice.

We prove two lemmata first.

Lemma 3.25. *Let Λ_1, Λ_2 be transitive modal logics and $\Lambda = \Lambda_1 \cap \Lambda_2$. Then $\mathrm{Fr}_{\mathrm{rooted}}(\Lambda) = \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_1) \cup \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_2)$.*

Proof. The right-to-left inclusion is obvious. For the other inclusion, assume $\mathfrak{F} \notin \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_1) \cup \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_2)$. We show that $\mathfrak{F} \notin \mathrm{Fr}_{\mathrm{rooted}}(\Lambda)$.

Let $i \in \{1, 2\}$. Since $\mathfrak{F} \notin \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_i)$, we find $\varphi_i \in \Lambda_i$ that can be refuted on $\mathfrak{F}$, say in a point $w_i \in \mathfrak{F}_{\mathrm{w}}$. Then $\overline{\Box}\varphi_i \in \Lambda_i$, as Λ_i is a modal logic. By transitivity, $\overline{\Box}\varphi_i$ can be refuted on $\mathfrak{F}$ in its root point.

By substitution we can guarantee that φ_1 and φ_2 do not share any atomic propositions. Then $\overline{\Box}\varphi_1$ and $\overline{\Box}\varphi_2$ can be refuted in the same model, both in the root. Hence $\overline{\Box}\varphi_1 \vee \overline{\Box}\varphi_2$ is refuted on $\mathfrak{F}$. But $\overline{\Box}\varphi_1 \vee \overline{\Box}\varphi_2 \in \Lambda_1 \cap \Lambda_2 = \Lambda$. Therefore $\mathfrak{F} \notin \mathrm{Fr}_{\mathrm{rooted}}(\Lambda)$. □

Lemma 3.26. *Let Λ_1, Λ_2 be logics and let $\mathcal{F}$ be a class of frames closed under taking generated subframes. Then $\mathcal{F}(\Lambda_1) \subseteq \mathcal{F}(\Lambda_2)$ iff $\mathcal{F}_{\mathrm{rooted}}(\Lambda_1) \subseteq \mathcal{F}_{\mathrm{rooted}}(\Lambda_2)$.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Let $\mathfrak{F}$ be a frame and suppose $\mathfrak{F} \notin \mathcal{F}(\Lambda_2)$. Then there exists a point $x \in \mathfrak{F}_{\mathrm{w}}$ and a model $\mathfrak{M}$ on $\mathfrak{F}$ such that $\mathfrak{M}, x \nvDash \Lambda_2$. Write $\mathfrak{M}'$ for the submodel generated by x, and $\mathfrak{F}'$ for its underlying Kripke frame. Then $\mathfrak{M}', x \nvDash \Lambda_2$, so $\mathfrak{F}' \nvDash \Lambda_2$. Since $\mathfrak{F}'$ is rooted and $\mathcal{F}_{\mathrm{rooted}}(\Lambda_1) \subseteq \mathcal{F}_{\mathrm{rooted}}(\Lambda_2)$, we conclude that $\mathfrak{F}' \nvDash \Lambda_1$. Then also $\mathfrak{F} \nvDash \Lambda_1$, so $\mathfrak{F} \notin \mathcal{F}(\Lambda_1)$. □

Proposition 3.27. *Let $\mathcal{X}$ be a set of logics, $\Lambda \in \mathcal{X}$ and let $\mathcal{F}$ be a class of frames closed under taking generated subframes. Let $\Lambda_1, \Lambda_2 \in \deg_{\mathcal{F}}^{\mathcal{X}}(\Lambda)$. If $\Lambda_1 \cap \Lambda_2 \in \mathcal{X}$, then $\Lambda_1 \cap \Lambda_2 \in \deg_{\mathcal{F}}^{\mathcal{X}}(\Lambda)$.*

Proof. Let us write $\mathcal{F}_{\mathrm{rooted}} := \mathcal{F} \cap \mathrm{Fr}_{\mathrm{rooted}}$. By Lemma 3.25,

$$\begin{aligned}\mathcal{F}_{\mathrm{rooted}}(\Lambda_1 \cap \Lambda_2) = \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_1 \cap \Lambda_2) \cap \mathcal{F} &= (\mathrm{Fr}_{\mathrm{rooted}}(\Lambda_1) \cup \mathrm{Fr}_{\mathrm{rooted}}(\Lambda_2)) \cap \mathcal{F} \\ &= \mathcal{F}_{\mathrm{rooted}}(\Lambda_1) \cup \mathcal{F}_{\mathrm{rooted}}(\Lambda_2).\end{aligned}$$

Since $\Lambda_i \in \deg_{\mathcal{F}}^{\mathcal{X}}(\Lambda)$,

$$\mathcal{F}_{\mathrm{rooted}}(\Lambda_i) = \mathcal{F}(\Lambda_i) \cap \mathrm{Fr}_{\mathrm{rooted}} = \mathcal{F}(\Lambda) \cap \mathrm{Fr}_{\mathrm{rooted}}.$$

We conclude that

$$\mathcal{F}_{\text{rooted}}(\Lambda_1 \cap \Lambda_2) = \mathcal{F}_{\text{rooted}}(\Lambda),$$

and hence, by the previous lemma, $\mathcal{F}(\Lambda_1 \cap \Lambda_2) = \mathcal{F}(\Lambda)$. □

In particular this proposition applies to $\mathcal{F} \in \{\mathrm{Fr}_{\mathrm{fin}}, \mathrm{Fr}_{\mathrm{wf}}, \mathrm{Fr}_{\mathrm{cwf}}, \mathrm{Fr}\}$; in fact these classes are closed under taking (not necessarily generated) subframes. For $\mathrm{Fr}_{\mathrm{fin}}$ a much stronger result was already proven by G. Bezhanishvili, N. Bezhanishvili and Moraschini [5]. They show that finite frame degrees have a bottom element [5, Theorem 10.3], so they are closed under arbitrary non-empty intersections.

Combining Proposition 3.27 with results from the previous section, we conclude that a degree forms a lattice under the inclusion-order.

Proposition 3.28. *Let $\mathcal{F}$ be a class of frames closed under taking generated subframes and $\mathcal{X}$ a lattice of logics closed under completion $\mathrm{Log}_{\mathcal{F}}(\mathcal{F}(-))$ taking binary intersections. Let $\Lambda \in \mathcal{X}$. Then $\langle \deg^{\mathcal{X}}_{\mathcal{F}}(\Lambda), \subseteq \rangle$ forms a sublattice of $\mathcal{X}$.*

Proof. By Proposition 3.27 it is closed under binary intersections, which acts as a meet. Since $\mathcal{X}$ is closed under completion $\mathrm{Log}_{\mathcal{F}}(\mathcal{F}(-))$, the degree contains $\mathrm{Log}_{\mathcal{F}}(\mathcal{F}(\Lambda))$. By Proposition 3.16 it is a top element.

For the join, suppose $\Lambda_1, \Lambda_2 \in \deg^{\mathcal{X}}_{\mathcal{F}}(\Lambda)$. Then $\mathrm{Log}_{\mathcal{F}}(\mathcal{F}(\Lambda))$ extends both Λ_1 and Λ_2, hence also their join in $\mathcal{X}$. By Proposition 3.15 this join is an element of $\deg^{\mathcal{X}}_{\mathcal{F}}(\Lambda)$. □

3.5 An Infinite WF-Frame Degree

In this section we construct an infinite WF-frame degree over the Kripke complete logics. In fact, the degree we will consider is the degree of **GL.3**.

Let $\mathfrak{F}_n := \langle W_n, R_n \rangle$ be the frame where $W_n := \omega + n$ and

$$R_n := \{\langle \alpha, \beta \rangle \in (W_n)^2 \mid \beta < \alpha\} \cup \{\langle \omega + i, \omega + j \rangle \mid i, j \in \{0, \dots, n\}\}.$$

Define $\Lambda_n := \mathrm{Log}(\mathfrak{F}_n)$.

Note the following about these logics.

Lemma 3.29. $\Lambda_n \supseteq \mathbf{K4.3} \oplus \{\neg gl \to \Diamond \delta_m \mid m \in \omega\}$.

Proof. Clearly $\mathfrak{F}_n$ is transitive and linear, hence $\mathrm{Log}(\mathfrak{F}_n) \supseteq \mathbf{K4.3}$. Note that in $\mathfrak{F}_n$, $\neg gl$ is only satisfiable on points $\omega + i$ for some $i \in \{0, \dots, n\}$. Therefore $\neg gl \to \Diamond \delta_m \in \mathrm{Log}(\mathfrak{F}_n) = \Lambda_n$ for all $m \in \omega$. □

In fact, one can show that $\Lambda_0 = \mathbf{GL.3}$. This fact follows from Fine's finite width theorem and Fine's selective filtration via maximal points method, which will be discussed later in this thesis.

We first show that these logics are all in a single WF-frame degree. Next, we show that they are in fact distinct logics, so the degree is infinite.

Lemma 3.30. *For any $n \in \omega$,* $\mathrm{Fr_{wf}}(\Lambda_n) = \mathrm{Fr_{wf}}(\Lambda_0)$.

Proof. ($\subseteq$) By Lemma 3.26 it suffices to prove the inclusion for rooted WF frames instead of all WF frames. Let $\mathfrak{F} \in \mathrm{Fr_{wf}}(\Lambda_n)$ be a rooted frame. Since $\mathbf{K4.3} \subseteq \Lambda_n$, $\mathfrak{F}$ is a transitive linear frame.

Suppose $\neg gl$ is satisfiable on some point x of $\mathfrak{F}$. By the previous lemma, x sees for each $m \in \omega$ a point y_m satisfying δ_m. But by linearity these form a descending sequence, contradicting the fact that $\mathfrak{F}$ is WF. Hence $\mathfrak{F} \vDash gl$.

Since $\mathfrak{F} \vDash gl$ it is CWF and irreflexive points. Then $\mathfrak{F}$ is WF, CWF and linear, hence finite. So $\mathfrak{F}$ is a finite irreflexive chain, hence a generated subframe of $\mathfrak{F}_0$. Hence $\mathfrak{F} \in \mathrm{Fr_{wf}}(\Lambda_0)$.

($\supseteq$) Define a function $f\colon \mathfrak{F}_{n,\mathrm{w}} \to \mathfrak{F}_{0,\mathrm{w}}$ being the identity on ω and sending each $\omega + i$ for $i \in \{0, \dots, n\}$ to ω. Clearly f is a surjective p-morphism. Therefore $\Lambda_n \subseteq \Lambda_0$, and hence $\mathrm{Fr_{wf}}(\Lambda_n) \supseteq \mathrm{Fr_{wf}}(\Lambda_0)$. ☐

The proof that the logics differ, is based on a generalised version of the formula δ from Chagrov and Zakharyaschev [11, Lemma 6.11].

Lemma 3.31. *For every $m, n \in \omega$, $\Lambda_m \neq \Lambda_n$.*

Proof. We construct a formula φ_n such that $\varphi_n \in \Lambda_i$ iff $i < n$. These formulas φ_n are a kind of generalisations of *grz* in that they are refutable on frames that are not CWF and on frames with a cluster of size at least n.

In order to make defining φ_n easier, we introduce a kind of 'counter' formulas $\gamma_\_$. We go for simplicity instead of minimising the (finite) number of atomic propositions used, by using a tally encoding. Let $p_\_$ be an ω-sequence of atomic propositions. Define

$$\gamma_n := p_0 \wedge \dots \wedge p_{n-1} \wedge \neg p_n.$$

These formulas γ_n have the property that always precisely one of them is true in a given point in some Kripke model.

Now define

$$\varphi_n := \neg(\gamma_0 \wedge \overline{\Box}(\gamma_0 \to \Diamond\gamma_1) \wedge \dots \wedge \overline{\Box}(\gamma_{n-2} \to \Diamond\gamma_{n-1}) \wedge \overline{\Box}(\gamma_{n-1} \to \Diamond\gamma_0)).$$

Clearly, if a transitive frame $\mathfrak{F}$ contains a cluster of size at least n or an infinite strictly-ascending sequence, then $\neg\varphi_n$ is satisfiable on $\mathfrak{F}$.

Conversely, suppose $\neg\varphi_n$ is satisfiable on $\mathfrak{F}$, say in a point x_0 in a model $\mathfrak{M}$ on $\mathfrak{F}$. Then $\mathfrak{M}, x_0 \vDash \gamma_0$ and $\mathfrak{M}, x_0 \vDash \gamma_0 \to \Diamond\gamma_1$. Hence there exists a successor x_1 of x_0 such that $\mathfrak{M}, x_1 \vDash \gamma_1$. Now $\mathfrak{M}, x_0 \vDash \Box(\gamma_1 \to \Diamond\gamma_2)$ so $\mathfrak{M}, x_1 \vDash \Diamond\gamma_2$, hence x_1 sees a point x_2 such that $\mathfrak{M}, x_2 \vDash \gamma_2$. Continuing like this, we find an ascending ω-sequence $x_\_$ such that $\mathfrak{M}, x_i \vDash \gamma_{i \bmod n}$, where $i \bmod n$ is the smallest element of the equivalence class of i modulo n in the natural numbers.

Suppose $\mathfrak{F}$ contains no cluster of size at least n. Let $k \in \omega$. Then the points $x_{k \cdot n+i}$ for $i \in \{0, \dots, n-1\}$ each satisfy a different $\gamma_\_$ formula. Since there is no cluster of this size, $x_{k\cdot n+n-1}$ does not see $x_{k\cdot n}$. Hence the subsequence of $x\_$ where we restrict to indices that are a multiple of n is strictly-ascending.

Now note that $\mathfrak{F}_i$ is CWF and it has a cluster of size at least n iff $n \leq i$. Therefore φ_n refutable on $\mathfrak{F}_i$ iff $n \leq i$. Hence $\varphi_n \in \Lambda_i$ iff $i < n$. □

We conclude:

Theorem 3.32. *There is an infinite* WF*-frame degree over the Kripke complete extensions of* **K4.3**.

3.6 WF-Model Completeness

In the setting of finite frames, the finite frame completeness (known as the finite frame property) and finite model completeness (known as the fmp) are equivalent [see e.g. 7, Theorem 3.28]. In this section we will see that the analogous statement is not true in the setting of WF completeness. In fact, we show that *every* normal modal logic is WF-model complete. The proof is very straightforward, using the path unravelling introduced in Section 2.12.

Theorem 3.33. *Let* $\Lambda \in \mathrm{NExt}(\mathbf{K})$*. Then* Λ *is* WF*-model complete, and has singleton* WF*-model degree over* $\mathrm{NExt}(\mathbf{K})$.

Proof. Suppose $\neg\varphi \notin \Lambda$, i.e. φ is Λ-consistent. Since any logic is complete w.r.t. its Kripke models (even descriptive frames), there exists a Kripke model $\mathfrak{M}$ that validates Λ and satisfies φ in some point x. By Proposition 2.103 there exists a surjective p-morphism f from the path unravelling $\vec{\mathfrak{M}}_{\mathrm{fr}}$ of $\mathfrak{M}_{\mathrm{fr}}$ to $\mathfrak{M}_{\mathrm{fr}}$. By Proposition 2.71 this induces a p-morphism from a model $\mathfrak{M}^*$ on $\vec{\mathfrak{M}}_{\mathrm{fr}}$ to $\mathfrak{M}$. Then $\mathfrak{M}^*$ satisfies a formula iff $\mathfrak{M}$ does so. Hence $\mathfrak{M}^*$ validates Λ and satisfies φ (in an f-preimage of x). By construction $\vec{\mathfrak{M}}_{\mathrm{fr}}$ is a strict tree, and hence WF.

For the latter part, note that since every logic is complete, no two logics can share a degree. □

This shows that for pre-well-foundedness the frame versus model degree story is completely different from that of finiteness. Every WF-model degree is singleton, while WF-frame degrees can be infinite, even over the Kripke complete extensions of **K4**, as Theorem 3.32 shows. In this regard the class of WF frames is more similar to that of all frames, where frame and model degrees differ wildly. This also contrasts strongly with CWF-model degrees, where continuum degrees exist, as we will see in the next chapter. In fact, for converse pre-well-foundedness we leave it as an open question whether CWF-frame and CWF-model completeness coincide.

In fact, regarding WF- and CWF-model completeness, we can show more. Using a transfinite inductive construction, one can show that any CWF-frame complete extension of **K4** is complete for models that are simultaneously WF and CWF. The constructed model will be of infinite width in general though. While this might seem unimportant compared to the above result, it does mean that simultaneous WF and CWF model degrees behave rather different from finite-model degrees, even though pre-well-foundedness and converse pre-well-foundedness conceptually correspond to, in a sense, 'downward and upward finiteness' respectively. Also note that this does not follow from a proof similar to the above, as taking any kind of tree-unravelling will destroy the converse pre-well-foundedness. We omit this result for brevity though.

Chapter 4

Degrees of Converse Pre-well-foundedness

In the previous chapter we defined degrees of completeness in a general setting and studied WF-degrees. In this chapter we turn to CWF-degrees.

4.1 Introduction

Compared to WF-degrees, that we briefly studied at the end of the previous chapter, proofs about CWF-frame degrees require somewhat more elaborate constructions. Therefore, we dedicate this entire chapter to the study of CWF-degrees.

Even though CWF-frame *degrees* have not been studied before, some results about CWF-frame *completeness* are already known. Particularly notable is Fine's finite width theorem [20]. A frame $\mathfrak{F}$ is said to have *finite width* iff every anti-chain in $\mathfrak{F}$ is finite. Now Fine's finite width theorem states the following.

Theorem 4.1 (Fine's finite width theorem; [20, Theorem 4])**.** *Let $\Lambda \supseteq \mathbf{K4}$ be a modal logic that is complete w.r.t. its general frames of finite width. Then Λ is CWF-frame complete (and hence Kripke complete).*

The proof can be found in the original paper of Fine [20] or in Chagrov and Zakharyaschev [11, Section 10.4].

Remark 4.2. Two remarks concerning the notion of finite width in this theorem are in order. First, note that it is not required that all (rooted) frames of Λ are of finite width. It is only required that there exists some class of general frames, each of which (individually) has finite width, with respect to which Λ is sound and complete. For example, the theorem applies to **S4**, since this logic has the fmp, so is complete w.r.t. its finite general frames.[1] Clearly a finite frame has finite width.

[1] Not that this would prove anything useful. Having the fmp is already stronger than the consequent of the theorem, CWF-frame completeness.

Second, our definition of *finite width* is already somewhat peculiar, in that it differs from having width *bounded* by a natural number $n \in \omega$. That is, a frame can have anti-chains of size n for any $n \in \omega$ and still be finite width, because it fails to have a single infinite anti-chain. These two features make our formulation of Fine's finite width theorem the most general one, equivalent to how it is stated in e.g. Theorem 10.43 in Chagrov and Zakharyaschev [11]. •

For CWF-frame degrees this theorem implies that every degree over NExt($\mathbf{K4BW}_n$) is singleton, for $n \in \omega$, where $\mathbf{K4BW}_n = \mathbf{K4} \oplus bw_n$ is the logic of transitive frames in which every rooted generated subframe contains no anti-chain of size $> n$. Note how this differs from WF-frame degrees, where **GL.3** has infinite degree over NExt(**K4.3**) = NExt($\mathbf{K4BW}_1$), as shown in Section 3.5.

The second known result we will mention, comes from the study of *super-intuitionistic* logics and modal companions. Shehtman [40, Theorem 1] shows that there exists a Kripke complete super-intuitionistic logic Λ which is CWF-frame incomplete, and whose largest modal companion (i.e. the unique modal companion extending **Grz**) is Kripke incomplete. The proof of the latter can also be found in Chagrov and Zakharyaschev [11, Theorem 6.27], and the former follows from it [33, Theorem 1]. Now it follows that the least modal companion of Λ is also Kripke complete but CWF-frame incomplete [11, Theorem 9.70].

For CWF-frame degrees this means that there is a non-singleton degree over Kripke $\cap$ NExt(**S4**), i.e. the Kripke complete extensions of **S4**. Later in this chapter, we will generalise this statement to infinitely many continuum sized degrees.

This chapter is organised as follows. In Section 4.2 we prove a result that is somewhat similar to Fine's finite width theorem, in the sense that we give a sufficient condition for a CWF-frame degree to be singleton. We will be working over Kripke $\cap$ NExt(**K4**), i.e. we show that the concerned degrees contain exactly one Kripke complete extension of **K4**. In particular, any extension of **GL** or **Grz** satisfies the sufficient condition, and hence has singleton degree over Kripke $\cap$ NExt(**K4**).

In Sections 4.3 and 4.4 we generalise the previously mentioned result that follows from the work of Shehtman and Litak. First, in Section 4.3 we show the existence of a Kripke complete but CWF-model incomplete logic. This proof also lays the foundations for Section 4.4, where we show that there exist infinitely many continuum sized CWF-model degrees over the extensions of **S4**, and also infinitely many continuum CWF-frame degrees over the *Kripke complete* extensions of **S4**. The techniques used in these two sections also turn out to be useful again in the next chapter, where we study the recently introduced notion of quasi-canonicity.

4.2 Extensions of GL and Grz

In this section we investigate the CWF-degrees of extension of **GL** and **Grz**. We work over Kripke complete extensions of **K4**. In particular we will prove both degrees to be singleton, as a corollary to a more general theorem.

The proof consist of several steps, where the goal is to turn a non-CWF frame into a CWF frame with an infinite cluster by means of repeatedly taking p-morphic images and generated subframes. Recall the definitions of the deep part and upper part of a frame from Definitions 2.93 and 2.94.

Lemma 4.3. *Let $\mathfrak{F}$ be a transitive frame such that $\mathfrak{F}^{\text{upper}}$ has a linear p-morphic image*[2] *and $\mathfrak{F}^{\text{deep}}$ is non-empty. Then there exists a generated subframe $\mathfrak{F}_3$ of a p-morphic image $\mathfrak{F}_2$ of $\mathfrak{F}$ such that*

(i) $\mathfrak{F}_3^{\text{deep}}$ is non-empty, and

(ii) any point in $\mathfrak{F}_3^{\text{deep}}$ sees all of $\mathfrak{F}_3^{\text{upper}}$.

Proof. By assumption there exists a linear p-morphic image $\mathfrak{G}$ of $\mathfrak{F}^{\text{upper}}$. Obviously, we can extend the p-morphism to a p-morphic image $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ of $\mathfrak{F}$ such that $\mathfrak{F}_2^{\text{upper}} = \mathfrak{G}$.

Write $\mathsf{Up}(\mathfrak{G})$ for the set of upsets of $\mathfrak{G}$. Define a function

$$\text{sp}\colon W_2 \to \mathsf{Up}(\mathfrak{G}) : x \mapsto R_2(x) \cap \mathfrak{G}_{\text{w}}.$$

Note that since $\mathfrak{F}_2$ is transitive, $R_2(x)$ is an upset in $\mathfrak{F}_2$, and hence also in the generated subframe $\mathfrak{G}$. Again by transitivity, sp is monotone as a function from $\mathfrak{F}_2^{\text{deep}}$ to $\langle \mathsf{Up}(\mathfrak{G}), \supseteq \rangle$.

Note that $\langle \mathsf{Up}(\mathfrak{G}), \supseteq \rangle$ is a linear frame since $\mathfrak{G}$ is linear. Since $\mathfrak{F}^{\text{upper}}$ is CWF and $\mathfrak{G}$ is a p-morphic image of it, by Proposition 2.98 $\mathfrak{G}$ is CWF too. Hence $\langle \mathsf{Up}(\mathfrak{G}), \supseteq \rangle$ is CWF.

With the axiom of choice one can construct an ascending sequence $x\colon \omega \to W_2^{\text{deep}}$ such that for any $i \in \omega$ and $y \in R_2(x(i)) \cap W_2^{\text{deep}}$, if $\text{sp}(y) \subsetneq \text{sp}(x(i))$ then $\text{sp}(x(i+1)) \subsetneq \text{sp}(x(i))$.

Now $\text{sp} \circ x\colon \langle \omega, \leq \rangle \to \langle \mathsf{Up}(\mathfrak{G}), \supseteq \rangle$ is a monotone function, ω is infinite and $\langle \mathsf{Up}(\mathfrak{G}), \supseteq \rangle$ is CWF. Then $\text{sp} \circ x$ is eventually constant, say from $m \in \omega$ onward. By construction of x this means that for $y \in R_2(x(m)) \cap W_2^{\text{deep}}$ we have $\text{sp}(y) = \text{sp}(x(m))$. Hence y is sp-minimal.

Let $\mathfrak{F}_3$ be the subframe of $\mathfrak{F}_2$ generated by $x(m)$. Note that by transitivity $\mathfrak{F}_{3,\text{w}} = R_2(x(m))$. Clearly $\mathfrak{F}_3^{\text{deep}} = \mathfrak{F}_2^{\text{deep}} \cap R_2(x(m))$. Hence $\mathfrak{F}_{3,\text{w}}^{\text{deep}}$ is non-empty and any $y \in \mathfrak{F}_{3,\text{w}}^{\text{deep}}$ has $\text{sp}(y) = \text{sp}(x(m)) = \mathfrak{F}_{3,\text{w}}^{\text{upper}}$. □

Lemma 4.4. *Let $\mathfrak{F}$ be a transitive frame such that $\mathfrak{F}^{\text{deep}}$ is non-empty, and any point in $\mathfrak{F}^{\text{deep}}$ sees all of $\mathfrak{F}^{\text{upper}}$. Then there exists a generated subframe $\mathfrak{F}_3$ of a p-morphic image $\mathfrak{F}_2$ of $\mathfrak{F}$ and a strictly-ascending sequence $x\colon \omega \to \mathfrak{F}_{3,\text{w}}^{\text{deep}}$ such that x is cofinal in $\mathfrak{F}_3^{\text{deep}}$ and any point in $\mathfrak{F}_3^{\text{deep}}$ sees all of $\mathfrak{F}_3^{\text{upper}}$.*

[2] A linear p-morphic image is a p-morphic image that is, as a frame, linear.

Proof. Let $x' \colon \omega \to \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$ be a strictly-ascending sequence in $\mathfrak{F}^{\mathrm{deep}}$. Set $X' := R^{\mathrm{op}}(\{x'(i) \mid i \in \omega\})$ and define an equivalence relation $\sim$ on W by $y \sim y'$ iff

- $y = y'$, or
- $y, y' \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$ and $y, y' \notin X'$.

We claim $\sim$ is a bisimulation equivalence. For assume $y, y', z \in W$ such that $y \sim y'$, $y \neq y'$ and $R(y, z)$. Then $y, y' \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$ and $y, y' \notin X'$. If $z \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{upper}}$ then $R(y', z)$, so assume $z \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$. Pick any successor $z' \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$ of y'. Since X' is a downset, $z, z' \notin X'$. Hence $z \sim z'$, as required.

Since $\sim$ is a bisimulation equivalence, it induces a surjective p-morphism $f \colon \mathfrak{F} \to \mathfrak{F}_2$, where $\mathfrak{F}_2 := \mathfrak{F}/{\sim}$. Note that f is injective on X'. We consider three cases:

Case 1. Suppose $\mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}} \subseteq X'$. Define $\mathfrak{F}_3 := \mathfrak{F}_2$, $x := x'$ and note that $\mathfrak{F}_3^{\mathrm{deep}} = f(X')$.

Case 2. Suppose there exists some $y \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}} \setminus X'$, and for all $i \in \omega$, $f(x(i))$ sees $f(y)$. Define $\mathfrak{F}_3 := \mathfrak{F}_2$ and note that $f(y) \in \mathfrak{F}_3^{\mathrm{upper}}$. Hence $\mathfrak{F}_3^{\mathrm{deep}} = f(X')$. Define $x := x'$.

Case 3. Suppose there exists some $y \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}} \setminus X'$, and there exists $m \in \omega$ such that $f(x'(m))$ does not see $f(y)$. Define $\mathfrak{F}_3$ to be the subframe of $\mathfrak{F}_2$ generated by $x'(m)$. Define for all $i \in \omega$, $x(i) := x'(m + i)$.

□

Lemma 4.5. *Let* Λ *be a Kripke complete extension of* **K4** *such that*

- *any* CWF Λ*-frame has a linear p-morphic image, and*
- *any* CWF Λ*-frame has no infinite clusters.*

Then any Λ*-frame is* CWF.

Proof. Let $\mathfrak{F}$ be any Λ-frame, and assume, in order to reach a contradiction, it is not CWF. Since $\mathfrak{F}^{\mathrm{upper}}$ is a generated subframe of $\mathfrak{F}$, it is a Λ-frame. By definition it is CWF. Hence, by assumption, there exists a linear p-morphic image of $\mathfrak{F}^{\mathrm{upper}}$. By Lemmata 4.3 and 4.4 there exists a Λ-frame $\mathfrak{G} = \langle W, R\rangle$ and a strictly-ascending sequence $x \colon \mathfrak{G} \to W$ such that x is cofinal in $\mathfrak{G}^{\mathrm{deep}}$ and any point in $\mathfrak{G}^{\mathrm{deep}}$ sees all points in $\mathfrak{G}^{\mathrm{upper}}$.

Define a function $f \colon \mathfrak{G}_{\mathrm{w}}^{\mathrm{deep}} \to \omega$ by setting $f(y)$ to be the smallest $m \in \omega$ such that $R(y, x(m))$. Note that by cofinality of x such m always exists. Since ω forms a wellorder, there exists a smallest. Since x is strictly-ascending, $f(x(i)) = i$ for all $i \in \omega$.

Let $g \colon \omega \to \omega$ be a surjective function such that for any $m \in \omega$, the preimage $g^{-1}(m)$ is infinite. Define an equivalence relation $\sim$ on W by setting $y \sim' y'$ iff

- $y = y'$, or
- $y, y' \in \mathfrak{G}_{\mathrm{w}}^{\mathrm{deep}}$ and $(g \circ f)(y) = (g \circ f)(y')$.

We claim this is a bisimulation equivalence. For assume $y, y', z \in W$ such that $y \sim y'$, $y \neq y'$ and $R(y, z)$. Then $y, y' \in \mathfrak{G}_{\mathrm{w}}^{\mathrm{deep}}$ and $f(y)$ and $f(y')$ are congruent modulo n. If $z \in \mathfrak{G}_{\mathrm{w}}^{\mathrm{upper}}$ then $R(y', z)$, so assume $z \in \mathfrak{G}_{\mathrm{w}}^{\mathrm{deep}}$. Since $g^{-1}((g \circ f)(z))$ is infinite, it contains some $m > f(y')$. Then y' sees $x(f(y'))$ sees $x(m)$, and

$$(g \circ f)(x_m) = g(m) = (g \circ f)(z).$$

Hence $z \sim x(m)$.

Since $\sim$ is a bisimulation equivalence, $\mathfrak{G}/{\sim}$ is a Λ-frame. Clearly it is a CWF frame and has an infinite cluster. This contradicts the assumption. □

We conclude that any logic as in the previous lemma has singleton CWF-degree:

Theorem 4.6. *Let Λ be a Kripke complete extension of* **K4** *such that*

- *any CWF Λ-frame has a linear p-morphic image, and*
- *any CWF Λ-frame has no infinite clusters.*

Then the CWF-frame degree of Λ equals its Kripke-frame degree,[3] i.e.

$$\deg_{\mathrm{Fr}_{\mathrm{cwf}}}^{\mathrm{NExt}(\mathbf{K4})}(\Lambda) = \deg_{\mathrm{Fr}}^{\mathrm{NExt}(\mathbf{K4})}(\Lambda).$$

In other words, the CWF-frame degree contains only a single Kripke complete logic:

$$\deg_{\mathrm{Fr}_{\mathrm{cwf}}}^{\mathrm{NExt}(\mathbf{K4})\cap\mathrm{Kripke}}(\Lambda) = \{\Lambda\}.$$

Proof. Let $\Lambda' \in \deg_{\mathrm{Fr}_{\mathrm{cwf}}}^{\mathrm{NExt}(\mathbf{K4})\cap\mathrm{Kripke}}(\Lambda)$. We show that Λ' is complete w.r.t. its CWF frames. Note that since Λ' is Kripke complete, it suffices to show that every Λ'-frame is CWF. By Lemma 4.5, it suffices to show that any CWF frame of Λ' has a linear p-morphic image and finite clusters. Note that Λ has these properties by assumption, and $\mathrm{Fr}_{\mathrm{cwf}}(\Lambda') = \mathrm{Fr}_{\mathrm{cwf}}(\Lambda)$. □

To apply this theorem, we need to establish linear p-morphic images. Two important classes of frames with this property are reflexive frames and irreflexive frames.

Proposition 4.7. *Let $\mathfrak{F}$ be a reflexive frame. Then $\mathfrak{F}$ has a linear p-morphic image.*

Proof. Trivial, since the reflexive point is a p-morphic image of $\mathfrak{F}$. □

[3] That is, the degree of incompleteness as Fine [19] introduced, but taking the set of logics instead of the cardinality.

Proposition 4.8. *Let $\mathfrak{F}$ be an irreflexive transitive* CWF *frame. Then $\mathfrak{F}$ has a linear p-morphic image.*

Proof. Write $\mathfrak{F} = \langle W, R\rangle$. Define an equivalence relation $\sim$ on W by $x \sim y$ iff $x = y$ or $x, y \in W$ and $\mathrm{depth}_{\mathfrak{F}}(x) = \mathrm{depth}_{\mathfrak{F}}(y)$. It is a bisimulation equivalence, for assume $x \sim x'$ and $y \in R(x)$. Assume w.l.o.g. $x \neq x'$. Then $x, x' \in W$ and $\mathrm{depth}_{\mathfrak{F}}(x) = \mathrm{depth}_{\mathfrak{F}}(x')$. Now $\mathrm{depth}_{\mathfrak{F}}(y) < \mathrm{depth}_{\mathfrak{F}}(x)$, and since $\mathrm{depth}_{\mathfrak{F}}(x') = \mathrm{depth}_{\mathfrak{F}}(x)$, there exists $y' \in R(x')$ with $\mathrm{depth}_{\mathfrak{F}}(y') = \mathrm{depth}_{\mathfrak{F}}(y)$. Hence $y \sim y'$ as required.

We show that $\mathfrak{F}/{\sim}$ is linear. Write f for the quotient map, and let $x, y \in \mathfrak{F}_{\mathrm{w}}$, and assume w.l.o.g. $\mathrm{depth}_{\mathfrak{F}}(x) \leq \mathrm{depth}_{\mathfrak{F}}(y)$. If $\mathrm{depth}_{\mathfrak{F}}(x) = \mathrm{depth}_{\mathfrak{F}}(y)$ then $x \sim y$ so $f(x) = f(y)$. So suppose $\mathrm{depth}_{\mathfrak{F}}(x) < \mathrm{depth}_{\mathfrak{F}}(y)$. Then y sees a point y' with $\mathrm{depth}_{\mathfrak{F}}(y') = \mathrm{depth}_{\mathfrak{F}}(x)$. Therefore $y' \sim x$, so $f(y)$ sees $f(y') = f(x)$. □

Since **GL**-frames are irreflexive and **Grz**-frames are reflexive, the requirement of linear p-morphic images is satisfied. Note that **GL**-frames have no non-degenerate clusters and **Grz**-frames have no clusters of size ≥ 2, so they do not contain infinite clusters. From Theorem 4.6 we conclude:

Corollary 4.9. *Let Λ be a Kripke complete extension of* **GL** *or* **Grz**. *Then*

$$\deg_{\mathrm{Fr}_{\mathrm{cwf}}}^{\mathrm{NExt}(\mathbf{K4})\cap\mathrm{Kripke}}(\Lambda) = \{\Lambda\}.$$

4.3 CWF-Model Incompleteness

In this section and the next we construct continuumly many continuum sized CWF-frame degrees over the Kripke complete extensions of **S4**. The approach is as follows. We start syntactically, by considering a kind of syntactic version of CWF-model incompleteness, as a set of axioms and a single 'non-axiom'. Next, we prove that Kripke complete logics with these properties exist, by giving a frame, or in fact continuumly many, that validates these axioms and refutes the non-axiom. We conclude that there is a Kripke complete but CWF-model incomplete logic. In the next section we show, using the same frames and the properties we established about them in this section, that there exist infinitely many continuum sized CWF-model degrees.

Cwf-incompleteness syntactically. We start by considering a kind of syntactic version of CWF-model incompleteness. Recall that a logic Λ being CWF-model incomplete means that there exists some formula φ_0 such that $\neg\varphi_0 \notin \Lambda$ but φ_0 is not satisfiable on a CWF-model of Λ. In other words, if φ_0 is satisfied in a model $\mathfrak{M}$, then there exists an infinite strictly-ascending sequence in $\mathfrak{M}_{\mathrm{fr}}$.

Each point in a ω-sequence can be modelled as a formula φ_n. The monotonicity of the sequence can then be described by implications $\varphi_n \to \Diamond\varphi_{n+1}$, i.e. each point in the sequence sees the next one. For the strictness, no point must see any of the previous points. This can globally be described by implications $\varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\}$. Alternatively, as a more local approach, one can change the implications describing the monotonicity to

$$\varphi_n \to \Diamond\big(\varphi_{n+1} \wedge \bigwedge\{\neg\Diamond\varphi_i \mid i \leq n\}\big).$$

Proposition 4.10. *Let $\varphi\_$ be a ω-sequence of modal formulas and Λ a modal logic such that for all $n \in \omega$,*

$$\varphi_n \to \Diamond\varphi_{n+1}, \varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\} \in \Lambda. \tag{4.1}$$

Then φ_0 is not satisfiable on a CWF *model of Λ. If $\neg\varphi_0 \notin \Lambda$ then Λ is* CWF*-model incomplete.*

Proof. Suppose φ_0 is satisfied on a Kripke model $\mathfrak{M}$ of Λ. We show that $\mathfrak{M}_{\mathrm{fr}}$ is not CWF.

We construct, by induction on $n \in \omega$, a strictly-ascending ω-sequence $x\_$ such that for all n, $\mathfrak{M}, x_n \vDash \varphi_n$. For the basis, pick x_0 in $\mathfrak{M}$ which satisfies φ_0.

For the inductive step, assume x_n is already defined, $x_0, \dots, x_n$ is strictly-ascending and $\mathfrak{M}, x_i \vDash \varphi_i$ for all $i \leq n$. Since $\mathfrak{M}$ is a Λ-model,

$$\mathfrak{M}, x_n \vDash \varphi_n \to \Diamond\varphi_{n+1}.$$

By the induction hypothesis the antecedent φ_n is satisfied, hence x_n sees a point x_{n+1} such that $\mathfrak{M}, x_{n+1} \vDash \varphi_{n+1}$. Then $x_0, \dots, x_{n+1}$ is an ascending sequence. It is strict since $\mathfrak{M}, x_{n+1} \vDash \bigwedge\{\neg\Diamond\varphi_i \mid i < n+1\}$, so it cannot see any x_i with $i < n+1$, as x_i satisfies φ_i.

Hence there exists a strictly-ascending sequence in $\mathfrak{M}_{\mathrm{fr}}$, so it is not CWF. □

Let us remark that this syntactic description of converse pre-well-foundedness really captures its essence, and can be used to define analogues of converse pre-well-foundedness in other semantics for modal logic, such as modal algebra semantics and topological semantics. For example, one can call a modal algebra $\mathfrak{A}$ CWF iff there exists *no* ω-sequence $a\_$ of elements of $\mathfrak{A}$ such that, for all $n \in \omega$,

$$a_n \leq \Diamond a_{n+1} \quad \text{and} \quad a_n \leq \bigwedge\{\neg\Diamond a_i \mid i < n\}.$$

As a second example, in joint work with Takapui [42] we explore the connection with scattered spaces in topological d-semantics.

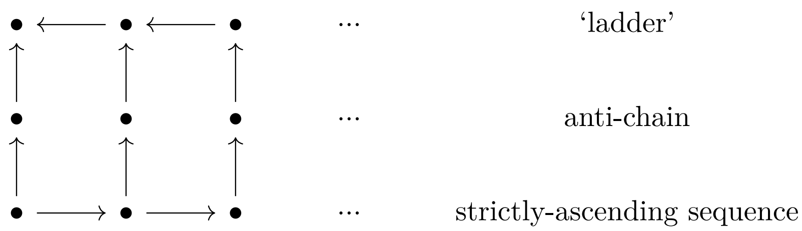

Figure 4.1: *Global idea for constructing a transitive frame whose logic satisfies the requirements of Proposition 4.10. It consists of an infinite strictly-ascending sequence at the bottom, an infinite anti-chain in the middle, and an infinite descending ladder at the top. Transitive arrows are omitted.*

Sketch of the frame construction. Now ‘all’ we have to do is find a logic Λ and a sequence of formulas $\varphi_\_$ that satisfy the assumptions of Proposition 4.10. Since we want a Kripke complete logic, we will define the logic Λ to be the logic of a frame $\mathfrak{F}$. Obviously this frame $\mathfrak{F}$ cannot be CWF, so it must contain an infinite strictly-ascending sequence. Moreover, by Fine’s finite width theorem, $\mathfrak{F}$ must contain an infinite anti-chain. These two requirements and the formulation of Proposition 4.10 will guide the construction of the frame $\mathfrak{F}$.

By Proposition 4.10 we want to have a strictly-ascending sequence in which each point is ‘characterised’ by a formula. Fine’s finite width theorem and its proof suggest that we should describe these points by the points that they see in an infinite anti-chain. The most natural way to characterize the points of the anti-chain by formulas, is to have them look into some kind of *ladder* at various depths. This leads to a construction like the one depicted in Figure 4.1.

The ladder. As announced at the start of the section, we want our CWF-incomplete logic to extend **S4**. Therefore, our frame $\mathfrak{F}$ needs to be reflexive. This means that we cannot ‘measure’ depth using formulas in a simple infinite strictly-descending sequence.[4] Hence, we need a more complicated ladder.

The intuition here is that we need a ladder where no two distinct points can be identified using a p-morphism, except when two final points in the ladder are also identified. This can be achieved using a ladder of ‘width 3’, in which we have a ‘layer’ of three final points, and then for each layer, a layer of three points below it, where the points each see a different subset of two points in the layer above.

More formally, define $\mathfrak{F}_1 := \langle W_1, R_1 \rangle$ as follows. Set

$$W_1 := \omega \times 3,$$

and name these points $a_n^i := \langle n, i \rangle$. Define to R be the reflexive transitive closure of

$$\left\{ \langle a_{n+1}^i, a_n^j \rangle \;\middle|\; n \in \omega, i, j \in 3, i \neq j \right\}.$$

[4]In the irreflexive setting this would be possible: the point at depth n precisely satisfies $\Box^{n+1}\bot \wedge \Diamond^n\top$.

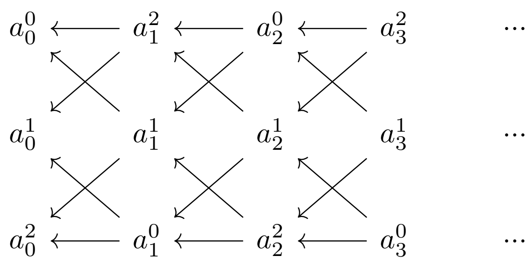

Figure 4.2: *The reflexive and transitive ladder $\mathfrak{F}_1$. Note that the direction in which the upper index increases alternates between top to bottom and bottom to top.*

This frame is depicted in Figure 4.2.

Suppose that each of the the final points a_0^- is characterised by an atomic proposition. Then each of the points a_-^- can be characterised by a formula α_-^-. For this, assume formulas α_0^i for $i \in 3$ are given. Define, by induction on $n \in \omega$,

$$\alpha_{n+1}^i := \neg \Diamond \alpha_n^i \wedge \bigwedge \left\{ \Diamond \alpha_n^j \mid j \in 3, j \neq i \right\}.$$

Proposition 4.11. *Let $\mathfrak{M}$ be a model on $\mathfrak{F}_1$ and α_0^i formulas for $i \in 3$ such that $\mathfrak{M}, w \vDash \alpha_0^i$ iff $w = a_0^i$. Then for all $\langle n, i \rangle, w \in W_1$,*

$$\mathfrak{M}, w \vDash \alpha_n^i \quad \textit{iff} \quad w = a_n^i.$$

Proof. By a trivial induction. □

A cwf-incomplete logic. We combine this ladder with an infinite anti-chain and an ascending sequence as described. In fact, we construct a family of frames by varying the ascending sequence at the bottom.

Let us call a function $f\colon \omega \to \omega + 1$ *admissible* iff it is nowhere 0 and $f(0) = 1$. Define, for a given admissible function f, the frame $\mathfrak{F}_f = \langle W_f, F_f \rangle$ by adding to $\mathfrak{F}_1$ new distinct points b_n for each $n \in \omega$ and c_n^i for each $n \in \omega$ and $i \in f(n)$. Let R_f be the reflexive transitive closure of

$$R_1 \cup \left\{ \langle c_n^i, b_n \rangle, \langle b_n, a_n^j \rangle \mid n \in \omega, i \in f(n), j \in 3 \right\} \cup \left\{ \langle c_n^i, c_n^j \rangle \mid n \in \omega, i, j \in f(n) \right\}.$$

An example of such a frame is shown in Figure 4.3.

The proof now consists of roughly three steps. First we define a valuation on $\mathfrak{F}_f$, and in a sense characterise this valuation by a modal formula. Second, under this valuation, we characterise the points of $\mathfrak{F}_f$ by formulas. This will give us a formula γ_n that precisely characterises the points c_n^-. Finally we combine the characterisations of the valuation and the points c_n^- into formulas φ_n as in Proposition 4.10.

We want to characterise each of the a_0^- by an atomic proposition, so let $p_0, \ldots, p_2$ be distinct atomic propositions. Since the points a_0^- are 'indistinguishable from each

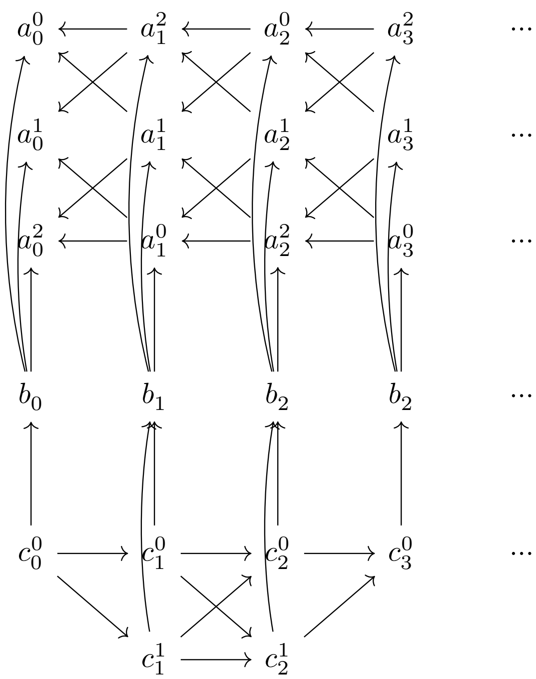

Figure 4.3: *The reflexive and transitive frame* $\mathfrak{F}_f$*, in case* $f(0) = f(3) = 1$ *and* $f(1) = f(2) = 2$.

other', we define a valuation for each permutation of the atomic propositions. For a permutation $\sigma\colon 3 \to 3$, define a valuation

$$\mathfrak{V}_\sigma(p_i) := \left\{a_0^{\sigma(i)}\right\}.$$

Write $\mathfrak{M}_{f,\sigma} := \left\langle \mathfrak{F}_f, \mathfrak{V}_\sigma \right\rangle$.

We characterise these valuations using a modal formula ψ_0. Define

$$\begin{aligned}
\psi_1 &:= \bigwedge\left\{\Box\left(p_i \to \neg p_j\right) \mid i, j \in 3, i \neq j\right\},\\
\psi_2 &:= \bigwedge\{\Box(p_i \to \Box p_i) \mid i \in 3\},\\
\psi_3 &:= \bigwedge\{\Diamond p_i \mid i \in 3\},\\
\psi_0 &:= \psi_1 \wedge \psi_2 \wedge \psi_3.
\end{aligned}$$

The 'characterisation' is given by the following two lemmata.

Lemma 4.12. *Let f be an admissible function and $\sigma\colon 3 \to 3$ a permutation. Then for all $n \in \omega$ and $j \in f(n)$,*

$$\mathfrak{M}_{f,\sigma}, c_n^j \vDash \psi_0.$$

Proof. Easy to check. □

Lemma 4.13. *Let f be admissible, $w \in W_f$ and $\mathfrak{F} = \langle W, R\rangle$ the subframe of $\mathfrak{F}_f$ generated by w. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a model on $\mathfrak{F}$ such that $\mathfrak{M}, w \vDash \psi_0$. Then $a_0^i \in W$ for all $i \in 3$. Moreover, there exists a permutation $\sigma\colon 3 \to 3$ such that for all $i \in 3$, $\mathfrak{V}(p_{\sigma(i)}) = \{a_0^i\}$.*

Proof. By ψ_1 the sets $\mathfrak{V}(p_i)$ are disjoint, by ψ_2 they are upsets and by ψ_3 they are non-empty. Since in $\mathfrak{F}_f$ every upset contains one of the three final points a_0^i, we see that $a_0^i \in W$ and $a_0^i \in \mathfrak{V}(p_{\sigma(i)})$ for some permutation $\sigma\colon 3 \to 3$. Now note that in $\mathfrak{F}_f$, any non-final point sees at least two final points. By the disjointness it follows that $\mathfrak{V}(p_{\sigma(i)}) = \{a_0^i\}$. □

Hence we can use $p_{\sigma(i)}$ as the formula α_0^i describing a_0^i. Now the formulas α_n^i defined earlier are satisfied precisely in the point a_n^i. Having these formulas dependent on the permutation σ however can be problematic. Therefore we define $\alpha_0^i := p_i$ independent of such permutation. For points outside the ladder of a_-^- points, the permutation is irrelevant, as they always see either all or none of the a_n^-.

Lemma 4.14. *Let f be admissible, $\sigma\colon 3 \to 3$ a permutation. Then for all $n \in \omega$, $i \in 3$, and $w \in W_f$ we have*

(i) $\mathfrak{M}_{f,\sigma}, w \vDash \alpha_n^{\sigma(i)}$ *iff* $w = a_n^i$,

(ii) $\mathfrak{M}_{f,\sigma}, w \vDash \bigwedge\{\Diamond \alpha_n^i \mid i \in 3\}$ *iff for all* $i \in 3$, $R(w, a_n^i)$, *and*

(iii) $\mathfrak{M}_{f,\sigma}, w \vDash \bigwedge\{\neg\Diamond\alpha_n^i \mid i \in 3\}$ *iff for all* $i \in 3$, $\neg R(w, a_n^i)$.

Proof. Clearly, the latter two statements trivially follow from the first.

When w is one of the a-points, this follows by induction on n, like in Proposition 4.11. Any other point sees a_m^j for some j iff it sees it for all j. Hence it cannot satisfy any α_{m+1}^j for any m and j. Since α_0^j is only satisfied in a final a-point, that concludes the proof. □

The notable fact about the latter two statements is that they are independent of σ. This allows us to characterise the $b_\_$ and $c_\_^-$ points of $\mathfrak{F}_f$ by formulas. Define, for $n \in \omega$,

$$\begin{aligned}
\beta_n &:= \bigwedge\{\alpha_n^i \mid i \in 3\} \wedge \bigwedge\{\neg\Diamond\alpha_{n+1}^i \mid i \in 3\},\\
\gamma_0 &:= \Diamond\beta_0,\\
\gamma_{n+1} &:= \Diamond\beta_{n+1} \wedge \neg\Diamond\beta_n,\\
\varphi_n &:= \psi_0 \wedge \gamma_n.
\end{aligned}$$

Lemma 4.15. *Let f be admissible, $\sigma\colon 3 \to 3$ a permutation. Then for all $n \in \omega$ and $w \in W_f$ we have*

(i) $\mathfrak{M}_{f,\sigma}, w \vDash \beta_n$ *iff* $w = b_n$, *and*

(ii) $\mathfrak{M}_{f,\sigma}, w \vDash \gamma_n$ *iff* $w \in \{c_n^i \mid i \in f(n)\}$.

Proof.

(i) Trivial, as b_n is the only point that sees all a_n^- but none of the a_{n+1}^-.

(ii) Follows from the first statement and the fact that the c_n^- are the only points seeing b_n but not b_{n-1} (when $n \geq 1$).

□

Finally we want to show that the formulas from Proposition 4.10 are valid on $\mathfrak{F}_f$. We first show this for $\mathfrak{M}_{f,\sigma}$, and then use the characterising property of ψ_0 to extend this to all models on $\mathfrak{F}_f$.

Lemma 4.16. *Let f be admissible, $\sigma\colon 3 \to 3$ a permutation, $n \in \omega$. Then*

$$\begin{aligned}
&\mathfrak{M}_{f,\sigma} \vDash \varphi_n \to \Diamond\varphi_{n+1}, \text{ and}\\
&\mathfrak{M}_{f,\sigma} \vDash \varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\}.
\end{aligned}$$

Proof. By Lemma 4.12, ψ_0 is satisfied in all $c_\_^-$, and by Lemma 4.15 γ_n is satisfied precisely on the c_n^- points. Hence φ_n is satisfied precisely on the c_n^- points, and the claim easily follows. □

Proposition 4.17. *Let f be an admissible function. Then $\neg\varphi_0 \notin \mathrm{Log}(\mathfrak{F}_f)$ and*

$$\left\{\varphi_n \to \Diamond\varphi_{n+1}, \varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\} \mid n \in \omega\right\} \subseteq \mathrm{Log}(\mathfrak{F}_f).$$

Proof. For $\neg\varphi_0 \notin \mathrm{Log}(\mathfrak{F}_f)$, note that $\varphi_0 = \psi_0 \wedge \gamma_0$ is satisfiable on $\mathfrak{F}_f$. By Lemma 4.12 $\mathfrak{M}_{f,\sigma}, c_0^0 \vDash \psi_0$ for some permutation $\sigma\colon 3 \to 3$, e.g. the identity on 3. By Lemma 4.15 $\mathfrak{M}_{f,\sigma}, c_0^0 \vDash \gamma_0$.

For the second claim, let $n \in \omega$. Then both formulas have $\varphi_n = \psi_0 \wedge \gamma_n$ as antecedent. Let $\mathfrak{M}$ be a model on $\mathfrak{F}_f$ and x a point of $\mathfrak{F}_f$ such that $\mathfrak{M}, x \vDash \varphi_n$. By Lemma 4.13 there exists a permutation $\sigma\colon 3 \to 3$ such that $\mathfrak{M} = \mathfrak{M}_{f,\sigma}$. Now applying the previous proposition finishes the proof. □

By this proposition, $\mathrm{Log}(\mathfrak{F}_f)$ satisfies all the requirements of Proposition 4.10. Hence $\mathrm{Log}(\mathfrak{F}_f)$ is CWF-model incomplete. Since it is the logic of a frame, it is Kripke complete.

4.4 Continuum CWF-Degrees

We continue to work towards our end-goal of proving the existence of continuum CWF-degrees. In fact we prove two similar statements. First, there exist infinitely many continuumly sized CWF-*model* degrees over the extensions of **S4**. Second, there exists infinitely many continuumly sized CWF-*frame* degrees over the *Kripke complete* extensions of **S4**. Note that neither statement implies the other. The proofs for these two statements are very similar. In fact, we will only prove the former in detail, and afterwards note the modifications to make to this proof in order to prove the second result.

Using the frames $\mathfrak{F}_f$ introduced in the previous section, we introduce continuumly many Kripke complete logics that share their CWF-model degree. Proving that these logics are all distinct actually turns out to be the hardest part. Here the results from the previous section will come to the rescue. Finally, we extend the result from a single degree to infinitely many degrees.

The logics. Let us endow the admissible functions with the pointwise ordering, denoted by $\leq$. Clearly, when $f \leq g$ then $\mathfrak{F}_f$ is a subframe of $\mathfrak{F}_g$. In fact it is also a p-morphic image.

Lemma 4.18. *Let f, g be admissible functions with $f \leq g$. Then $\mathfrak{F}_f$ is a p-morphic image of $\mathfrak{F}_g$.*

Proof. Trivial, since for all $i, j \in g(n)$,

$$R_g(c_n^i) \setminus \{c_n^i\} = R_g\left(c_n^j\right) \setminus \left\{c_n^j\right\},$$

so they can be identified as needed by a p-morphism. □

We define the logics that will serve as elements for the degrees. For f, g admissible with $f \leq g$, define

$$L_f^g := \operatorname{Log}(\mathfrak{F}_f) \cap \operatorname{Log}\Big(\operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)\Big).$$

This is the logic of the frame $\mathfrak{F}_f$ and all CWF general frames of $\operatorname{Log}(\mathfrak{F}_g)$.

These logics are constructed so that for a constant g and varying f, they have the same CWF general frames (and hence Kripke models):

Lemma 4.19. *Let f, g be admissible functions with $f \leq g$. Then*

(i) $\operatorname{gFr}_{\mathrm{cwf}}(L_g^g) = \operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)$,

(ii) $\operatorname{gFr}\big(L_f^g\big) \subseteq \operatorname{gFr}(L_g^g)$, *and*

(iii) $\operatorname{gFr}_{\mathrm{cwf}}\big(L_f^g\big) = \operatorname{gFr}_{\mathrm{cwf}}(L_g^g)$.

Proof.

(i) By Proposition 3.9 (vi), $\operatorname{Log}(\mathfrak{F}_g) \subseteq \operatorname{Log}\Big(\operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)\Big)$. Therefore

$$\begin{aligned} L_g^g = \operatorname{Log}(\mathfrak{F}_g) \cap \operatorname{Log}\Big(\operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)\Big) &\subseteq \operatorname{Log}(\mathfrak{F}_g) = \operatorname{Log}(\mathfrak{F}_g) \cap \operatorname{Log}(\mathfrak{F}_g) \\ &\subseteq \operatorname{Log}(\mathfrak{F}_g) \cap \operatorname{Log}\Big(\operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)\Big) = L_g^g, \end{aligned}$$

so $L_g^g = \operatorname{Log}(\mathfrak{F}_g)$.

(ii) Clearly, it suffices to show $\operatorname{Log}(\mathfrak{F}_f) \subseteq \operatorname{Log}(\mathfrak{F}_g)$. This is the case since, by Lemma 4.18, $\mathfrak{F}_f$ is a p-morphic image of $\mathfrak{F}_g$.

(iii) The $\subseteq$ inclusion immediately follows from (ii). For the $\supseteq$ inclusion, note that by (i), $\operatorname{gFr}_{\mathrm{cwf}}(L_g^g) = \operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)$. Now let $\mathfrak{f} \in \operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)$. Then it is a general frame of $\operatorname{Log}\Big(\operatorname{gFr}_{\mathrm{cwf}}\big(\operatorname{Log}(\mathfrak{F}_g)\big)\Big)$, and hence of L_f^g.

□

Hence for a given admissible function g, the logics $L_{\_}^g$ are in a single CWF-model degree. For $g \colon \omega \to \omega + 1$ that sends 0 to 1 and all other numbers to 2, there are continuumly many admissible functions $f \leq g$. It is tempting to conclude that the CWF-model degree of L_g^g is continuumly sized. However, this requires one more property: all $L_{\_}^g$ are distinct. We will prove this using Propositions 4.10 and 4.17.

Differentiating the logics. Let $q_\_$ be an injective sequence of atomic propositions all distinct from $p_0, \dots, p_2$. Define, for $m, n \in \omega$, formulas

$$\chi_n^m := \Big(\Box\Big(\bigwedge\{q_i \to \neg q_j \mid i, j \in m, i \neq j\}\Big) \wedge \bigwedge\{\Diamond(\gamma_n \wedge q_i) \mid i \in m\}\Big) \to \Box\Big(\gamma_n \to \bigvee\{q_i \mid i \in m\}\Big).$$

A rooted transitive frame validates χ_n^m iff in any model on it, there are at most m points satisfying γ_n.

Lemma 4.20. *Let f be admissible, $m, n \in \omega$. Then $\mathfrak{F}_f \vDash \varphi_0 \to \chi_n^m$ iff $f(n) \leq m$.*

Proof. Easy, since φ_0 can be satisfied precisely in and only in c_0^0 with one of the valuations $\mathfrak{V}_\sigma$. Under this valuation precisely the points c_n^- satisfy γ_n, so there are precisely $f(n)$ distinct points satisfying γ_n. □

Lemma 4.21. *Let f, g be admissible, $m, n \in \omega$. Then $\varphi_0 \to \chi_n^m \in L_f^g$ iff $f(n) \leq m$.*

Proof. ($\Rightarrow$) Since $\mathfrak{F}_f$ is a frame of L_f^g this follows immediately from the previous lemma.

($\Leftarrow$) Assume $f(n) \leq m$. By the previous lemma $\varphi_0 \to \chi_n^m \in \mathrm{Log}(\mathfrak{F}_f)$. Hence it suffices to show $\varphi_0 \to \chi_n^m \in \mathrm{Log}\big(\mathrm{gFr}_{\mathrm{cwf}}(\mathrm{Log}(\mathfrak{F}_g))\big)$. But by Propositions 4.10 and 4.17 $\neg\varphi_0$ is valid on any CWF model of $\mathrm{Log}(\mathfrak{F}_g)$, so $\neg\varphi_0 \in \mathrm{Log}\big(\mathrm{gFr}_{\mathrm{cwf}}(\mathrm{Log}(\mathfrak{F}_g))\big)$. It follows that any implication with φ_0 as antecedent is also in the logic. □

Hence these logics differ, i.e. L_f^g is injective in f. We conclude:

Proposition 4.22. *Let g be an admissble function such that $g(n) \geq 2$ for infinitely many $n \in \omega$. Then $\mathrm{Log}(\mathfrak{F}_g)$ has a continuumly sized CWF-model degree over* NExt(**S4**).

Proof. Clearly there are continuumly many admissible functions f under g. By Lemma 4.19 all L_f^g have the same class of CWF general frames and hence CWF Kripke models, so they are in a single CWF-model degree.

The frames $\mathfrak{F}_f$ are preorders so $\mathrm{Log}(\mathfrak{F}_f)$ extends **S4**. By Proposition 3.9 (vi), $\mathrm{Log}\big(\mathrm{gFr}_{\mathrm{cwf}}(\mathrm{Log}(\mathfrak{F}_g))\big)$ extends $\mathrm{Log}(\mathfrak{F}_f)$. Hence $L_f^g := \mathrm{Log}(\mathfrak{F}_f) \cap \mathrm{Log}\big(\mathrm{gFr}_{\mathrm{cwf}}(\mathrm{Log}(\mathfrak{F}_g))\big)$ extends **S4**. Finally, by Lemma 4.21 the map $f \mapsto L_f^g$ is injective. □

Cwf-frame degrees. For the analogous result about CWF-frame degrees over Kripke complete extensions of **S4**, we define slightly different logics. For f, g admissible with $f \leq g$, define

$$L_f'^g := \mathrm{Log}(\mathfrak{F}_f) \cap \mathrm{Log}\big(\mathrm{Fr}_{\mathrm{cwf}}(\mathrm{Log}(\mathfrak{F}_g))\big).$$

The only difference with L^g_f is that we take CWF Kripke frames instead of CWF general frames.

An analogue of Lemma 4.21 where L^-_- is replaced by L'^-_- follows with the exact same proof. In Lemma 4.19 all mentions of general frame classes need to be replaced by Kripke frame classes. The analogue of Proposition 4.22 becomes the following.

Proposition 4.23. *Let g be an admissble function such that $g(n) \geq 2$ for infinitely many $n \in \omega$. Then* $\mathrm{Log}(\mathfrak{F}_g)$ *has a continuumly sized* CWF*-frame degree over* Kripke $\cap$ NExt(**S4**).

Again, the proof goes analogous to that of Proposition 4.22. As an extra step, we need to show that L'^g_f is Kripke complete. For this, note that it is defined as the intersection of two logics of frame classes. As the intersection of Kripke complete logics is Kripke complete, we are done.

Infinitely many degrees. Finally we show that there exists infinitely many of these CWF-degrees. Note that if two logics have the same CWF-frame degree then they also have the same CWF-model degree. Therefore, by Propositions 4.22 and 4.23, it suffices to show that there exist infinitely many admissible functions g with $g(n) \geq 2$ for infinitely many n, such that each of the $\mathrm{Log}(\mathfrak{F}_g)$ has a different CWF-frame degree.

Define a sequence of admissible functions g_- by

$$g_n(i) := \begin{cases} 1 & \text{if } i = 0, \\ n+2 & \text{if } i = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Then every g_n is admissible and $g_n(i) \geq 2$ for infinitely many i.

We use Fine-Rautenberg formulas, as introduced in Theorem 2.76, to show that for $m \neq n$, the logics $\mathrm{Log}(\mathfrak{F}_m)$ and $\mathrm{Log}(\mathfrak{F}_n)$ have different classes of CWF frames. Let $\mathfrak{G}_n$ a reflexive transitive frame consisting of a root d_0, above it an anti-chain of $n+2$ points $e_0, \dots, e_{n+1}$, above this a two point cluster d_1, d_2 and finally a single point d_3 on top, as depicted in Figure 4.4.

Lemma 4.24. *Let $m, n \in \omega$. Then $\mathfrak{G}_m$ is a p-morphic image of (a generated subframe of) $\mathfrak{F}_{g_n}$ iff $m \leq n$.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{G}_m$ is a p-morphic image of a generated subframe of $\mathfrak{F}_{g_n}$. Note that only points in or below a proper cluster or deep points can be mapped to points in a proper cluster by a p-morphism. Since $\mathfrak{F}_{g_n}$ contains no proper clusters, only c^-_- points can be sent to d_0, d_1, d_2 and the e_-. By monotonicity of the p-morphism, the anti-chain of e_- points can only be the image of an anti-chain in $\mathfrak{F}_{g_n}$. But $\mathfrak{F}_{g_n}$ contains only anti-chains of size up to $n+2$ in the c^-_- points, and the anti-chain in $\mathfrak{G}_n$ has size $m+2$. Hence $m \geq n$.

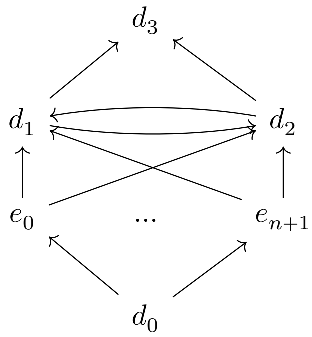

Figure 4.4: *The reflexive transitive frame $\mathfrak{G}_n$, consisting of a root d_0, an anti-chain of $n+2$ points $e_\_$, a cluster d_1, d_2 and a top d_3.*

($\Leftarrow$) Suppose $m \leq n$. Define a p-morphism h from $\mathfrak{F}_{g_n}$ to $\mathfrak{G}$ by

$$h(c_0^0) := d_0, \qquad h(c_1^i) := e_{\min(i,m)},$$
$$h(c_{2k+2}^i) := d_1, \qquad h(c_{2k+3}^i) := d_2,$$
$$h(b_k) := d_3, \qquad h(a_k^i) := d_3,$$

where k ranges over ω and $\min(i, m)$ denotes the minimum of i and m. It is easy to check that h is indeed a p-morphism as claimed. □

Using Fine-Rautenberg formulas for the frames $\mathfrak{G}_n$, it follows that each $\mathrm{Log}\left(\mathfrak{F}_{g_n}\right)$ has a different set of *finite* (hence in particular CWF) Kripke frames.

Lemma 4.25. *Let $m, n \in \omega$ such that $n < m$. Then $\mathfrak{G}_m$ is a finite frame of* $\mathrm{Log}\left(\mathfrak{F}_{g_m}\right)$ *but not of* $\mathrm{Log}\left(\mathfrak{F}_{g_n}\right)$.

Proof. The former claim is trivial, since by the previous lemma $\mathfrak{G}_m$ is a p-morphic image of $\mathfrak{F}_{g_m}$. For the latter, we use Fine-Rautenberg formulas. By Theorem 2.76 there exists a formula $\chi(\mathfrak{G}_m)$ such that $\mathfrak{F}_g \not\models \chi(\mathfrak{G}_m)$ iff $\mathfrak{G}_m$ is a p-morphic image of a generated subframe of $\mathfrak{F}_g$. By the previous lemma, since $m > n$, $\mathfrak{G}_m$ is not a p-morphic image of a generated subframe of $\mathfrak{F}_{g_n}$. We conclude that $\mathfrak{F}_{g_n} \models \chi(\mathfrak{G}_m)$.

Obviously $\mathfrak{G}_m$ is a p-morphic image of a generated subframe of itself. Hence $\mathfrak{G}_m \not\models \chi(\mathfrak{G}_m)$. Therefore, $\mathfrak{G}_m$ is not a frame of $\mathrm{Log}\left(\mathfrak{F}_{g_n}\right)$. □

We conclude:

Theorem 4.26. *There exists infinitely many continuumly sized CWF-model degrees over* NExt(**S4**) *and infinitely many continuumly sized CWF-frame degrees over* Kripke ∩ NExt(**S4**).

Proof. Define $\Lambda_n := \mathrm{Log}\left(\mathfrak{F}_{g_n}\right)$ for each $n \in \omega$. By Propositions 4.22 and 4.23 Λ_n has a continuum sized CWF-model degree over NExt(**S4**) and a continuum sized CWF-frame degree over Kripke $\cap$ NExt(**S4**). By the previous lemma, no two $\Lambda_\_$ are in a single finite-frame degree, so in particular also not in a single CWF-frame or CWF-model degree. □

We leave it as an open problem whether there exist countably infinite or finite but non-singleton CWF-frame degrees, either over all extensions of **K4** or **S4**, or over the Kripke complete such extensions. These questions seem significantly more difficult to answer than the continuum case, but answering any one of them would greatly improve our understanding of CWF-frame degrees. A potentially more tractable question that we leave open, is whether CWF-frame completeness and CWF-model completeness differ, as in the WF-case (see Section 3.6).

Chapter 5

Quasi-Canonicity

In this short chapter we apply the techniques from Section 4.3 in a different setting, to establish results about the recently introduced notion of *quasi-canonicity*. In particular, we answer the question posed by Takapui [41] of whether **GL** is quasi-canonical negatively.

5.1 Introduction

While studying topological d-semantics for **GL** and its extensions, Takapui [42] introduced the following property for modal logics, weaker than canonicity but strong enough for proving some interesting results [42]. As such it is called *quasi-canonicity*. Recall that for a general frame $\mathfrak{f}$, the underlying Kripke frame is denoted by $\mathfrak{f}_{\#}$.

Definition 5.1 (Quasi-canonicity)**.** A logic Λ is called *quasi-canonical* iff for every extension $\Lambda' \in \mathrm{NExt}(\Lambda)$, Λ' is complete w.r.t.

$$\big\{\mathfrak{f} \text{ general frame} \bigm| \mathfrak{f} \vDash \Lambda', \mathfrak{f}_{\#} \vDash \Lambda\big\}. \tag{5.1}$$

It is easily seen that this notion lies in-between canonicity and Kripke completeness [42], the proofs for which are also included in the next section. In fact, we improve this claim by showing that quasi-canonicity is strictly weaker than canonicity.

Takapui [41] poses the question whether **GL** is quasi-canonical. We answer this question negatively using methods developed in Section 4.3, and prove the same for **Grz**. The former is used by Takapui [42] to show that a particular construction does not apply to all extensions of **GL**. In joint work with Takapui [42, Section 6.2], we prove an analogue of this result for **GL** in the setting of topological d-semantics.

The chapter is organised as follows. In the next section we show that quasi-canonicity lies strictly in-between canonicity and Kripke completeness, strengthening a result of Takapui [42]. In Sections 5.3 and 5.4 we show that **Grz** and **GL** respectively are not quasi-canonical, thus negatively answering the question posed by Takapui [41].

5.2 Quasi-Canonicity and Canonicity

It is easily seen that quasi-canonicity lies in-between canonicity and Kripke completeness in strength, as is shown by Takapui [42]. In this section we strengthen this by showing that quasi-canonicity is strictly weaker than canonicity. We first state the former fact in the following two propositions. As Takapui [42] is as of yet to appear, we do provide proofs for these simple propositions, but note that both are due to Takapui [42].

Proposition 5.2 ([42, Theorem 6.1])**.** *Any canonical logic is quasi-canonical.*

Proof. Let Λ be a canonical modal logic, and $\Lambda' \in \mathrm{NExt}(\Lambda)$. Then Λ' is complete w.r.t. its descriptive frames, i.e.

$$\{\mathfrak{f} \text{ descriptive frame} \mid \mathfrak{f} \vDash \Lambda'\}. \tag{5.2}$$

But any such descriptive frame $\mathfrak{f}$ is then a descriptive frame of Λ, and since Λ is canonical, $\mathfrak{f}_\# \vDash \Lambda$. □

Proposition 5.3 ([42, Theorem 6.2])**.** *Any quasi-canonical logic is Kripke complete.*

Proof. Let Λ be a quasi-canonical modal logic. Since $\Lambda \in \mathrm{NExt}(\Lambda)$ is an extension of itself, it is complete w.r.t.

$$\left\{\mathfrak{f} \text{ general frame} \;\middle|\; \mathfrak{f} \vDash \Lambda, \mathfrak{f}_\# \vDash \Lambda\right\}. \tag{5.3}$$

Obviously $\mathfrak{f}_\# \vDash \Lambda$ implies $\mathfrak{f} \vDash \Lambda$, so Λ is complete w.r.t. general frames over Kripke Λ-frames. But then it is complete w.r.t. only its Kripke frames as well. □

We now show that quasi-canonicity is *strictly* weaker than canonicity. Recall the bounded width axiom bw_n from Section 2.11. We show, using Fine's finite width theorem, that $\mathbf{GLBW}_n$ and $\mathbf{GrzBW}_n$ are quasi-canonical for all $n \in \omega$, even though they are known not to be canonical. We start with the following observation:

Proposition 5.4. *Let Λ be a logic for which every extension is Kripke complete. Then Λ is quasi-canonical.*

Proof. Let $\Lambda' \in \mathrm{NExt}(\Lambda)$. Then Λ' is complete w.r.t. its Kripke frames, so also w.r.t. general frames on Λ'-frames. Note that every frame of Λ' is a frame of Λ. □

Recall Fine's finite width theorem, as stated in Theorem 4.1. It states that any extension of **K4**, complete w.r.t. its general frames of finite width is CWF-frame complete. In particular, this applies to $\mathbf{GLBW}_n$ and $\mathbf{GrzBW}_n$.

Proposition 5.5. *Let $n \in \omega$ and $\Lambda \in \mathrm{NExt}(\mathbf{K4BW}_n)$. Then Λ is quasi-canonical.*

Proof. Let $\Lambda' \in \mathrm{NExt}(\Lambda)$. Every rooted refined frame of Λ' is of width at most n [11, Proposition 10.32]. Since any logic is complete w.r.t. its rooted descriptive frames, Λ' is complete w.r.t. its general frames of finite width. By Theorem 4.1 it is Kripke complete. Hence every extension of Λ is Kripke complete, so by Proposition 5.4 Λ is quasi-canonical. □

We conclude that quasi-canonicity does not imply canonicity.

Proposition 5.6. **Grz.3** *is quasi-canonical but not canonical.*

Proof. Note that $\mathbf{Grz.3} = \mathbf{GrzBW}_1$, and hence it is quasi-canonical by Proposition 5.5. It is a well-known fact that **Grz.3** is not canonical, and this also follows from Theorem 6.19. □

In the next two sections we will show that **Grz** and **GL** are not quasi-canonical. Since both of these logics are Kripke complete, we can conclude that (unsurprisingly) quasi-canonicity is strictly stronger than Kripke completeness. Therefore, quasi-canonicity is strictly in-between canonicity and Kripke completeness.

5.3 Grz is not Quasi-Canonical

In this section we show that **Grz** is not quasi-canonical, using techniques from Chapter 4. We prove this by constructing an extension of **Grz** that is not complete w.r.t. its CWF general frames.

Recall that in Chapter 4 we developed a technique to create logics which are incomplete w.r.t. their CWF general frames. The problem for our current application is that these logics were Kripke complete and hence cannot be extensions of **Grz**. However, this is easily solved by restricting the set of admissibles on the frame used to define the logic.

We first derive a sufficient condition for satisfying **Grz**.

Definition 5.7. Let $\mathfrak{F}$ be a frame and $X \subseteq \mathfrak{F}_{\mathrm{w}}$. Then a sequence $x \colon \omega \to \mathfrak{F}_{\mathrm{w}}$ is said to *eventually decide* X iff there exists $n_0 \in \omega$ such that either

- for all $n \geq n_0$, $x(n) \in X$, or
- for all $n \geq n_0$, $x(n) \notin X$.

For a model $\mathfrak{M}$ on $\mathfrak{F}$ and a formula φ, x is said to *eventually decide* φ iff it eventually decides $[\![\varphi]\!]_{\mathfrak{M}}$.

Lemma 5.8. *Let $\mathfrak{F}$ be a preorder and $\mathfrak{M}$ a model on it. If for every formula φ and every ascending sequence $x \colon \omega \to \mathfrak{F}$, x eventually decides φ, then $\mathfrak{M} \vDash$ **Grz**.*

Proof. Suppose, in order to reach a contradiction, that $\mathfrak{M} \not\models \mathbf{Grz}$. Then $\mathfrak{M}$ refutes some substitution of *grz*, say

$$\mathfrak{M}, x_0 \not\models \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$$

for some point $x_0 \in \mathfrak{F}_{\mathrm{w}}$ and formula φ. Then x_0 refutes φ and for every successor y of x_0,

$$\mathfrak{M}, y \models \Box(\varphi \to \Box\varphi) \to \varphi.$$

Hence y either satisfies φ, or has a successor refuting $\varphi \to \Box\varphi$.

By induction we construct an ascending sequence $x \colon \omega \to \mathfrak{F}$ such that, for all $n \in \omega$, $x(2n)$ refutes φ and $x(2n+1)$ satisfies φ and refutes $\Box\varphi$. Define $x(0) := x_0$, and note that it refutes φ. Next, assume $x(n)$ is already defined, and note that since the sequence is ascending, it is a successor of x_0.

Case 1. If n is even, then $x(n)$ refutes φ so it has a successor z refuting $\varphi \to \Box\varphi$. Then z satisfies φ and refutes $\Box\varphi$. Set $x(n+1) := z$.

Case 2. If n is odd, then $x(n)$ refutes $\Box\varphi$ by assumption, so there is a successor z of it refuting φ. Set $x(n+1) := z$.

Now clearly x is an ascending sequence which does not eventually decide φ, contradicting the assumption. □

Recall the family of frames $\mathfrak{F}_f$ defined in Section 4.3. Let us write $\mathfrak{F}_1$ for the frame where f is set to the constant 1 function. Recall that there existed a formula φ_0 with the following properties:

- $\mathfrak{M}_1, c_0^0 \models \varphi_0$, where $\mathfrak{M}_1$ is the model on $\mathfrak{F}_1$ with three atomic propositions p_0, p_1, p_2 which sets p_i true precisely at a_0^i, and
- φ_0 is not satisfied in any CWF model of $\mathrm{Log}(\mathfrak{F}_1)$ (by Propositions 4.10 and 4.17).

We claim that this $\mathfrak{M}_1$ validates **Grz**.

Lemma 5.9. $\mathfrak{M}_1 \models \mathbf{Grz}$.

Proof. By Lemma 5.8 it suffices to show that any formula φ is eventually decided on every ascending sequence in $\mathfrak{M}_1$. We prove this by induction on the formula φ.

For atomic propositions it is clear, since every atomic proposition is only satisfied at a single point. The cases for conjunction, disjunction and negation are trivial.

So we are only left with the modal box. If no point in the sequence satisfies $\Box\psi$ then we are obviously done. So assume some point in the sequence satisfies $\Box\psi$. Then so do all successors, in particular all later points in the ascending sequence. Then clearly the sequence eventually decides $\Box\psi$. □

Hence so does the general frame $\mathfrak{f}_1$ induced by $\mathfrak{M}_1$, so $\mathbf{Grz} \subseteq \mathrm{Log}(\mathfrak{f}_1)$.

Lemma 5.10. *Let* Λ *be a logic such that* $\mathrm{Log}(\mathfrak{F}_1) \oplus \mathbf{Grz} \subseteq \Lambda \subseteq \mathrm{Log}(\mathfrak{f}_1)$*. Then* Λ *is not complete w.r.t. its general frames on* **Grz***-frames.*

Proof. Since φ_0 is satisfied in $\mathfrak{M}_1$, we conclude $\neg\varphi_0 \notin \mathrm{Log}(\mathfrak{f}_1)$. Therefore it suffices to show that φ_0 is not satisfied on any general frame of $\mathrm{Log}(\mathfrak{F}_1) \oplus \mathbf{Grz}$ on a **Grz**-frame. Now let $\mathfrak{M}$ be a $\mathrm{Log}(\mathfrak{F}_1)$-model on a **Grz**-frame. Then it is CWF. By Propositions 4.10 and 4.17 it does not satisfy φ_0. □

We conclude:

Theorem 5.11. **Grz** *is not quasi-canonical. In particular there exists an extension of* **Grz** *which is not complete w.r.t. general frames whose underlying Kripke frames are* CWF*.*

5.4 GL is not Quasi-Canonical

In this section we show, using a proof similar to that in the last section, that **GL** is not quasi-canonical, thereby answering the question raised by Takapui [41]. We again start by giving a sufficient condition for Kripke models to validate **GL**.

Definition 5.12 (Upper limit property)**.** An irreflexive transitive general frame $\mathfrak{f} = \langle \mathfrak{F}, A\rangle$ is said to have the *upper limit property* iff every $a \in A$ has the property that, whenever $x_\_ : \omega \to a$ is an ascending sequence in $\mathfrak{F}{\restriction}a$, then $\mathfrak{F}_{\mathrm{w}}^{\mathrm{upper}} \cap a$ contains a successor of x_0. An irreflexive transitive Kripke model is said to have the *upper limit property* iff its induced general frame has the upper limit property.

Lemma 5.13. *Let* $\mathfrak{f} = \langle \mathfrak{F}, A\rangle$ *be an irreflexive transitive general frame. If* $\mathfrak{f}$ *has the upper limit property then* $\mathfrak{f} \vDash \mathbf{GL}$*.*

Proof. Suppose, in order to reach a contradiction, that $\mathfrak{f} \nvDash \mathbf{GL}$. Then $\mathfrak{f}$ refutes a substitution of *gl*, say

$$\mathfrak{M}, x_0 \nvDash \Box(\Box\varphi \to \varphi) \to \Box\varphi$$

for some point $x_0 \in \mathfrak{F}_{\mathrm{w}}$ and formula φ. Then $\mathfrak{M}, x_0 \nvDash \Box\varphi$ and for all successors y of x_0, $\mathfrak{M}, y \vDash \Box\varphi \to \varphi$. Hence if y refutes φ then it has a successor that does so too.

By induction we construct an ascending ω-sequence $y_\_ : \omega \to \mathfrak{F}$ such that each $y_\_$ refutes φ in $\mathfrak{M}$. Note that x_0 refutes $\Box\varphi$, so it sees a point y_0 refuting φ. For the inductive step, assume x_n is defined and refutes φ. Then x_0 sees y_0 and $y_0 = y_n$ (if $n = 0$) or y_0 sees y_n (if $n > 0$), so by transitivity x_0 sees y_n. Since y_n refutes φ, it follows that it has a successor y_{n+1} which refutes φ.

Define $a := [\![\neg\varphi]\!]_{\mathfrak{M}}$. Then the sequence $y_\_$ is contained in a, and a is admissible. Moreover, a similar induction to the previous one shows that above any point in $a \cap R(x_0)$, there exists an ascending ω-sequence in a. Since $\mathfrak{F}$ is irreflexive and transitive, such sequence is strictly-ascending, so $a \cap R(x_0) \subseteq \mathfrak{F}_{\mathrm{w}}^{\mathrm{deep}}$. Hence $\mathfrak{f}$ does not have the upper limit property. □

The analogous result for Kripke models obviously follows.

We would like to use the frame $\mathfrak{F}_1$ from the previous section again. However, applying the previous lemma requires an irreflexive frame. Therefore, we take the irreflexivisation of the frame $\mathfrak{F}_1$ instead, call it $\mathfrak{F}_1'$. Define $\mathfrak{f}_1'$ to be the general frame on $\mathfrak{F}_1'$ generated by the sets $\{a_0^i\}$ for $i \in 3$. We claim that $\mathfrak{f}_1' \vDash \mathbf{GL}$.

Lemma 5.14. *$\mathfrak{f}_1' \vDash \mathbf{Grz}$.*

Proof. Note that $\mathfrak{F}_1$ is irreflexive and transitive. By Lemma 5.13 it suffices to show that $\mathfrak{f}_1'$ has the upper limit property.

We prove by induction on the generation of admissible sets a that if an admissible a contains infinitely many $b_\_$ or $c_\_$ points then there exists $n \in \omega$ such that for all $m \geq n$ and $i \in 3$, $a_m^i \in a$. For $\varnothing$ and the three generators, this is trivial as they do not contain any $b_\_$ or $c_\_$ points, and for $\mathfrak{F}_{1,\mathrm{w}}'$ it is trivial since it contains all $a_\_^-$ points.

For binary union, assume a_1 and a_2 are admissible and $a_1 \cup a_2$ contains infinitely many $b_\_$ or $c_\_$ points. Then clearly one of a_1 and a_2 does so too, so by induction hypothesis it contains all a_m^- from some m onwards. Hence so does $a_1 \cup a_2$.

For binary intersection, assume a_1 and a_2 are admissible and $a_1 \cap a_2$ contains infinitely many $b_\_$ or $c_\_$ points. Then clearly both a_1 and a_2 do so too, and hence by the induction hypthesis contain all a_m^- for $m \geq n_1$ and $m \geq n_2$ respectively. Define n to be the maximum of n_1 and n_2, and we are done.

For $\Box$, note that $\Box a$ is always an upset, so the claim trivially holds. For $\Diamond$, suppose a is admissible and $\Diamond a$ contains infinitely many $c_\_$ points. If a already contained infinitely many $b_\_$ or $c_\_$ points, then it contains a_m^- for $m \geq n$ for some n, and then $\Diamond a$ contains these for $m \geq n+1$. If a contains finitely many $b_\_$ and $c_\_$ then it must contain a_n^i for some $n \in \omega$ and $i \in 3$. Then $\Diamond a$ contains a_m^- for $m \geq n+2$.

It clearly follows that $\mathfrak{f}_1'$ has the upper limit property, for assume a is some admissible and $x_\_ : \omega \to a$ and ascending sequence in it. Then $x_\_$ must contain infinitely many $c_\_$, hence by the inductive argument there exists $n \in \omega$ such that $a_m^- \in a$ for all $m \geq n$. Then $a_n^0 \in \mathfrak{F}_{\mathrm{w}}^{\mathrm{upper}} \cap a$ and a_n^0 is seen by x_0. Hence $\mathfrak{f}_1' \vDash \mathbf{GL}$. ☐

Similar to in the previous section, we want to show that $\mathrm{Log}(\mathfrak{f}_1')$ is not complete w.r.t. its general frames on **Grz** frames. We again use Propositions 4.10 and 4.17 but since we took the irreflexivisation of the frame, we need to use the translation from Definition 2.12. We first prove a lemma that we can use $\overline{\Diamond}$ instead of $\Diamond$ in the hypotheses of Proposition 4.17.

Lemma 5.15. *Let $\varphi_\_$ be a ω-sequence of modal formulas and Λ a modal logic such that for all $n \in \omega$,*

$$\varphi_n \to \overline{\Diamond}\varphi_{n+1}, \varphi_n \to \bigwedge\{\neg\overline{\Diamond}\varphi_i \mid i < n\} \in \Lambda.$$

Then

$$\varphi_n \to \Diamond\varphi_{n+1}, \varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\} \in \Lambda.$$

Proof. From
$$\varphi_{n+1} \to \bigwedge\{\neg\overline{\Diamond}\varphi_i \mid i < n+1\}$$
one easily derives $\varphi_{n+1} \to \neg\overline{\Diamond}\varphi_n$. Combining this with $\varphi_n \to \overline{\Diamond}\varphi_{n+1}$ proves $\varphi_n \to \Diamond\varphi_{n+1}$. This proves the first formula.

For the second formula, note that $\neg\overline{\Diamond}\psi$ is a stronger statement than $\neg\Diamond\psi$, for any ψ. More formally, in any modal algebra $\mathfrak{A}$, $\neg\overline{\Diamond}a \leq \neg\Diamond a$ for any $a \in \mathfrak{A}$, where $\overline{\Diamond}a$ is a shorthand for $a \vee \Diamond a$. Hence $\varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\}$ follows from the version with $\overline{\Diamond}$. □

Lemma 5.16. *Let* Λ *be a logic such that* $\mathrm{Log}(\mathfrak{F}_1') \oplus \mathbf{GL} \subseteq \Lambda \subseteq \mathrm{Log}(\mathfrak{f}_1')$. *Then* Λ *is not complete w.r.t. its general frames on* **GL***-frames.*

Proof. Write $\mathfrak{M}_1'$ for the irreflexivisation of $\mathfrak{M}_1$ and note it is a Kripke model on $\mathfrak{f}_1'$. By Proposition 2.54 a formula φ is satisfied respectively valid on $\mathfrak{M}_1$ iff $\overline{\varphi}$ is satisfied respectively valid on $\mathfrak{M}_1'$.

Since φ_0 is satisfied in $\mathfrak{M}_1$, $\overline{\varphi_0}$ is so in $\mathfrak{M}_1'$, hence $\neg\overline{\varphi_0} \notin \mathrm{Log}(\mathfrak{f}_1')$. Therefore, it suffices to show that $\overline{\varphi_0}$ is not satisfied on any general frame of $\mathrm{Log}(\mathfrak{F}_1') \oplus \mathbf{GL}$ on a **GL**-frame. Now let $\mathfrak{M}$ be a $\mathrm{Log}(\mathfrak{F}_1')$-model on a **GL**-frame. Then it is CWF.

By Proposition 4.17
$$\left\{\varphi_n \to \Diamond\varphi_{n+1}, \varphi_n \to \bigwedge\{\neg\Diamond\varphi_i \mid i < n\} \;\middle|\; n \in \omega\right\} \subseteq \mathrm{Log}(\mathfrak{F}_1),$$
so by Proposition 2.54
$$\left\{\overline{\varphi_n} \to \overline{\Diamond}\overline{\varphi_{n+1}}, \overline{\varphi_n} \to \bigwedge\{\neg\overline{\Diamond}\overline{\varphi_i} \mid i < n\} \;\middle|\; n \in \omega\right\} \subseteq \mathrm{Log}(\mathfrak{F}_1').$$
Applying the previous lemma to the sequence of formulas $\overline{\varphi_-}$ proves the hypothesis of Proposition 4.17. Hence φ_0 is not satisfied on a CWF model of $\mathrm{Log}(\mathfrak{F}_1')$, so in particular not on $\mathfrak{M}$. □

We conclude:

Theorem 5.17. **GL** *is not quasi-canonical. In particular there exists an extension of* **GL** *which is not complete w.r.t. general frames whose underlying Kripke frames are* CWF.

Thus we answered the question posed by Takapui [41] whether **GL** is quasi-canonical. The negative answer might be disappointing; Takapui [42] uses it to show that a particular construction of him in the area of topological d-semantics does not apply to all extensions of **GL**. However, the negative answer is also not entirely unexpected, as **GL** is quite non-canonical, in several ways. For example, it is not 0-canonical, but then, neither is **GL.3**,[1] which *is* quasi-canonical, as we saw in Section 5.2. In joint work with Takapui [42, Section 6.2] we also give a topological analogue of Theorem 5.17, which again relies heavily on our work in Section 4.3.

[1]The 0-canonical frame of **GL.3**, and hence also **GL**, contains a reflexive point.

Chapter 6

Canonical Approximations

In this chapter we have another look at canonicity. This time we consider approximating logics with canonical ones.

6.1 Introduction

Many interesting classes of logics form *complete lattices*, i.e. lattices that have not only binary meets and joins, but meets and joins for arbitrary sets of elements. As such, it is possible to approximate arbitrary modal logics with logics from the class under consideration. Two specific instances of these approximations have already been studied in the literature. In the setting of super-intuitionistic logics, G. Bezhanishvili, N. Bezhanishvili and Ilin [4] and Ilin [24] study approximations for the complete lattices of subframe logics and stable logics, which they named *subframizations* and *stabilizations* respectively. We develop some general theory about approximations, and study approximations for the complete lattice of canonical modal logics.

Similar to the general theory of semantics and degrees studied in Sections 3.2 and 3.3, we develop a general theory of approximations in Section 6.2. In the following sections we apply this general notion of approximations to the class of canonical logics. In Section 6.3 we show that the class of canonical logics forms a complete lattice, but not a complete sublattice of the lattice of all normal modal logics. The former suffices for the theory of approximations to apply. Next, in Section 6.4 it is shown that the canonical approximation from above and the one from below equal for logics with the fmp. Finally, the last two sections compute the canonical approximations of **Grz.3** and **Grz.2** respectively.

6.2 Approximations

Many interesting classes of logics form complete lattices w.r.t. the subset-order. Then a logic outside the class under consideration can be approximated by logics in the class. This can be done *from above*, by approximating with extensions of the logic, or *from below*, by approximating with weakenings. Because we were considering a complete lattice, both approaches give rise to a closest approximant, which we will call the *approximation* from above or from below respectively.

Note that any of this is generic over the language under consideration, and what constitutes a *logic*, as long as the set of all logics $\mathcal{U}$ forms a complete lattice under some order $\leq$. We will always instantiate $\mathcal{U}$ with the set of normal modal logics and $\leq$ with the subset-order. In cases where it might be ambiguous in which complete lattice a meet or join is taken, we will indicate the lattice in the subscript, as in $\bigwedge_{\mathcal{U}}$.

Definition 6.1 (Approximation)**.** Let $\mathcal{X} \subseteq \mathcal{U}$ be a set of logics, which forms a complete lattice w.r.t. $\leq$. Define

$$\mathcal{X}_{\downarrow} : \mathcal{U} \to \mathcal{X} : \Lambda \mapsto \bigwedge\nolimits_{\mathcal{X}} \{\Lambda' \in \mathcal{X} \mid \Lambda \subseteq \Lambda'\}, \text{ and}$$
$$\mathcal{X}_{\uparrow} : \mathcal{U} \to \mathcal{X} : \Lambda \mapsto \bigvee\nolimits_{\mathcal{X}} \{\Lambda' \in \mathcal{X} \mid \Lambda' \subseteq \Lambda\}.$$

The former is called the *$\mathcal{X}$-approximation from above* and the latter the *$\mathcal{X}$-approximation from below*. Both $\mathcal{X}_{\downarrow}(\Lambda)$ and $\mathcal{X}_{\uparrow}(\Lambda)$ are called *$\mathcal{X}$-approximations* of Λ.

Remark 6.2. It is known from order theory that if $\mathcal{X}$ has either meets for all subsets or joins for all subsets, then it forms a complete lattice with, for any $\mathcal{Y} \subseteq \mathcal{X}$,

$$\bigwedge \mathcal{Y} = \bigvee \{\Lambda \in \mathcal{X} \mid \forall \Lambda' \in \mathcal{Y}.\ \Lambda \subseteq \Lambda'\}, \text{ and}$$
$$\bigvee \mathcal{Y} = \bigwedge \{\Lambda \in \mathcal{X} \mid \forall \Lambda' \in \mathcal{Y}.\ \Lambda' \subseteq \Lambda\}.$$

When $\mathcal{X}$ is a complete sublattice of $\mathcal{U}$ these approximations behave nicely.

Proposition 6.3. *Let $\mathcal{X} \subseteq \mathcal{U}$ be a complete meet-semi-sublattice of $\mathcal{U}$, i.e. $\bigwedge_{\mathcal{X}} = \bigwedge_{\mathcal{U}}$. Then*

$$\forall \Lambda \in \mathcal{U}.\ \Lambda \subseteq \mathcal{X}_{\downarrow}(\Lambda). \tag{6.1}$$

Similarly, if it is a complete join-semi-sublattice, i.e. $\bigvee_{\mathcal{X}} = \bigvee_{\mathcal{U}}$, then

$$\forall \Lambda \in \mathcal{U}.\ \mathcal{X}_{\uparrow}(\Lambda) \subseteq \Lambda.$$

Proof. Trivial, for let $\Lambda \in \mathcal{U}$. Then

$$\bigvee\nolimits_{\mathcal{U}} \{\Lambda' \in \mathcal{X} \mid \Lambda' \subseteq \Lambda\} \subseteq \Lambda \subseteq \bigwedge\nolimits_{\mathcal{U}} \{\Lambda' \in \mathcal{X} \mid \Lambda \subseteq \Lambda'\}.$$

□

Remark 6.4. The assumption that $\mathcal{X}$ is a complete semi-*sub*lattice is really necessary here. For example in the case that $\mathcal{X} := \text{Can}$ is the set of canonical logics, eq. (6.1) does not hold, as we will see in the next section.

The approximations from above and below can be expressed in terms of each other, as follows.

Proposition 6.5. *Let $\mathcal{X} \subseteq \mathcal{U}$ be a set of logics which forms a complete lattice w.r.t. $\leq$, and let $L \in \mathcal{U}$. Then*

$$\mathcal{X}_{\downarrow}(L) = \mathcal{X}_{\uparrow}\Big(\bigwedge\nolimits_{\mathcal{U}}\{\Lambda \in \mathcal{X} \mid L \subseteq \Lambda\}\Big), \tag{6.2}$$

and similarly

$$\mathcal{X}_{\uparrow}(L) = \mathcal{X}_{\downarrow}\Big(\bigvee\nolimits_{\mathcal{U}}\{\Lambda \in \mathcal{X} \mid \Lambda \subseteq L\}\Big).$$

Proof. This is a matter of writing out the definitions.

$$\begin{aligned}
\mathcal{X}_{\downarrow}(L) &= \bigwedge\nolimits_{\mathcal{X}}\{\Lambda \in \mathcal{X} \mid L \subseteq \Lambda\} \\
&= \bigvee\nolimits_{\mathcal{X}}\{\Lambda' \in \mathcal{X} \mid \forall \Lambda \in \{\Lambda \in \mathcal{X} \mid L \subseteq \Lambda\}.\ \Lambda' \subseteq \Lambda\} \\
&= \bigvee\nolimits_{\mathcal{X}}\Big\{\Lambda' \in \mathcal{X} \;\Big|\; \Lambda' \subseteq \bigwedge\nolimits_{\mathcal{U}}\{\Lambda \in \mathcal{X} \mid L \subseteq \Lambda\}\Big\} \\
&= \mathcal{X}_{\uparrow}\Big(\bigwedge\nolimits_{\mathcal{U}}\{\Lambda \in \mathcal{X} \mid L \subseteq \Lambda\}\Big).
\end{aligned}$$

The first equality is by definition of $\mathcal{X}_{\downarrow}$. For the second equality we use one of the equations from Remark 6.2. For the third, note that Λ' is below all elements of a set iff it is below the intersection. The final equality is the definition of $\mathcal{X}_{\uparrow}$.

The derivation of formula for $\mathcal{X}_{\uparrow}(L)$ is analogous. □

6.3 The Lattice of Canonical Logics

In order to apply the theory from the previous section to canonicity, we need to show that the canonical modal logics form a complete lattice. We prove this by showing that the canonical logics are closed under joins in $\mathcal{U}$, so they form a complete join-semi-sublattice of $\mathcal{U}$. We also show that the canonical logics are closed under binary intersections, but not under arbitrary ones. Even though the closure under intersections is stated as Problem 10.2 in Chagrov and Zakharyaschev [11], the proof turns out to be a relatively easy exercise.[1]

Let us write Can for the set of canonical modal logics. We first prove that it is closed under joins in $\mathcal{U}$, as this is the easiest to show.

Proposition 6.6. *Let $\mathcal{X} \subseteq \mathrm{Can}$. Then $\bigvee_{\mathcal{U}} \mathcal{X} \in \mathrm{Can}$. In other words, $\mathcal{X}$ forms a complete join-semi-sublattice of $\mathcal{U}$.*

[1]'Problem's in Chagrov and Zakharyaschev [11] are intended as major open problems in the field, and a significant number of them is still open.

Proof. Recall that by Proposition 2.88 a logic Λ is canonical iff for every descriptive frame $\mathfrak{f}$ of Λ, the underlying Kripke frame $\mathfrak{f}_\#$ is a frame of Λ. Let $\mathfrak{f}$ be a descriptive frame $\mathfrak{f}$ of $\bigvee_{\mathcal{U}} \mathcal{X}$. Then it is a descriptive frame of Λ for each $\Lambda \in \mathcal{X}$, and hence by the canonicity of these Λ, the underlying frame $\mathfrak{f}_\#$ is a frame of Λ for all $\Lambda \in \mathcal{X}$. Since $\bigvee_{\mathcal{U}} \mathcal{X}$ is by definition a least upper bound for these Λ, $\mathfrak{f}_\#$ is a frame of $\bigvee_{\mathcal{U}} \mathcal{X}$. □

By Remark 6.2 it follows that Can forms a complete lattice.

Next we consider the intersections of canonical logics. Problem 10.2 in Chagrov and Zakharyaschev [11] asks for 'canonicity' and '$\mathcal{D}$-persistence' separately, where their 'canonicity' coincides with what we call ω-canonicity and '$\mathcal{D}$-persistence' is precisely property (ii) in Proposition 2.88, hence equivalent to our notion of canonicity. To answer both questions at once, we show for all cardinals κ that the κ-canonical logics are closed under binary intersections.

Lemma 6.7. *Let Λ, Λ' be modal logics such that $\Lambda \subseteq \Lambda'$, and let P be some set meant to be used as atomic propositions. Then the P-canonical frame $\mathbb{F}_P^{\Lambda'} = \langle W', R' \rangle$ of Λ' is a generated subframe of the P-canonical frame $\mathbb{F}_P^{\Lambda} = \langle W, R \rangle$ of Λ.*

Proof. Let $\Gamma \in W'$ be a Λ'-MCS. Clearly it is also Λ-MCS. Therefore $W' \subseteq W$. Since the frame relation for canonical frames is purely defined in terms of the formulas in the MCSs, we see that $\mathbb{F}_P^{\Lambda'}$ is a subframe of $\mathbb{F}_P^{\Lambda}$.

We show that it is a generated subframe. Let $\Gamma \in W'$, $\Delta \in W$ such that $R(\Gamma, \Delta)$.

We claim that Δ extends Λ', for let $\varphi \in \Lambda'$. Then by necessitation $\Box\varphi \in \Lambda'$, and Γ is a Λ'-MCS, hence $\Box\varphi \in \Gamma$. Since $R(\Gamma, \Delta)$ and the definition of canonical frames, it follows that $\varphi \in \Delta$.

So Δ indeed extends Λ', as claimed. Since it is maximally consistent, it follows that it is a Λ'-MCS. Therefore $\Delta \in W'$. Hence W' forms an upset in $\mathbb{F}_P^{\Lambda}$. □

Lemma 6.8. *Let Λ_1, Λ_2 be modal logics and set $\Lambda_0 := \Lambda_1 \cap \Lambda_2$ their intersection. Then a Λ_0-MCS is either a Λ_1-MCS or a Λ_2-MCS.*

Proof. Let Γ be a Λ_0-MCS, and suppose it is neither Λ_1-consistent nor Λ_2-consistent. Then there exists formulas $\varphi_1, \varphi_2 \in \Gamma$ such that $\neg\varphi_i \in \Lambda_i$ for $i \in \{1, 2\}$. Clearly also $\neg(\varphi_1 \wedge \varphi_2) \in \Lambda_i$. Hence

$$\neg(\varphi_1 \wedge \varphi_2) \in \Lambda_1 \cap \Lambda_2 = \Lambda_0 \subseteq \Gamma.$$

But Γ is a Λ_0-MCS and $\varphi_1, \varphi_2 \in \Gamma$ so also $(\varphi_1 \wedge \varphi_2) \in \Gamma$. Hence Γ is inconsistent, contradiction. □

Theorem 6.9. *Let κ be a cardinal and Λ_1, Λ_2 be κ-canonical logics. Then $\Lambda_1 \cap \Lambda_2$ is κ-canonical.*

Proof. Write $\Lambda_0 := \Lambda_1 \cap \Lambda_2$, and recall that $\mathbb{F}_\kappa^{\Lambda_i}$ is the κ-canonical frame of Λ_i.

Suppose $\varphi \in \Lambda_0$ and Γ is a Λ_0-MCS (over the atomic propositions κ). By Lemma 6.8 it is a Λ_i-MCS for some $i \in \{1, 2\}$. By Lemma 6.7 $\mathbb{F}_\kappa^{\Lambda_i}$ is a generated subframe of $\mathbb{F}_\kappa^{\Lambda_0}$, and since Γ is a Λ_i-MCS, it point of this generated subframe. Since $\varphi \in \Lambda_i$ and Λ_i is κ-canonical, φ is valid on $\mathbb{F}_\kappa^{\Lambda_i}$. As generated subframes preserve validity, $\mathbb{F}_\kappa^{\Lambda_0}, \Gamma \vDash \varphi$. Since Γ and $\varphi \in \Lambda_0$ were arbitrary, Λ_0 is canonical.

□

It obviously follows that also the canonical logics are closed under binary intersection. Hence, besides forming a complete lattice, Can also forms a sublattice of the lattice of all modal logics $\mathcal{U}$, i.e. the binary meet and join of Can coincide with the ones from $\mathcal{U}$.

We finally show that the canonical logics are not closed under countable intersections. In particular, we show that **GL** is, in a rather strong sense, not canonical, but is the intersection of countably many logics that *are* canonical.

Proposition 6.10. *The set of canonical logics* Can *is not closed under countable intersections, so not a complete sublattice of* $\mathcal{U}$*.*

Proof. Define, for $n \in \omega$, $\Lambda_n := \mathbf{K4} \oplus \Box^n \bot$. It is easy to see that Λ_n extends **GL**, either via syntactic, algebraic or frame theoretic methods. Let us take the syntactic method. We need to prove that

$$gl = \Box(\Box p \to p) \to \Box p \in \Lambda_n.$$

For $n = 0$ or $n = 1$ this is trivial, as $\Box^0 \bot = \bot$ derives everything and $\Box\bot$ immediately derives $\Box p$. So suppose $n \geq 2$.

Recall that **GL** extends **K4**, so we can start reasoning in **K4**. First, repeatedly using the transitivity axiom and modus ponens shows that, for all $i \in \omega$,

$$\Box(\Box p \to p) \to \Box^{i+1}(\Box p \to p) \in \mathbf{K4}.$$

Repeatedly applying the K-axiom using modus ponens one obtains

$$\Box(\Box p \to p) \to (\Box^{i+2} p \to \Box^{i+1} p) \in \mathbf{K4}.$$

Repeatedly applying these, for i decreasing from $n - 2$ to 0, one derives,

$$\Box(\Box p \to p) \to (\Box^n p \to \Box p) \in \mathbf{K4}.$$

Now in Λ_n, $\Box^n \bot$ is a theorem, so by monotonicity of $\Box$, $\Box^n p$ is so too. Using some classical reasoning, we derive

$$\Box(\Box p \to p) \to \Box p \in \Lambda_n,$$

as required.

Therefore indeed $\mathbf{GL} \subseteq \Lambda_n$ for all n, and hence

$$\mathbf{GL} \subseteq \bigcap \{\Lambda_n \mid n \in \omega\}.$$

We want to prove the converse inclusion as well.

It is well-known that **GL** has the fmp; in fact it is even sound and complete w.r.t. finite irreflexive transitive trees [see e.g. 7, Theorem 4.45]. In fact the completeness w.r.t. finite frames also follows from Fine's selective filtration via maximal points method that we discuss in Section 8.4. Recall that **GL** frames are transitive, CWF and irreflexive. Therefore a finite **GL** frame with n points does not contain any ascending sequence of length $> n$. Hence it is a frame of Λ_{n+1}. In particular, every finite **GL**-frame is a frame of the intersection $\bigcap \{\Lambda_n \mid n \in \omega\}$. By the completeness of **GL** w.r.t. these frames, it follows that

$$\bigcap \{\Lambda_n \mid n \in \omega\} \subseteq \mathbf{GL}.$$

Now it suffices to show that **GL** is not canonical while every Λ_n is canonical. Both of these claims hold in strong ways. First, **GL** is well-known to be not ω-canonical [7, Theorem 4.43], but in fact, as we saw in the last chapter, it is not even quasi-canonical (see Theorem 5.17), and quasi-canonicity is strictly weaker than canonicity (see Section 5.2). In addition **GL** is also not even 0-canonical; even the 0-canonical frame of **GL.3** contains a reflexive point.

On the other hand, each Λ_n is canonical. It is well-known that **K4** is canonical [7, proof of Theorem 4.27], and Λ_n is axiomatised over **K4** by a formula free of atomic propositions. Hence Λ_n is canonical. In fact it is Sahlqvist, from which the canonicity also follows [7, Theorem 5.91, 11, Theorem 10.31]. □

Note that since **GL** is not canonical in such a strong sense, while the Λ_n are even Sahlqvist, this single proof gives a wealth of results similar to this proposition: the κ-canonical logics are not closed under countable intersections for any κ and neither are the Sahlqvist, ω-strongly Kripke complete or quasi-canonical logics.

6.4 Canonical Approximations from Above and Below

Since the canonical logics form a complete join-semi-sublattice of $\mathcal{U}$, by Proposition 6.3, any logic extends its canonical approximation from below. However, since the canonical logics are not closed under arbitrary intersections, the analog for the canonical approximation from above does not follow. In this section we will show that there are in fact logics that strictly extend their canonical approximation from above, and in particular non-canonical logics where both canonical approximations equal.

Using the proof of Proposition 6.10 this is easy to see, as **GL** is the intersection of canonical extensions of it. Then from eq. (6.2) it follows that $\mathrm{Can}_{\downarrow}(\mathbf{GL}) = \mathrm{Can}_{\uparrow}(\mathbf{GL})$. Since **GL** is not canonical, $\mathrm{Can}_{\uparrow}(\mathbf{GL})$ is strictly below **GL**.

However, it turns out this is only an instance of a much more general result. In fact we can show, using the Fine-van Benthem theorem, that for any logic with the fmp, i.e. every logic which is complete w.r.t. its finite frames, the canonical approximations equal. To state the Fine-van Benthem theorem, we need to introduce *elementary classes* first. These are defined as in model theory.

Definition 6.11 (Elementary)**.** A class of frames $\mathcal{F}$ is called *elementary* iff there exists a set Φ of first-order formulas (over the language containing binary relation symbols for equality $=$ and and the relation of a frame R) such that $\mathfrak{F} \in \mathcal{F}$ iff for all $\varphi \in \Phi$, $\mathfrak{F} \vDash \varphi$ (where $\vDash$ denotes validation in the model theoretic sense).

The Fine-van Benthem theorem now states that the logic of an elementary class of frames is canonical. It was first proven by Fine [21, Theorem 3], using model theoretic methods. It was proven again by van Benthem [45, Corollary 3.7], using the theory of ultrafilter extensions. The proof can also be found in Chagrov and Zakharyaschev [11, Theorem 10.19].

Theorem 6.12 (Fine-van Benthem theorem; [21, Theorem 3])**.** *Let Λ be a modal logic that is sound and complete w.r.t. an elementary class of Kripke frames. Then Λ is canonical.*

Now we can show that for logics with the fmp, the approximations from above and below equal. The proof is essentially due to G. Bezhanishvili.

Proposition 6.13. *Let Λ be a logic with the fmp. Then* $\mathrm{Can}_{\uparrow}(\Lambda) = \mathrm{Can}_{\downarrow}(\Lambda)$.

Proof. Because Λ has the fmp, we have

$$\Lambda = \mathrm{Log}(\{F \mid F \in \mathrm{Fr}_{\mathrm{fin}}(\Lambda)\}) = \bigcap \{\mathrm{Log}(F) \mid F \in \mathrm{Fr}_{\mathrm{fin}}(\Lambda)\}.$$

Note that every (singleton of a) finite frame is an elementary class, hence by the Fine-van Benthem theorem their logics are canonical. So each of the $\mathrm{Log}(F)$ is canonical. Obviously $\Lambda \subseteq \mathrm{Log}(F)$. We conclude that

$$\Lambda = \bigcap \{\Lambda' \mid \Lambda' \in \mathrm{Can}, \Lambda \subseteq \Lambda'\},$$

so $\mathrm{Can}_{\downarrow}(\Lambda) = \mathrm{Can}_{\uparrow}(\Lambda)$. □

It should be pointed out that from the proof it also follows that any logic with the fmp is the intersection of its canonical extensions.

6.5 The Canonical Approximations of Grz.3

Now that we have established the basic theory of canonical approximations, we will look at some examples. In particular, in this section we will compute the canonical approximations of **Grz.3**, and in the next section those of **Grz.2**.

Recall the axioms and frame properties of the logics **S4.3**, **S4.3.1** and **Grz.3** from Section 2.11. We list some well-known properties about these logics.

Proposition 6.14.

(i) *Any extension of* **K4.3** *is complete w.r.t. its* CWF *frames, so in particular Kripke complete,*

(ii) **S4.3***,* **S4.3.1** *and* **Grz.3** *are complete w.r.t. their finite frames,*

(iii) **Grz.3** *extends* **S4.3.1***, and*

(iv) **S4.3.1** *is canonical.*

Proof sketch. (i) By Fine's finite width theorem (Theorem 4.1).

(ii) By (i) all these logics are complete w.r.t. their CWF frames. Clearly their frame classes are closed under taking finite subframes. By Fine's selective filtration via maximal points method it follows that they are complete w.r.t. their finite frames, see Section 8.4 and Lemma 8.32. In fact, it was shown by Bull [10] and later Fine [17] that every extension of **S4.3** has the fmp, much earlier than the two general results of Fine used above.

(iii) By (ii) it suffices to show that every finite frame of **Grz.3** is a frame of **S4.3.1**. But a **Grz**-frame is CWF so it contains a point x that is maximal for the entire frame, and does not contain any proper clusters, so the cluster of x is singleton. Then x is final, and by transitivity and linearity every point sees x.

(iv) By (i) **S4.3.1** is Kripke complete, and clearly it has an elementary frame class. It follows by the Fine-van Benthem theorem (Theorem 6.12) that it is canonical. □

From the last two facts, we conclude that **S4.3.1** is contained in the canonical approximation of **Grz.3** from below. In the rest of this section we show that it actually is the canonical approximation.

The proof essentially consists of two parts. First, we derive that **S4.3.1** is (sound and) complete w.r.t. certain ordinals. Second, we show that any ω-strongly Kripke complete logic contained in **Grz.3** has at least those ordinals as frames. Hence **S4.3.1** extends such ω-strongly Kripke complete logic, in particular the canonical approximation from below.

Lemma 6.15. **S4.3.1** *is complete with respect to the class of rooted linear orders with a top element.*

Proof. Let $\neg\varphi \notin$ **S4.3.1**. As noted above, **S4.3.1** is sound and complete w.r.t. its finite frames. Hence φ is satisfiable on a finite frame $\mathfrak{F}_0$ of **S4.3.1**, i.e. a finite upward linear preorder with top element. Say φ is satisfiable in some point w_0. Taking the subframe $\mathfrak{F}_1$ of $\mathfrak{F}_0$ generated by w_0 gives a rooted linear preorder with top element on which φ is still satisfiable.

To turn this into a linear order, we use the technique known as *bulldozing* [11, Theorem 3.20]. The technique is to replace any proper cluster by an infinite ascending sequence of points. More formally, define $\mathfrak{F}_2$ to be the frame consisting of points $\langle C, i\rangle$ for any cluster C in $\mathfrak{F}_1$ and $i \in \omega$ such that $i \neq 0$ implies that C is a proper cluster. For the relation of $\mathfrak{F}_2$, define $\langle C_1, i\rangle$ sees $\langle C_2, j\rangle$ iff either

- $C_1 = C_2$ and $i \leq j$, or
- $C_1 \neq C_2$ and the points in C_1 see the points in C_2.

Clearly $\mathfrak{F}_2$ is still rooted (with as root $\langle C_{\text{root}}, 0\rangle$ where C_{root} is the cluster of the root of $\mathfrak{F}_1$) and a linear preorder. Since we replaced every proper cluster of $\mathfrak{F}_1$ with something anti-symmetric, $\mathfrak{F}_2$ is anti-symmetric, hence a linear order. It still has a top element since by definition a top element has a singleton cluster, hence it was not replaced by an ascending sequence.

We claim that $\mathfrak{F}_2$ is a p-morphic image of $\mathfrak{F}_1$. For any proper cluster C of $\mathfrak{F}_1$ let $|C|$ denote the number of elements of C. Note that since $\mathfrak{F}_0$ is finite, so is $\mathfrak{F}_1$ and hence the cluster C. Pick a bijection $g_C\colon |C| \hookrightarrow\!\!\!\!\rightarrow C$.

Now define a function $f\colon \mathfrak{F}_{2,\mathrm{w}} \to \mathfrak{F}_{1,\mathrm{w}}$ by sending $\langle\{x\}, 0\rangle$ to x for any singleton cluster $\{x\}$ in $\mathfrak{F}_1$, and $\langle C, i\rangle$ to $g_C(i \bmod |C|)$, where $i \bmod n$ denotes the smallest natural number that equals i modulo n. Clearly, f is surjective, and it is easy to check that it is a p-morphism. Hence $\mathfrak{F}_1$ is a p-morphic image of $\mathfrak{F}_2$, so $\mathfrak{F}_2$ satisfies φ. □

To prove completeness of **S4.3.1** w.r.t. certain ordinals, we employ an ad-hoc selection method.

Lemma 6.16. **S4.3.1** *is sound and complete with respect to the set of frames*

$$S := \{\langle \alpha + 1, \leq\rangle \mid \alpha \in \omega^2\}.$$

Proof. Note that S is a subset of the frame class of **S4.3.1**, hence the soundness. For the completeness, let $\varphi' \notin$ **S4.3.1**, say refuted in a model $\mathfrak{M}$ on a linear order with top element $\langle W, \leq\rangle$. Let φ be a formula equivalent to $\neg\varphi'$ with only negations on atomic propositions (push negations down through box, diamond, conjunction and disjunction). Let w_0 be a point of $\mathfrak{M}$ that satisfies φ (so refutes φ') and let w_1 be the top point of $\mathfrak{M}$. Let $D := \{\psi \mid \Diamond\psi \in \mathrm{Sub}(\varphi)\}$. Note that D is finite.

We construct a subset $X \subseteq W$ inductively. Define $X_0 := \{w_0, w_1\}$. Given a finite $X_i \subseteq W$ we define W_{i+1}. Let $\psi \in D$. Define $X_{i,\psi} := \{x \in X_i \mid \mathfrak{M}, x \vDash \Diamond\psi\}$. If this set is non-empty then it has a maximal element, say x. Since $\mathfrak{M}, x \vDash \Diamond\psi$, there is a point $x_{i,\psi}$ above x (hence by maximality above all elements of $X_{i,\psi}$) such that $\mathfrak{M}, x_{i,\psi} \vDash \psi$. Now define $X_{i+1} := X_i \cup \{x_{i,\psi} \mid \psi \in D, X_{i,\psi} \neq \varnothing\}$, and note that it is finite again. Finally we define $X := \bigcup\{X_i \mid i \in \omega\}$.

Consider the submodel $\mathfrak{M}'$ of $\mathfrak{M}$ consisting of precisely the points in X. We prove by induction on $\psi \in \mathrm{Sub}(\varphi)$ that for $x \in X$, if $\mathfrak{M}, x \vDash \psi$ then $\mathfrak{M}', x \vDash \psi$. For atomic propositions and their negations this is trivial, since the valuation is inherited from $\mathfrak{M}$.

For conjunction, assume $\psi = \psi_1 \wedge \psi_2$ and the claim holds for ψ_1 and ψ_2 (IH). If $\mathfrak{M}, x \vDash \psi$ then $\mathfrak{M}, x \vDash \psi_1$ and $\mathfrak{M}, x \vDash \psi_2$ so by the induction hypothesis $\mathfrak{M}', x \vDash \psi_1$ and $\mathfrak{M}', x \vDash \psi_2$. We conclude that $\mathfrak{M}', x \vDash \psi$.

For disjunction, assume $\psi = \psi_1 \vee \psi_2$ and the claim holds for ψ_1 and ψ_2 (IH). If $\mathfrak{M}, x \vDash \psi$ then $\mathfrak{M}, x \vDash \psi_i$ for some $i \in \{1, 2\}$. Then by IH $\mathfrak{M}', x \vDash \psi_i$, hence $\mathfrak{M}', x \vDash \psi$.

For box, assume $\psi = \Box\psi_1$ and the claim holds for ψ_1 (IH). If $\mathfrak{M}, x \vDash \psi$ then for any successor y of x, $\mathfrak{M}, y \vDash \psi_1$. If $y \in X$ then by IH $\mathfrak{M}', y \vDash \psi_1$. Hence all successors of x in $\mathfrak{M}'$ satisfy ψ_1, so $\mathfrak{M}', x \vDash \psi$.

For diamond, assume $\psi = \Diamond\psi'$ and the claim holds for ψ' (IH). Note that $\psi' \in D$. Since $x \in X$, we can find $i \in \omega$ such that $x \in X_i$. If $\mathfrak{M}, x \vDash \psi$ then $x \in X_{i,\psi'}$, which is therefore non-empty. Hence $x_{i,\psi'} \in X_{i+1} \subseteq X$ exists. By definition $\mathfrak{M}, x_{i,\psi'} \vDash \psi'$ and $x_{i,\psi'}$ is a successor of x. Hence $\mathfrak{M}', x \vDash \psi$.

That concludes the proof by induction. From $x_0 \in X$ and $\mathfrak{M}, x \vDash \varphi$ it now follows that $\mathfrak{M}', x_0 \vDash \varphi$. Since φ is equivalent to $\neg\varphi'$, $\mathfrak{M}', x_0 \nvDash \varphi'$.

Looking again at the construction of X, we note that

$$X = X_0 \cup \{x_{i,\psi} \mid i \in \omega, \psi \in D, X_{i,\psi} \neq \varnothing\}.$$

Now, for a fixed ψ, $X_{i,\psi}$ is a monotone increasing in i, and hence so are the $x_{i,\psi}$ (when they exist, i.e. when $X_{i,\psi} \neq \varnothing$). Since D is finite, this shows that X is the union of finitely many countable chains. It follows that $\langle X, \leq\rangle$ is isomorphic to $\langle \omega \cdot m + n, \leq\rangle$ for some $m, n \in \omega$. Since $\langle X, \leq\rangle$ has a top element, namely w_1, $n \neq 0$. Therefore $\omega \cdot m + n = \alpha + 1$ for some $\alpha \in \omega^2$. So φ' is also refuted on a frame in S. □

The next part of the proof is to show that any ω-strongly Kripke complete logic contained in **Grz.3** has at least those ordinals as frames.

Lemma 6.17. *Let $\Lambda \subseteq$ **Grz.3** be a logic extending **S4** which is ω-strongly Kripke complete. Then there exists a frame $\mathfrak{F}$ of Λ such that $\langle \omega^2, \leq\rangle$ is a subframe of $\mathfrak{F}$.*

Proof. The idea of the proof is to define a **Grz.3**-consistent set of formulas over the atomic propositions ω. By ω-strong Kripke completeness this set of formulas will be satisfied in a single point in some model on a Λ-frame. We will use the valuation of that model to extract the required subframe from this Λ-frame.

Fix some injection $p\colon \omega^2 \to \omega$, embedding the ordinal ω^2 in our set of atomic propositions ω. We define a set of formulas

$$\Gamma := \{\Diamond p(\alpha) \mid \alpha \in \omega^2\} \cup \{\Box(p(\alpha) \to \Box\neg p(\beta)) \mid \alpha, \beta \in \omega^2, \alpha > \beta\}.$$

We claim that Γ is **Grz.3**-consistent, for let $\Gamma' \subsetneq \Gamma$ be a finite subset. Then Γ' references only finitely many atomic propositions, say $p(X)$ for a finite set $X \subsetneq \omega^2$. Clearly $\langle X, \leq\rangle$ is a frame of **Grz.3**. Endowed with the valuation $\mathfrak{V}$ that sets $\mathfrak{V}(p(\alpha)) := \{\alpha\}$ for all $\alpha \in X$ and $\mathfrak{V}(q) := \varnothing$, this turns into a model that satisfies Γ' in its bottom point. Therefore every finite subset of Γ is **Grz.3**-consistent, hence so is Γ.

Since $\Lambda \subseteq$ **Grz.3**, we conclude that Γ is Λ-consistent. Now Λ is ω-strongly Kripke complete, so Γ is satisfied in a single point w of a frame $\mathfrak{F}$ of Λ. Then for any $\alpha \in \omega^2$, w sees a point w_α satisfying $p(\alpha)$. For $\beta < \alpha$, w_α does not see any point satisfying $p(\beta)$, so in particular is does not see w_β. We conclude that $\{w_\alpha \mid \alpha \in \omega^2\}$ gives rise to a subframe isomorphic to $\langle \omega^2, \leq\rangle$. Obviously then there is also a frame of Λ such that $\langle \omega^2, \leq\rangle$ is an actual subframe (not just isomorphic). □

Lemma 6.18. *Let $\mathfrak{F} = \langle W, \leq\rangle$ be a frame of* **S4.3.1**, *and $\mathfrak{F}' = \langle W', R'\rangle$ a* WF *frame of* **S4.3.1** *that is a subframe of $\mathfrak{F}$ and includes the top element of it. Then $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{F}$.*

Proof. For $w \in W$, let $W'_w \subseteq W'$ denote the set of successors of w that are in the subframe $\mathfrak{F}'$. Since at least the top element is a successor of w and a point of $\mathfrak{F}'$, the set W'_w is non-empty. As $\mathfrak{F}'$ is transitive and WF, it follows that there exists a minimal point $f(w)$ for W'_w. Clearly, this defines a frame morphism $f\colon \mathfrak{F} \to \mathfrak{F}'$. Note that we can chose f such that the restriction of f to $\mathfrak{F}'$ is the identity map, as by reflexivity every point is a minimal successor of itself. Hence f is surjective.

To see that f is a p-morphisms, assume $u \in W$, $w \in W'$ and $R'(f(u), w)$. Then $u \leq f(u) \leq w$ since f is monotone (and R' is a restriction of $\leq$), so $u \leq w$. Since f restricts to the identity on W', and $w \in W'$, we find $f(w) = w$ as required. □

We conclude:

Theorem 6.19. $\mathrm{Can}_\uparrow(\textbf{Grz.3}) = \mathrm{Can}_\downarrow(\textbf{Grz.3}) = \textbf{S4.3.1}$.

Proof. Since **Grz.3** has the fmp, by Proposition 6.13, $\mathrm{Can}_\uparrow(\textbf{Grz.3}) = \mathrm{Can}_\downarrow(\textbf{Grz.3})$. Hence it suffices to prove $\mathrm{Can}_\uparrow(\textbf{Grz.3}) = \textbf{S4.3.1}$.

Define $\Lambda := \mathrm{Can}_\uparrow(\textbf{Grz.3})$. Then Λ is canonical, hence in particular ω-strongly Kripke complete. Because **S4.3.1** $\subseteq$ **Grz.3** is canonical, **S4.3.1** $\subseteq \Lambda$. Therefore we are left to prove $\Lambda \subseteq$ **S4.3.1**. Since we are considering the canonical approximation from below, $\Lambda \subseteq$ **Grz.3**. Hence the two previous lemmata are applicable to Λ.

By Lemma 6.17, we find a frame $\mathfrak{F}$ of Λ which has $\langle \omega^2, \leq \rangle$ as a subframe. Let $\alpha \in \omega^2$. Then $\langle \alpha, \leq \rangle$ is a subframe of $\langle \omega^2, \leq \rangle$ and hence of $\mathfrak{F}$. Note that $\langle \omega^2, \leq \rangle$ as a subframe of $\mathfrak{F}$ cannot include the top element, so neither can the smaller subframe $\langle \alpha, \leq \rangle$. Let us denote by $\mathfrak{F}_\alpha$ the subframe of $\mathfrak{F}$ consisting of α and the top element of $\mathfrak{F}$. Clearly, this is isomorphic to $\langle \alpha + 1, \leq \rangle$. Since $\mathfrak{F}_\alpha$ is WF, it follows from Lemma 6.18 that it is a frame of Λ. $\mathfrak{F}_\alpha$ is isomorphic to $\langle \alpha + 1, \leq \rangle$, so $\langle \alpha + 1, \leq \rangle$ is also a frame of Λ. But by Lemma 6.16, **S4.3.1** is sound and complete w.r.t. these frames. Hence $\Lambda \subseteq$ **S4.3.1**. □

Note that in the proof we only really used the fact that the canonical approximation is ω-strongly Kripke complete instead of its full canonicity. In fact the ω-canonical logics form a complete lattice, and using the proof above it follows that the ω-canonical approximations of **Grz.3** are **S4.3.1**.

6.6 The Canonical Approximations of Grz.2

Now that we have established that both canonical approximations of **Grz.3** are **S4.3.1**, we turn to the canonical approximations of **Grz.2**, the logic of confluent **Grz**-frames. We will show both are **S4.2.1**.

Recall the axioms and frame properties of the logics **S4.2.1** and **Grz.2** from Section 2.11. We list some well-known properties about these logics first.

Proposition 6.20.

(i) **S4.2.1** *and* **Grz.2** *have the fmp,*

(ii) **S4.2.1** *is canonical, and*

(iii) **Grz.2** *extends* **S4.2.1**.

Proof sketch. (i) Call a subframe, sub-general frame or submodel *cofinal* iff it contains, for every point of the frame x, a point y above x. Note that for both logics, the general frames of it are closed under taking cofinal sub-general frames (or equivalently the models are closed under taking cofinal submodels). By the Fine-Zakharyaschev selective filtration theorem [11, Theorem 9.34], based on work by Fine [22], it follows that they have the fmp.

(ii) By (i) it is Kripke complete, and clearly has an elementary frame class. Hence by the Fine-van Benthem theorem (Theorem 6.12) it is canonical.

(iii) By (i) it suffices to show that every finite frame of **Grz.2** is a frame of **S4.2.1**. A **Grz**-frame is CWF so it contains a point x that is maximal for the entire frame, and does not contain any proper clusters, so the cluster of x is singleton. Then x is final, and by transitivity and confluence every point sees x.

□

From the last two facts, we conclude that **S4.2.1** is contained in the canonical approximation of **Grz.2** from below. In the rest of this section we show that it actually is the canonical approximation.

Compared to the proof in the previous section, the first completeness part of the proof is simpler and more standard. Again, as a second part of the proof we show that any strongly Kripke complete logic contained in **Grz.2** has at least those frames.

Let us first introduce the following notation.

Definition 6.21. Let $\mathfrak{F} = \langle W, R \rangle$ be frame. Define $\mathfrak{F}^\top$ to be the of $\mathfrak{F}$ with a single reflexive top point added to it. More formally define $\mathfrak{F}^\top := \langle W^\top, R^\top \rangle$ where $W^\top := W \cup \{\top\}$ and $R^\top := R \cup \{\langle w, \top \rangle \mid w \in W^\top\}$, where $\top$ is some point distinct from all points in $\mathfrak{F}$.

An important property of this construction that we will use is that any p-morphism induces a p-morphism between the frames with top elements added, mapping the top point to the top point.

Lemma 6.22. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames and $f\colon \mathfrak{F} \to \mathfrak{G}$ a p-morphism. Then there exists a p-morphism $f^\top : \mathfrak{F}^\top \to \mathfrak{G}^\top$ with $f^\top(\top) = \top$ and for any point x of $\mathfrak{F}$, $f^\top(x) = f(x)$.*

Proof. Say $\mathfrak{F}^\top = \langle W, R \rangle$ and $\mathfrak{G}^\top = \langle W', R' \rangle$. Define f as in the lemma statement. Clearly it is a frame morphism, since f is. To see that it is a bounded morphism, let $x \in W$ and $y' \in W'$ such that $R'(f^\top(x), y)$. We want to find a preimage $y \in W$ of y' with $R(x, y)$.

- If neither x nor y' is $\top$ then this follows from the fact that f is a bounded morphism.
- If $y' = \top$ take $y = \top$. Then $f^\top(y) = \top = y'$ by definition and $R(x, y)$ since y is a top point.
- If $x = \top$ then $f^\top(x) = \top$. Since $R'(f^\top(x), y)$, it follows that y is the top element $\top$ as well, so we are in the previous case.

□

Now we can state and prove the completeness result that goes into our canonical approximation computation. It is a standard application of the usual tree unraveling technique.

Lemma 6.23. **S4.2.1** *is sound and complete w.r.t. the class of frames $\mathfrak{F}^\top$ where $\mathfrak{F}$ is a reflexive transitive tree.*

Proof sketch. Let $\mathfrak{F}^\top$ be a preorder with a top element satisfying a formula φ in some point w_0. Taking the subframe generated by w_0 gives a rooted preorder with top element $\mathfrak{G}^\top$ still satisfying φ in w_0.

Since $\mathfrak{G}$ is a preorder, by Proposition 2.103 it is a p-morphic image of the reflexive and transitive closure $\vec{\mathfrak{G}}^*$ of the path unravelling $\vec{\mathfrak{G}}$ of $\mathfrak{G}$. By Lemma 6.22 this surjective p-morphism extends into a surjective p-morphism from $\vec{\mathfrak{G}}^{*\top}$ to $\mathfrak{G}^\top$. Hence $\vec{\mathfrak{G}}^{*\top}$ also satisfies φ (in fact still in its root). Since $\vec{\mathfrak{G}}$ is a strict tree, $\vec{\mathfrak{G}}^*$ is reflexive transitive tree. $\square$

Remark 6.24. In fact, using a selection technique, somewhat similar to the proof of Lemma 6.16, one can improve the previous lemma to countable $\mathfrak{F}$. Doing this would in fact improve the final result of the section from computing the canonical approximations to the ω-canonical approximations of **Grz.2**. .

As a kind of work-around to save doing double work later, we note that we can in fact make the frames non-linear.

Lemma 6.25. **S4.2.1** *is sound and complete w.r.t. the class of frames* $\mathfrak{F}^\top$ *where* $\mathfrak{F}$ *is a* non-linear *reflexive transitive tree.*

Proof idea. It is easy to define a non-linear reflexive transitive tree $\mathfrak{F}'$ and a surjective p-morphism from $\mathfrak{F}'$ to $\mathfrak{F}$, while preserving all the above properties. For example, pick any point $x \in \mathfrak{F}_\mathrm{w}$, then 'add a duplicate copy of the frame generated by x to $\mathfrak{F}$, with the root seeing all these new points' to obtain such $\mathfrak{F}'$. $\square$

The next, and more interesting part of the proof is to show that any strongly Kripke complete logic contained in **Grz.2** has at least these non-linear reflexive transitive trees with tops added as frames.

Lemma 6.26. *Let κ be a cardinal,* **S4.2.1** $\subseteq \Lambda \subseteq$ **Grz.2** *a logic which is κ-strongly Kripke complete, and let $\mathfrak{F}$ be a reflexive transitive tree of cardinality κ. Then there exists a frame $\mathfrak{F}'$ of Λ such that one of $\mathfrak{F}^\top$ and $\mathfrak{F}$ is a p-morphic image of $\mathfrak{F}'$.*

Proof. Say $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{F}^\top = \langle W \amalg \{\top\}, R^\top\rangle$. Fix some injection $p\colon W \to \kappa$, and note such exists since $\mathfrak{F}$ has cardinality κ. Let r be the root of $\mathfrak{F}$. We define a set of formulas Γ to be the least set containing the following formulas:

- $\Box(p(x) \to \Diamond p(y))$ for any $x, y \in W$ with $R(x, y)$,
- $\Box(p(x) \to \neg\Diamond p(y))$ for any $x, y \in W$ with $\neg R(x, y)$,
- $\Box(p(x) \to \neg p(y))$ for any $x, y \in W$ with $x \neq y$,
- $p(r)$, and
- $\Box(\bigwedge\{\neg p(x) \mid x \in W, R(x, y)\} \to \neg\Diamond p(y))$ for any $y \in W$.

For the last formula we use the fact that since $\mathfrak{F}$ is a reflexive transitive tree, the downset generated by y is finite.

We claim that Γ is **Grz.2**-consistent, for let $\Gamma' \subseteq \Gamma$ be a finite subset. Then Γ' references only finitely many atomic propositions, say $p(X)$ for a finite set $X \subseteq W$. Clearly the subframe of $\mathfrak{F}^{\top}$ with the points in $X \cup \{\top\}$ is a finite poset with a top element, hence a frame of **Grz.2**. Endowed with the valuation $\mathfrak{V}$ that sets $\mathfrak{V}(p(x)) := \{x\}$ for all $x \in X$ and $\mathfrak{V}(q) := \varnothing$ for $q \notin p(X)$, this turns into a model that satisfies Γ' in its bottom point. Therefore every finite subset of Γ is **Grz.2**-consistent, hence so is Γ.

Since $\Lambda \subseteq$ **Grz.2**, we conclude that Γ is Λ-consistent. Now Λ is κ-strongly Kripke complete, so Γ is satisfied in a single point w_0 in a model $\mathfrak{M}'$ on a frame $\mathfrak{F}' = \langle W', R' \rangle$ of Λ, i.e. $\mathfrak{M}', w_0 \vDash \Gamma$. By taking a generated submodel we can assume that $\mathfrak{F}'$ is rooted with root w_0.

We show that every point of $\mathfrak{F}'$ satisfies at most one atomic proposition $p(x)$ in the model $\mathfrak{M}'$. So assume $w' \in W'$. Since w_0 is a root of $\mathfrak{F}'$, $R(w_0, w')$. For any $x, y \in W$ with $x \neq y$ we know that $\mathfrak{M}', w_0 \vDash \Box(p(x) \to \neg p(y))$. Hence $\mathfrak{M}', w' \vDash p(x) \to \neg p(y)$. We conclude that w' can satisfy at most one of the $p(x)$.

Let us call any point $w' \in W'$ that does not satisfy any atomic proposition of the form $p(x)$ *strange*. We show that any strange point sees only strange points. For let $w' \in W'$ be strange. Note that

$$\mathfrak{M}', w_0 \vDash \Box\Big(\bigwedge\{\neg p(x) \mid x \in W, R(x,y)\} \to \neg\Diamond p(y)\Big)$$

for any $y \in W$, so $\mathfrak{M}', w' \vDash \bigwedge\{\neg p(x) \mid x \in W, R(x,y)\} \to \neg\Diamond p(y)$. But the antecedent is trivially satisfied since w' is strange. Hence $\mathfrak{M}', w' \vDash \neg\Diamond p(y)$, for any $y \in W$. So indeed strange points see only strange points.

Now if $\mathfrak{F}'$ does not have strange points, it is easy to construct a surjective p-morphism from $\mathfrak{F}'$ to $\mathfrak{F}$. (This is highly analogous to the surjective p-morphism construction below, but without the strange points which are sent to $\top$.) So assume $\mathfrak{F}'$ does have at least one strange point, call it s. Since $\mathfrak{F}'$ is a frame of $\Lambda \supseteq$ **S4.2.1**, it does have a top point, say $t \in W'$. Then s sees t, since t is top. But s is strange, so only sees strange points. We conclude that t is strange.

We construct a surjective p-morphism $f\colon \mathfrak{F}' \to \mathfrak{F}^{\top}$. Define f as follows:

- If $w' \in W'$ is not strange, find the unique $x \in W$ such that $\mathfrak{M}', w' \vDash p(x)$. Set $f(w') := x$.
- If $w' \in W'$ is strange, define $f(w') := \top$.

It is surjective, for assume $x \in W$. Note that $R(r, x)$, so

$$\mathfrak{M}', w_0 \vDash \Box(p(r) \to \Diamond p(x)).$$

Since $\mathfrak{M}', w_0 \vDash p(r)$ and $\mathfrak{F}'$ is reflexive, we get $\mathfrak{M}', w_0 \vDash \Diamond p(x)$. Hence there exists $w' \in W'$ with $\mathfrak{M}', w' \vDash p(x)$, so $f(w') = x$ as required. Finally note that $f(t) = \top$, so also $\top$ has a preimage.

To see that it is a morphism assume $w', x' \in W'$ with $R'(w', x')$. If x' is strange, then $f(x') = \top$ which is clearly above $f(w')$. If x' is not strange, then neither is w', as strange points only see strange points. Assume, for sake of contradiction, $\neg R(f(w'), f(x'))$. Then $\mathfrak{M}', w_0 \vDash \Box(p(f(w')) \to \neg\Diamond p(f(x')))$. Since $R(w_0, w')$ and $\mathfrak{M}', w' \vDash p(f(w'))$ by definition, we deduce $\mathfrak{M}', w' \vDash \neg\Diamond p(f(x'))$. But $\mathfrak{M}', x' \vDash p(f(x'))$ and $R'(w', x')$, leading to a contradiction.

Finally, f is a p-morphism, for let $w' \in W'$ and $x \in W \amalg \{\top\}$ such that $R^\top(f(w'), x)$.

- If $x = \top$ then we can take $x' := t$ as the preimage for x, since $f(t) = \top$ and $R'(w', t)$ since t is top.
- If w' is strange then $f(w') = \top$, so x must be $\top$ as well, so the previous case applies.
- If w' is not strange and $x \in W$ (not $\top$) then $\mathfrak{M}', w_0 \vDash \Box(p(f(w')) \to \Diamond p(x))$. Since w' is a successor of w_0 satisfying $p(f(w'))$, we conclude $\mathfrak{M}', w' \vDash \Diamond p(x)$. Hence there exists $x' \in W'$ above w' satisfying $p(x)$, so $f(x') = x$ as required.

$\Box$

Lemma 6.27. *Let* $\mathbf{S4.2.1} \subseteq \Lambda \subseteq \mathbf{Grz.2}$ *be a logic which is strongly Kripke complete. Then* Λ *is sound w.r.t. the class of frames* $\mathfrak{F}^\top$ *where* $\mathfrak{F}$ *is a non-linear reflexive transitive tree.*

Proof. Let $\mathfrak{F}$ be a frame as in the lemma statement. By Lemma 6.26, we find a frame $\mathfrak{F}'$ of Λ such that either $\mathfrak{F}^\top$ or $\mathfrak{F}$ is a p-morphic image of $\mathfrak{F}'$. Hence either $\mathfrak{F}^\top$ or $\mathfrak{F}$ is a frame of Λ. Note that $\mathfrak{F}$ is non-linear and a reflexive transitive tree, so it does not have a top element. Therefore $\mathfrak{F}$ is not a frame of Λ. We conclude that $\mathfrak{F}^\top$ is a frame of Λ. $\Box$

Theorem 6.28. $\mathrm{Can}_\uparrow(\mathbf{Grz.2}) = \mathrm{Can}_\downarrow(\mathbf{Grz.2}) = \mathbf{S4.2.1}$.

Proof. Since $\mathbf{S4.2.1}$ is canonical and $\mathbf{S4.2.1} \subseteq \mathbf{Grz.2}$, we know that $\mathbf{S4.2.1} \subseteq \mathrm{Can}_\uparrow(\mathbf{Grz.2}) \subseteq \mathbf{Grz.2}$. Note that $\mathrm{Can}_\uparrow(\mathbf{Grz.2})$ is canonical, hence strongly Kripke complete. By Lemma 6.27 it follows that it must be sound w.r.t. the class of frames $\mathfrak{F}^\top$ where $\mathfrak{F}$ is a non-linear reflexive transitive tree. But $\mathbf{S4.2.1} \subseteq \mathrm{Can}_\uparrow(\mathbf{Grz.2})$, and by Lemma 6.25, $\mathbf{S4.2.1}$ is complete w.r.t. this frame class. Hence $\mathbf{S4.2.1} = \mathrm{Can}_\uparrow(\mathbf{Grz.2})$.

Since $\mathbf{Grz.2}$ has the fmp, by Proposition 6.13, $\mathrm{Can}_\uparrow(\mathbf{Grz.2}) = \mathrm{Can}_\downarrow(\mathbf{Grz.2})$. $\Box$

As noted in Remark 6.24, the completeness lemma for $\mathbf{S4.2.1}$ can be improved to countable frames. Then the strong Kripke completeness condition in Lemma 6.27 can be weakened to ω-strong Kripke completeness, and, similar to the previous section, we can conclude that $\mathbf{S4.2.1}$ is actually the ω-canonical approximation.

Concluding, we 'computed' the canonical approximations of the logics $\mathbf{Grz.3}$ and $\mathbf{Grz.2}$. However, even though $\mathbf{Grz.3}$ is an extension of $\mathbf{Grz.2}$, we needed separate proofs for these two results. This naturally raises the question of whether both are instances of a more general theorem, say which would apply to all extensions of **Grz.2** that satisfy certain conditions.

In both proofs, but especially the one for **Grz.2**, the unique top point played an important role. One losses this uniqueness of (the) final point(s) when moving to **Grz**. Whether and how this can be worked around is left for future research. In addition, the situation for the extensions of **GL** is yet to be explored.

Chapter 7

Computation, Trees and Dynamic Topological Logic

This chapter provides short introductions to concepts from computability theory, dynamic topological logic and Kruskal's tree theorem. It can be thought of as a kind of preliminaries for the next chapter, and does not contain any new results.

7.1 Introduction

In the next chapter, we prove computable enumerability of some dynamic topological logics. In preparation of that, this chapter introduces various preliminaries for it. In particular computable enumerability is defined and dynamic topological logics are introduced.

The next section starts with giving a very short introduction to computability theory. In particular computable enumerability is introduced, and its relation with axiomatisability is discussed. Next, Section 7.3 provides an introduction to dynamic topological logic and its semantics. Finally, Section 7.4 discusses two important theorems about trees, namely Kőnig's lemma and Kruskal's tree theorem.

The reader who is already familiar with these topics can skip the respective sections. It is advised however, to quickly review the frame semantics for dynamic topological logic in Section 7.3.

7.2 Computability and Logic

Given a logic, a very natural question is to ask for a proof system for the logic, i.e. a formal system which describes precisely which formulas are tautologies. In fact, very often logics are *defined* in terms of a proof system, or *axiomatisation*. Similarly, one can ask whether there exists an algorithmic way to determine whether a formula is a tautology or not. The second question leads to the field of computability theory, and turns out to be strongly related to the former question.

In this section we give a very brief introduction to computability theory and its relation with logic and axiomatisations. More information and details can be found introductory and reference works on the topic, notably Rogers [39] and Odifreddi [37].

Computability theory. Giving a precise description of computability theory is difficult, but in essence it studies the ability of algorithms to describe functions or sets. Classically this has been restricted to functions over and sets of natural numbers [37]. By describing other finite objects, like formulas, as numbers, the classical theory is still widely applicable, and suffices for our purposes in logic. This technique of representing objects as natural numbers is also known as *arithmetisation*.

The first central question obviously is how to formalise the notion of an 'algorithm'. There are many possible formalisations. Notable examples include general recursive functions [27],[1] λ-definable functions [12] and Turing machines [44]. However, all these definitions, and many others, turn out to be equivalent [26, 43], in the sense that any function of natural numbers that is computable according to one of the definitions, is computable according to all definitions. Hence the precise definition of computability of functions is of minor importance.

Definition 7.1 (Computable function)**.** An n-ary function $f\colon \omega^n \to \omega$ is called *computable* iff it is computable in one of the equivalent formalisms of computability. Informally this means that f is computable iff there exists an algorithm that, given inputs $\vec{x} \in \omega^n$ terminates and gives as output $f(\vec{x})$.

Definition 7.2 (Decidable set)**.** A set $X \subseteq \omega^n$ is called *decidable* iff the characteristic function of X is computable. Informally this means that X is decidable iff there exists an algorithm that, given inputs $\vec{x} \in \omega^n$ terminates and gives as output 0 if $\vec{x} \notin X$ and 1 if $\vec{x} \in X$.

Instead of *deciding* whether $\vec{x}$ is an element of X, one can also consider a (strictly) weaker definition, where there exists an algorithm that precisely lists all elements of X (in arbitrary order). More formally, this means precisely that X is either the empty set or the image of some computable function $f\colon \omega \to \omega$. Such a set X is called computably enumerable. Often more convenient though, is the following equivalent definition using existential quantification.

Definition 7.3 (Computably enumerable)**.** A set $X \subseteq \omega^n$ is called *computably enumerable* iff there exists a decidable set $Y \in \omega^{n+1}$ such that $\vec{x} \in X$ iff $\exists y \in \omega.\ \langle y, x\rangle \in Y$. Informally this means that X is computably enumerable iff there exists an algorithm that, given inputs $\vec{x} \in \omega^n$ terminates if $\vec{x} \in X$ and does not terminate if $\vec{x} \notin X$.

[1] In modern days an equivalent definition called μ-recursive functions is more commonly used.

The analogous definition but with a universal instead of existential quantifier is called co-computably enumerable. These are also precisely the complements of computably enumerable sets.

Definition 7.4 (Co-computably enumerable)**.** A set $X \subseteq \omega^n$ is called *co-computably enumerable* iff there exists a decidable set $Y \in \omega^{n+1}$ such that $\vec{x} \in X$ iff $\forall y \in \omega.\ \langle y, x \rangle \in Y$.

Remark 7.5. General recursive functions formed the first formalisation of computable functions. Due to this, to this day, computable functions are also called recursive functions, and computably enumerable sets are also called recursively enumerable. Computably enumerable is often abbreviated to c.e. or r.e. (for recursively enumerable). •

Computable enumerability and axiomatisability. As noted earlier, logical formulas can easily be coded as natural numbers, thus allowing to apply notions of computability theory to logics. This obviously requires a countable set of formulas, so the language needs to be finitary. Now computable enumerability of logics turns out to be strongly related to axiomatisability.

Let us fix a finitary language $\mathscr{L}$. An instance of an n-ary proof rule on $\mathscr{L}$ is a sequence $\langle \varphi_0, \dots, \varphi_n \rangle$ of $n+1$ formulas. It is interpreted as 'from φ_0 to φ_{n-1} derive φ_n'. Using the coding of formulas, rule instances are easily coded as natural numbers. Let us fix a decidable[2] set of such rule instances. Note that the instances of many common rules, for example *modus ponens*, form a decidable set.

A set of formulas Λ is now called a *logic* iff it is closed under the proof rule instances, i.e. whenever $\langle \varphi_0, \dots, \varphi_n \rangle$ is a rule instance and $\varphi_0, \dots, \varphi_{n-1} \in \Lambda$ then $\varphi_n \in \Lambda$. A logic Λ is called *axiomatised* by a set of formulas $A \subseteq \Lambda$ iff Λ is the least logic extending A.

Proposition 7.6. *Let Λ be a logic over $\mathscr{L}$ axiomatised by a computably enumerable set A. Then Λ is computably enumerable.*

Proof sketch. Intuitively one enumerates proofs using as axioms the formulas in A. Then one can enumerate Λ by giving for each proof in the previous enumeration the conclusion of that proof.

More formally, find decidable X such that $\varphi \in A$ iff $\exists m \in \omega.\ \langle m, \varphi \rangle \in X$. Let us code derivations using the proof rules by natural numbers. Now given a formula φ, (a code of) a derivation p and a finite list of pairs $\langle \langle m_1, \psi_1 \rangle, \dots \langle m_n, \psi_n \rangle \rangle$, it is decidable whether

- whether each of the $\langle m_i, \psi_i \rangle \in X$,

[2] The proposition generalises to a computably enumerable set of rule instances, but the proof becomes a bit more tedious.

- whether p codes a valid derivation from its axioms,
- whether p has as its conclusion the formula φ, and
- whether p has as its axioms only formulas ψ_i for some $i \in \{0, \dots, n\}$.

Note that $\varphi \in \Lambda$ iff there exist such p, n and $\langle\langle m_1, \psi_1\rangle, \dots \langle m_n, \psi_n\rangle\rangle$. Coding all these in a single natural number, we thus see that Λ can be written as an existential over a decidable set. Hence it is computably enumerable. □

In particular, any logic axiomatised by a decidable set of formulas is computably enumerable. Maybe surprisingly, if $\mathscr{L}$ contains an operator that behaves like the usual conjunction, then the converse also holds. This result, known as *Craig's theorem* or *Craig's trick*, is due to Craig [13]. The original formulation is more general than the one here, as it uses a far weaker condition than having a conjunction in the logic.

Proposition 7.7 (Craig's theorem, [13]). *Suppose $\mathscr{L}$ has an operator $\wedge$ with proof rules $\varphi \wedge \psi \vdash \varphi$, $\varphi \wedge \psi \vdash \psi$ and $\varphi, \psi \vdash \varphi \wedge \psi$. Let Λ be a computably enumerable logic over $\mathscr{L}$. Then Λ is axiomatised by a decidable set A.*

Proof sketch. Define a function $f\colon \omega \times \mathscr{L} \to \mathscr{L}$ by setting

$$f(0, \varphi) := \varphi \quad \text{and} \quad f(n+1, \varphi) := \varphi \wedge f(n, \varphi).$$

Note that f for all $n \in \omega$, $f(n, \varphi)$ and φ are inter-derivable. Moreover there exists a computable function which sends a formula ψ to a finite list of all its f-preimages. Instead of having a conjunction in $\mathscr{L}$, the existence of a function f with these properties actually suffices.

Find a decidable set X such that $\varphi \in \Lambda$ iff $\exists m \in \omega.\ \langle m, \varphi\rangle \in X$. Define $A := \{f(x) \mid x \in X\}$. This A is decidable, for given a formula ψ one can compute all finitely many f-preimages of it, and check each for being an element of X.

Note that A axiomatises Λ, for suppose $\varphi \in \Lambda$. Then there exists $m \in \omega$ such that $\langle m, \varphi\rangle \in X$. Then $f(m, \varphi) \in A$, and $f(m, \varphi)$ derives φ.

Conversely suppose $\varphi \in A$. Then there exist a number m and a formula ψ such that $\langle m, \psi\rangle \in X$ and $f(m, \psi) = \varphi$. Then by definition of X, $\psi \in \Lambda$, but by construction of f, ψ derives φ. Hence $\varphi \in \Lambda$. □

7.3 Dynamic Topological Logic

Dynamic topological logic refers to a family of multimodal logics that combine a unimodal logic interpreted over topological spaces with a linear temporal logic. The logic, in its current form, was introduced by Kremer and Mints [31] to study 'the confluence of three research areas: the topological semantics for S4, topological dynamics, and temporal logic.' [31]. It has three primitive modal operators: a box $\Box$

that is interpreted topologically, a temporal next operator $\bigcirc$ and a temporal box $\Box_F$ that is the transitive closure of $\bigcirc$. Formulas are then interpreted over a topological space equipped with a continuous function. In addition to this topological semantics, formulas can be interpreted over a weakly-transitive Kripke frame equipped with a monotone function.

Definition 7.8 (Dynamic topological structure)**.** A *dynamic topological structure*, or *DTS* for short, is a tuple $\langle \mathfrak{X}, f \rangle$ such that $\mathfrak{X}$ is a topological space and $f\colon \mathfrak{X} \to \mathfrak{X}$ is a continuous function.

Definition 7.9 (Dynamic topological model)**.** A *dynamic topological model*, or *DTM* for short, with atomic propositions P, is a tuple $\langle \mathfrak{X}, f, \mathfrak{V} \rangle$ such $\langle \mathfrak{X}, f \rangle$ is a DTS and $\mathfrak{V}\colon P \to \mathcal{P}(\mathfrak{X}_{\mathrm{w}})$.

Definition 7.10 (Topological semantics for DTL)**.** Let $\mathfrak{M} = \langle \mathfrak{X}, f, \mathfrak{V} \rangle$ be a DTM, $x \in \mathfrak{X}_{\mathrm{w}}$ and φ a DTL-formula. Define, by induction on φ, a subset $[\![\varphi]\!]_{\mathfrak{M}} \subseteq \mathfrak{X}_{\mathrm{w}}$:

- for an atomic proposition $p \in P$, $[\![p]\!]_{\mathfrak{M}} := \mathfrak{V}(p)$,
- Boolean connectives as usual,
- $[\![\Box\varphi]\!]_{\mathfrak{M}} := \mathsf{Int}([\![\varphi]\!]_{\mathfrak{M}})$,
- $[\![\bigcirc\varphi]\!]_{\mathfrak{M}} := f^{-1}([\![\varphi]\!]_{\mathfrak{M}})$, and
- $[\![\Box_F\varphi]\!]_{\mathfrak{M}} := \bigcap \{[\![\bigcirc^n\varphi]\!]_{\mathfrak{M}} \mid n \in \omega \setminus \{0\}\}$.

We write $\mathfrak{M}, x \vDash \varphi$ iff $x \in [\![\varphi]\!]_{\mathfrak{M}}$ and $\mathfrak{M} \vDash \varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} = \mathfrak{X}_{\mathrm{w}}$.

For the frame-based semantics we change the topological space to a weakly-transitive Kripke frame and f to a monotone function. The interpretation of $\Box$ then changes to usual Kripke interpretation.

Definition 7.11 (Dynamic frame structure)**.** A *dynamic frame structure*, or *DFS* for short, is a tuple $\langle \mathfrak{F}, f \rangle$ such that $\mathfrak{F}$ is a Kripke frame and $f\colon \mathfrak{F} \to \mathfrak{F}$ is a monotone function.

Definition 7.12 (Dynamic frame model)**.** A *dynamic frame model*, or *DFM* for short, with atomic propositions P, is a tuple $\langle \mathfrak{F}, f, \mathfrak{V} \rangle$ such $\langle \mathfrak{F}, f \rangle$ is a DFS and $\mathfrak{V}\colon P \to \mathcal{P}(\mathfrak{F}_{\mathrm{w}})$.

Definition 7.13 (Kripke semantics for DTL)**.** Let $\mathfrak{M} = \langle \mathfrak{F}, f, \mathfrak{V} \rangle$ be a DFM, $x \in \mathfrak{F}_{\mathrm{w}}$ and φ a DTL-formula. Define, by induction on φ, a subset $[\![\varphi]\!]_{\mathfrak{M}} \subseteq \mathfrak{F}_{\mathrm{w}}$:

- for an atomic proposition $p \in P$, $[\![p]\!]_{\mathfrak{M}} := \mathfrak{V}(p)$,
- Boolean connectives as usual,

- $[\![\Box\varphi]\!]_{\mathfrak{M}} := \Box[\![\varphi]\!]_{\mathfrak{M}}$,
- $[\![\bigcirc\varphi]\!]_{\mathfrak{M}} := f^{-1}([\![\varphi]\!]_{\mathfrak{M}})$, and
- $[\![\Box_F\varphi]\!]_{\mathfrak{M}} := \bigcap\{[\![\bigcirc^n\varphi]\!]_{\mathfrak{M}} \mid n \in \omega \setminus \{0\}\}$.

We write $\mathfrak{M}, x \vDash \varphi$ iff $x \in [\![\varphi]\!]_{\mathfrak{M}}$ and $\mathfrak{M} \vDash \varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} = \mathfrak{X}_{\mathrm{w}}$.

We introduce the following notation for defining logics over the DTL-language.

Definition 7.14. Let $\mathcal{S}$ be a class of topological spaces or weakly-transitive Kripke frames. We write DTL($\mathcal{S}$) for the logic of all DTSs over topological spaces in $\mathcal{S}$ or the logic of DFSs over frames in $\mathcal{S}$ respectively.

Kripke semantics as topological semantics. It should be noted that the Kripke semantics and topological semantics for DTL are related. In fact, it is known that DTL over **S4**-frames is equivalent to DTL over topological spaces called Alexandrov spaces. These spaces were introduced by Alexandrov [2].

Definition 7.15 (Alexandrov space)**.** A topological space is called an *Alexandrov space* (sometimes spelled *Alexandroff space*) iff the set of opens is closed under arbitrary intersections.

The relation between T_0 Alexandrov space and partial orders was already noted by Alexandrov [3, Section 1]. In fact, this extends to a category theoretic isomorphism between Alexandrov spaces with continuous functions and preorders with monotone functions [36, Proposition 2.4.6]. The preorder induced by an Alexandrov space is the so called *specialisation preorder*.

Definition 7.16 (Specialisation preorder)**.** Let $\mathfrak{X}$ be an Alexandrov space. Then the *specialisation preorder* of $\mathfrak{X}$ is $\langle \mathfrak{X}_{\mathrm{w}}, \leq \rangle$ where, for $x, y \in \mathfrak{X}_{\mathrm{w}}$, $x \leq y$ iff $\mathsf{Cl}(\{x\}) \subseteq \mathsf{Cl}(\{y\})$.

It is obvious from the definition that the specialisation preorder is a preorder. One can also note that in the specialisation preorder, $x \leq y$ iff $x \in \mathsf{Cl}(\{y\})$.

For the converse direction, a preorder induces an Alexandrov space by taking the *upset space*.

Lemma 7.17 (Upset space)**.** *Let $\mathfrak{F} = \langle X, R \rangle$ be a preorder. Then the collection of precisely all upsets $\mathsf{Up}(\mathfrak{F})$ of $\mathfrak{F}$ forms a topology on X. This topological space is an Alexandrov space and called the* upset space *of $\mathfrak{F}$. A set is closed iff it is a downset and the topological closure of a set Y is the downset $R^{\mathrm{op}}(Y)$.*

Proof. The set $\mathsf{Up}(\mathfrak{F})$ forms a topology on X since $\varnothing$ and X are upsets and the union or intersection of arbitrarily many upsets is again an upset. This immediately shows that the resulting topological space is an Alexandrov space. Since the complements of all upsets are precisely all downsets, a set is closed iff it is a downset. The least downset extending a set Y is $R^{\mathrm{op}}(Y)$. □

Taking specialisation preorders and upset spaces are inverse to each other.

Proposition 7.18. *Let $\mathfrak{F}$ be a preorder and $\mathfrak{X}$ a Alexandrov space. Then the specialisation preorder of the upset space of $\mathfrak{F}$ equals $\mathfrak{F}$ and the upset space of the specialisation preorder of $\mathfrak{X}$ equals $\mathfrak{X}$.*

Proof. For the former claim, write $\mathfrak{F} = \langle X, R\rangle$ and denote the specialisation preorder of the upset space of $\mathfrak{F}$ by $\langle X, S\rangle$. Let $x, y \in X$. Then $S(x, y)$ iff $x \in \mathsf{Cl}(\{y\}) = R^{\mathrm{op}}(y)$, i.e. $R(x, y)$.

For the latter claim, write $\mathfrak{F} = \langle X, \leq\rangle$ for the specialisation preorder of $\mathfrak{X}$ and let $Y \subseteq X$. We use the topological closure notation Cl for closure in $\mathfrak{X}$ only. Since $\mathfrak{X}$ is Alexandrov, the union of closed sets

$$\bigcup \{\mathsf{Cl}(\{y\}) \mid y \in Y\} \tag{7.1}$$

is closed. Since $\mathsf{Cl}(\{y\}) \subseteq \mathsf{Cl}(Y)$, the union equals $\mathsf{Cl}(Y)$.

Now Y is closed in the upset space of $\mathfrak{F}$ iff it is a downset in $\mathfrak{F}$. In $\mathfrak{F}$ a set Y is a downset iff whenever $x \in X$ and there exists $y \in Y$ such that $x \in \mathsf{Cl}(\{y\})$, then $x \in Y$. The existence of such y is equivalent to x being an element of the set in eq. (7.1), i.e. $x \in \mathsf{Cl}(Y)$. So Y is a downset in $\mathfrak{F}$ iff $\mathsf{Cl}(Y) \subseteq Y$, which is equivalent to Y being closed in $\mathfrak{X}$. □

In a similar way, it can be seen that a monotone function of preorders is continuous on the induced upset spaces, and conversely a continuous function of Alexandrov spaces is monotone on the induced specialisation preorders [36]. Finally it should be noted that, since taking downsets in the preorders corresponds to taking topological closures in the Alexandrov spaces, the interpretation of the unimodal $\Diamond$, and hence $\Box$, is invariant under the transforms. The analogous statements for the temporal operators $\bigcirc$ and $\Box_F$ are trivial since the function is not modified. Hence we get an equivalence between DTL over Alexandrov spaces and preorders.

Proposition 7.19. *Let $\mathfrak{X}$ be an Alexandrov space and $\mathfrak{F}$ a preorder such that $\mathfrak{F}$ is the specialisation preorder of $\mathfrak{X}$ or (equivalently) $\mathfrak{X}$ is the upset space of $\mathfrak{F}$. Then a DTL-formula is satisfied on a DTM $\langle \mathfrak{X}, f, \mathfrak{V}\rangle$ on $\mathfrak{X}$ iff it is satisfied on the DFM $\langle \mathfrak{F}, f, \mathfrak{V}\rangle$ on $\mathfrak{F}$.*

For this reason, the topological semantics can be seen as more general then the Kripke semantics, as it allows for spaces which are not Alexandrov. However, in Chapter 8 we will use techniques that apply naturally to the Kripke semantics, which we hence focus on.

7.4 Tree Theorems

In the next chapter we will use two major well-known theorems about trees: Kőnig's lemma [29] and Kruskal's tree theorem [32]. Various versions in the setting of undirected graphs exists, but we restrict our attention to the directed setting. Recall the various notions of trees from Section 2.12.

Kőnig's lemma. There are various formulations of Kőnig's lemma, but all essentially state that every infinite finitely branching tree has an infinite path in it. Here finitely branching means that every node in the tree has only finitely many children. The lemma was originally proven by Kőnig [29, Lemma E], and Kőnig [30] expands on the topic with various undirected graph versions of the lemma.

More formally, in the setting of strict trees, a strict tree is called *finitely branching* iff every node has only finitely many successors. Then Kőnig's lemma states the following:

Theorem 7.20 (Kőnig's lemma, [29, Lemma E]). *Let $\mathfrak{F}$ be an infinite finitely branching strict tree. Then there exists an ascending ω-sequence in $\mathfrak{F}$.*

Proof sketch. Construct the sequence by induction. Let r be the root of $\mathfrak{F}$. Since $\mathfrak{F}$ is infinite and finitely branching, there exists a successor x of r which generates an infinite subframe. Clearly this subframe is again an infinite finitely branching strict tree. Append x to the sequence and recurse with the subframe generated by x. □

Well-quasi-orders. Kruskal's tree theorem involves the notion of well-quasi-orders. Well-quasi-orders generalise the notion of well-orders from linear orders to preorders. The terminology comes from the fact that preorders are also called quasi-orders. It should not be confused with the stronger notion of a *prewellorder.*

Definition 7.21 (Well-quasi-order). Let $\mathfrak{F} = \langle X, R \rangle$ be a preorder. Then it is called a *well-quasi-order* iff every upset Y in $\mathfrak{F}$ is finitely generated, that is, there exists $n \in \omega$ and $y_0, \dots, y_{n-1} \in Y$ such that $Y = \bigcup \{R(y_i) \mid i < n\}$.

Proposition 7.22. *Let $\mathfrak{F} = \langle X, \leq \rangle$ be a preorder. Then the following are equivalent:*

(i) *$\mathfrak{F}$ is well-quasi-order,*

(ii) *any nowhere ascending sequence in $\mathfrak{F}$ is finite, that is, whenever $x_\_ : \omega \to \mathfrak{F}_{\mathrm{w}}$ is a ω-sequence then there exist $i, j \in \omega$ with $i < j$ such that $x_i \leq x_j$, and*

(iii) *$\mathfrak{F}$ is* WF *and every anti-chain in $\mathfrak{F}$ is finite.*

Kruskal's tree theorem. Now Kruskal's tree theorem, originally proven by Kruskal [32], states that finite trees in which the points are labelled with elements from a well-quasi-order, again form a well-quasi-order under a certain ordering. Defining this order requires us to introduce two more notions: that of tree-embedding and label-monotonicity.

Definition 7.23 (Tree-embedding)**.** Let $\mathfrak{F} = \langle X, R\rangle, \mathfrak{G} = \langle Y, S\rangle$ be strict trees. Then an injection $f\colon X \hookrightarrow Y$ is called a *tree-embedding* iff for every $x, y_1, y_2 \in X$ such that $R(x, y_i)$ for $i \in \{1, 2\}$ then there exist distinct $z_i \in Y$ such that $S(f(x), z_i)$ and $S^*(z_i, f(y_i))$.

Definition 7.24 (Labelled frame)**.** An X-labelled frame is a pair $\langle \mathfrak{F}, l\rangle$ where $\mathfrak{F}$ is a Kripke frame and $l\colon \mathfrak{F}_{\mathrm{w}} \to X$ is a function, called the *labelling function*.

Definition 7.25 (Label-monotone function)**.** Let $\langle Q, \leq\rangle$ be a preorder and $\langle \mathfrak{F}_1, l_1\rangle$ and $\langle \mathfrak{F}_2, l_2\rangle$ Q-labelled frames. Then a function $f\colon \mathfrak{F}_{1,\mathrm{w}} \to \mathfrak{F}_{2,\mathrm{w}}$ is called $\langle Q, \leq\rangle$*-label-monotone* iff for all $x \in \mathfrak{F}_{1,\mathrm{w}}$, $l_1(x) \leq (l_2 \circ f)(x)$. When $\langle Q, \leq\rangle$ is evident from the context, we omit it in our notation.

Theorem 7.26 (Kruskal's tree theorem, [32])**.** *Let $\langle Q, \leq\rangle$ be a well-quasi-order. Then the set of finite Q-labelled strict trees forms a well-quasi-order under label-monotone embeddability.*

While Kruskal [32] aims at the version for undirected tree graphs, the theorem is proven in the directed setting as a form of induction loading. A much simpler proof for the undirected setting was later given by Nash-Williams [35].

The transitive setting. We will want to consider Kruskal's tree theorem for transitive notions of trees as well. For irreflexive transitive trees the theorem easily adapts by taking transitive closures everywhere. The notion of a tree-embedding then turns in to the usual notion of an embedding of frames, as defined in Definition 2.45.

Theorem 7.27 (Kruskal's tree theorem for irreflexive transitive trees)**.** *Let $\langle Q, \leq\rangle$ be a well-quasi-order. Then the set of finite Q-labelled irreflexive transitive trees forms a well-quasi-order under label-monotone embeddability.*

This is the version of Kruskal's tree theorem is commonly used in logic [23, p. 253, 28, Theorem 16].

Chapter 8

Computably Enumerable DTL-Logics

In this chapter we show, using methods developed by Konev et al. [28], that the DTL-logics of certain classes of CWF Kripke frames are computably enumerable.

8.1 Introduction

DTL-logics are often defined semantically, i.e. as the set of formulas valid on a class of DTSs or DFSs. Hence such logics then do not automatically come with an axiomatisation. In fact, DTL over all topological spaces and DTL over **S4**-frames are known to be not finitely axiomatisable [16]. For many DTL-logics no natural decidable axiomatisation is known.[1] Hence computable enumerability is non-trivial.

Konev et al. [28] show that for the fragment DTL_1 of DTL, where no $\Box_F$ occurs under a $\Box$, the logic of **S4**-frames is computably enumerable. For this DTL_1 fragment, the logic of topological spaces equals the logic of **S4**-frames, hence also giving a computable enumerability result in the topological setting. In this chapter we apply their techniques to the full DTL-logics of classes of weakly-transitive CWF frames, slightly generalising these techniques in the process.

Notably, we employ a stronger version of Kruskal's tree theorem, which accounts for clusters, both to simplify the proofs and to allow more functions between quasi-states. The latter allows for weaker unravelling assumptions on the classes of frames we consider.

We perform this strengthening of Kruskal's tree theorem in the next section. Next, in Section 8.3 we introduce sequences of labelled frames called quasi-segments and quasi-models. These notions are used in the sections following it. We also develop 'conversion theorems' for inducing quasi-models from DTMs and conversely. In Section 8.4 we develop the selective filtration technique for quasi-segments. We apply

[1] An exception is the logic of topological spaces in a language DTL^* extending the DTL-language [15].

this to prove a local finiteness result.

The selective filtration for quasi-segments is reused and combined with Kruskal's tree theorem in Section 8.5, to prove co-computable enumerability of satisfiability on quasi-models over finite forest-like frames. In Section 8.5 we combine the main results of the three preceding sections together with a form of tree unravelling to prove computable enumerability for many DTL-logics of classes of CWF frames. Examples of frame classes for which this main result applies are given.

8.2 Strengthening Kruskal's Tree Theorem

Recall Kruskal's tree theorem from Section 7.4 and our different notions of trees from Section 2.12. In order to develop the theory of quasi-models in the largest generality, we need a version Kruskal's tree theorem which applies to forests instead of trees and allows for clusters and mixed reflexive and irreflexive points in the trees, i.e. forest-like frames. We generalize Kruskal's tree theorem to this general setting in several steps.

As a first step, we go from trees to forests. The proof technique, enriching the label well-quasi-order and transforming the labelled frames, is illustrative for the other steps as well. Each time we 'enrich' the label well-quasi-order Q to some well-quasi-order Q' in a way that allows it to describe some of the structure that the 'new' frames have but the 'old' frames lack. Then we transform new Q-labelled frames into old Q'-labelled frames. On these, the 'previous' Kruskal's tree theorem applies, giving a label-monotone embedding. Using the structure that the Q'-labelling provides, we turn this into a label-monotone embedding of the new Q-labelled frames prior to the transformation.

Lemma 8.1 (Kruskal's tree theorem for forests)**.** *Let $\langle Q, \leq_Q \rangle$ be a well-quasi-order. Then the set of finite Q-labelled irreflexive transitive forests forms a well-quasi-order under label-monotone embeddability.*

Proof. Define $\langle Q', \leq_{Q'} \rangle$ by adding to $\langle Q, \leq_Q \rangle$ a new point q_0 incomparable to all points in Q. Clearly $\langle Q', \leq_{Q'} \rangle$ is still a well-quasi-order.

We define a transformation T from finite Q-labelled irreflexive transitive forests to finite Q'-labelled irreflexive transitive trees as follows. Given a Q-labelled irreflexive transitive forest $\langle \mathfrak{F}, l \rangle$, define $T_0(\mathfrak{F})$ to be the frame that results from adding an irreflexive point r to $\mathfrak{F}$ that sees all points of $\mathfrak{F}$. Note that $T_0(\mathfrak{F})$ is an irreflexive transitive tree. Define $T_1(l)$ to be l extended by mapping r to q_0, and $T(\langle \mathfrak{F}, l \rangle) := \langle T_0(\mathfrak{F}), T_1(l) \rangle$.

Now let $\langle \mathfrak{F}_-, l_- \rangle$ be a sequence of Q-labelled irreflexive transitive forests. Then $T(\langle \mathfrak{F}_-, l_- \rangle)$ is a sequence of Q'-labelled irreflexive transitive trees, so by Theorem 7.27 there exist $i, j \in \omega$ with $i < j$ and a label-monotone embedding $f' : T(\langle \mathfrak{F}_i, l_i \rangle) \to T(\langle \mathfrak{F}_j, l_j \rangle)$. We will construct from it a label-monotone embedding $f \colon \langle \mathfrak{F}_i, l_i \rangle \to \langle \mathfrak{F}_j, l_j \rangle$.

Note that since f' is label-monotone, $q_0 = T_1(l_i)(r) \leq_{Q'} \big(T_1(l_j) \circ f'\big)(r)$. Since q_0 is incomparable to any other point of Q', $\big(T_1(l_j) \circ f'\big)(r) = q_0$. But $T_1(l_j)$ only maps the point r to q_0, so $f'(r) = r$. It follows that the restriction of f' to $\mathfrak{F}_i$ maps to $\mathfrak{F}_j$, and it is easy to check that this forms a label-monotone embedding from $\langle \mathfrak{F}_i, l_i \rangle$ to $\langle \mathfrak{F}_j, l_j \rangle$. □

Next, we generalise to forest-like frames. For simplicity this is done in two steps. First we allow irreflexive clusters only.

Lemma 8.2 (Kruskal's tree theorem with clusters)**.** *Let $\langle Q, \leq_Q \rangle$ be a well-quasi-order. Then the set of finite Q-labelled irreflexive forest-like frames forms a well-quasi-order under label-monotone embeddability.*

Proof. The basic idea is as follows. Given a Q-labelled irreflexive forest-like frame $\langle \mathfrak{F}, l \rangle$, we transform this to its skeleton $\mathfrak{F}/{\sim}$ with the following labelling. To a point of this skeleton representing a cluster in $\mathfrak{F}$, we assign as the label the multiset of all the labels of the points in this cluster. These multisets are ordered, informally, by $A \leq B$ iff B can be obtained from A by adding elements to the multiset and replacing elements by $\leq_Q$-larger ones. Both operations are allowed on 'single' elements, so they need not be applied to an element with its full multiplicity. For example $\{1, 1\} \leq \{1, 2\}$ over the natural numbers ordered as usual.

Multisets over Q however can be seen as Q-labelled frames where every point is isolated. For an element q with multiplicity m in the multiset, this labelled frame contains precisely m points with label q. The order $\leq$ defined above then corresponds precisely with label-monotone embeddability. Since $\leq_Q$ is a well-quasi-order, it follows by Lemma 8.1 that $\leq$ is so too.

Now given a label-monotone embedding between transformed frames, we can lift it to a label-monotone embedding of the original Q-labelled irreflexive forest-like frames. The embedding between skeleta already maps clusters to clusters. Within a cluster, we use the label-monotonicity to choose the mapping. By the definition of the order on multisets, there exists a label-monotone injection from the source to the target cluster. The disjoint union of one such injection for every cluster, gives the required 'lifted' label-monotone embedding.

More formally, let Q' be the set of finite Q-labelled irreflexive transitive forests, and $\leq_{Q'}$ the label-monotone embeddability order on it. By Lemma 8.1 $\langle Q', \leq_{Q'} \rangle$ is a well-quasi-order.

We will transform finite Q-labelled irreflexive forest-like frames $\langle \mathfrak{F}, l \rangle$ into finite Q'-labelled irreflexive transitive forests $T(\langle \mathfrak{F}, l \rangle) = \langle \mathfrak{F}', l' \rangle$. For the underlying frames we just take the skeleton, i.e. $\mathfrak{F}' := \mathfrak{F}/{\sim}$. For the labellings, let c be a point $\mathfrak{F}'$, i.e. a cluster of $\mathfrak{F}$. Define $l'(c)$ to be the Q-labelled forest $\langle c, \varnothing, l{\restriction}c \rangle$, i.e. the forest with the elements of c as its points, all isolated, and labelled as in $\langle \mathfrak{F}, l \rangle$.

Let $\langle \mathfrak{F}_{\_}, l_{\_} \rangle$ be a sequence of Q-labelled irreflexive forest-like frames. By Lemma 8.1 the finite Q'-labelled irreflexive transitive forests form a well-quasi-order under label-monotone embeddability. Hence there exists $i, j \in \omega$ with $i < j$ and a label-monotone embedding $f' : T(\langle \mathfrak{F}_i, l_i \rangle) \to T(\langle \mathfrak{F}_j, l_j \rangle)$. We will construct from it a label-monotone embedding $f \colon \langle \mathfrak{F}_i, l_i \rangle \to \langle \mathfrak{F}_j, l_j \rangle$.

For any point c of $\mathfrak{F}'_i$, we know that $l'_i(c) \leq_{Q'} l'_j(f'(c))$, so there exists a label-monotone embedding f_c from the the Q-labelled forest $l'_i(c)$ to $l'_j(c)$. Hence f_c is a function from c to $f'(c)$. For $x \in c$, define $f(x) = f_c(x)$.

We show that f is an embedding. Clearly f is injective, since each f_c is injective and, since f' is injective, have disjoint co-domains from each other. It preserves and reflects order, since f' does so and the f_c functions map points from a cluster to a cluster.

To see that f is label-monotone, let x be a point of $\mathfrak{F}_i$, say in cluster c. Since f_c was a label-monotone embedding between trees that inherit their labelling from l_i and l_j respectively,

$$l_i(x) = l_i{\restriction}c(x) \leq_Q l_j{\restriction}f'(c)(f_c(x)) = l_j(f(x)).$$

□

Finally we lift the irreflexivity requirement on the forests, giving the theorem we actually need in the next section. The proof is rather trivial compared to the previous steps, so we only give a rough sketch.

Theorem 8.3 (Kruskal's tree theorem for forest-like frames)**.** *Let $\langle Q, \leq_Q \rangle$ be a well-quasi-order. Then the set of finite Q-labelled forest-like frames forms a well-quasi-order under label-monotone embeddability.*

Proof. Let $Q' := \{0, 1\} \times Q$, and equip it with the order defined by $\langle b_1, q_1 \rangle \leq_{Q'} \langle b_2, q_2 \rangle$ iff $b_1 = b_2$ and $q_1 \leq_Q q_2$. Obviously $\langle Q', \leq_{Q'} \rangle$ forms a well-quasi-order.

Transform finite Q-labelled forest-like frames $\langle \mathfrak{F}, l \rangle$ into finite Q'-labelled irreflexive forest-like frames by taking the irreflexivisation of the underlying frame and tagging the label of each originally reflexive point with 1 and each originally irreflexive point with 0.

By Lemma 8.2 the finite Q'-labelled irreflexive forest-like frames form a well-quasi-order under label-monotone embeddability. Now obviously, any label-monotone embedding from some $T(\langle \mathfrak{F}_i, l_i \rangle)$ to some $T(\langle \mathfrak{F}_j, l_j \rangle)$ is also a label-monotone embedding from $\langle \mathfrak{F}_i, l_i \rangle$ to $\langle \mathfrak{F}_j, l_j \rangle$. Hence the Q-labelled frames form a well-quasi-order under label-monotone embeddability too. □

8.3 Quasi-Models

In most of the remainder of the chapter, instead of DFMs, we will work with quasi-segments and quasi-models, which we introduce in this section. They are similar to DFMs, but instead of having a valuation, we track truth using a *labelling function.*

This labelling indicates not only the truth of atomic proposition in a given world, but also the truth of more complex formulas including modal operators. Having this information locally available simplifies for example the selection technique we employ in the next section. Being *labelled* structures also allows us to apply Kruskal's tree theorem later. Moreover, we discuss how to convert between DFMs and quasi-models.

We will define quasi-segments to be sequences of labelled frames with certain morphisms in between. Let us first note that DFSs and DFMs can be viewed in a similar way, which we call their stratified form. This will allow us later to convert between DFMs and quasi-models.

Definition 8.4 (Stratified)**.** A DFS $\langle \mathfrak{F}, f\rangle$ is called *stratified* iff $\mathfrak{F}$ is the disjoint union of frames $\mathfrak{F}_i$ for $i \in \omega$, and f is the disjoint union of maps $f_i \colon \mathfrak{F}_i \to \mathfrak{F}_{i+1}$. In this case we write $\langle \mathfrak{F}_\_, f_\_\rangle$ for the DFS.

A DFM $\langle \mathfrak{F}, f, \mathfrak{V}\rangle$ is called *stratified* iff the underlying DFS is stratified. In this case we write $\langle \mathfrak{M}_\_, f_\_\rangle$ for the DFM, where $\mathfrak{M}_i$ is $\langle \mathfrak{F}, \mathfrak{V}\rangle$ restricted to $\mathfrak{F}_i$.

Remark 8.5. One can turn any DFS $\langle \mathfrak{F}, f\rangle$ into an stratified one $\langle \mathfrak{G}_\_, g_\_\rangle$ by setting $\mathfrak{G}_i := \mathfrak{F}$ and $g_i := f$. Formally, some tagging is involved to make the $\mathfrak{G}_i$ disjoint, but when writing a DFS as $\langle \mathfrak{G}_\_, g_\_\rangle$, we will leave all tagging implicit. ▪

Quasi-states and quasi-segments. Next, we define quasi-states, quasi-segments and quasi-models. These structures are generic over a subformula closed set of DTL-formulas Σ. For the rest of this section, we suppose such set Σ is given.

We say a subset $\Gamma \subseteq \Sigma$ is a Σ-*type* iff

- if $\varphi_1 \wedge \varphi_2 \in \Sigma$ then $\varphi_1 \wedge \varphi_2 \in \Gamma$ iff $\{\varphi_1, \varphi_2\} \subseteq \Gamma$,
- if $\neg\varphi \in \Sigma$ then $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$,
- if $\Box_F \varphi \in \Gamma$ then for all $n \in \omega \setminus \{0\}$ such that $\bigcirc^n \varphi \in \Sigma$, $\bigcirc^n \varphi \in \Gamma$, and
- if $\bigcirc^n \varphi \in \Gamma$ for all $n \in \omega \setminus \{0\}$ then $\Box_F \varphi \in \Gamma$.

Note that the final requirement is void if Σ is finite, which we will restrict to in later sections. We write ty_Σ for the set of Σ-types.

Definition 8.6 (Quasi-state)**.** A Σ-*quasi-state* is a ty_Σ-labelled Kripke frame $\mathfrak{S} = \langle X, R, l\rangle$ such that, for all $x \in X$ and $\Box\varphi \in \Sigma$,

$$\Box\varphi \in l(x) \quad \text{iff} \quad \forall y \in R(x).\ \varphi \in l(y).$$

If $\mathcal{F}$ is a class of Kripke frames and $\mathfrak{F} \in \mathcal{F}$, we call it a $(\Sigma, \mathcal{F})$-quasi-state.

Since we have a single fixed set Σ in this section, we will mostly omit it from the notations.

For a quasi-state $\mathfrak{S} = \langle \mathfrak{F}, l \rangle$, a formula $\varphi \in \Sigma$ and a point $x \in \mathfrak{F}_{\mathrm{w}}$, we say that x *satisfies* φ iff $\varphi \in l(x)$. We write $[\![\varphi]\!]_{\mathfrak{S}} := \{x \in \mathfrak{F}_{\mathrm{w}} \mid \varphi \in l(x)\}$. Note that since points are labelled with Σ-types, satisfaction of Boolean connectives follows the usual rules in Kripke models, and by eq. (8.6) so does $\Box$.

If a quasi-state is an analogue of a Kripke model $\mathfrak{M}_i$ in an stratified DFM then a quasi-state-morphism is an analogue of the function f_i. It is a monotone function that also satisfaction for the temporal operators behaves similar to in an stratified DFM.

Definition 8.7 (Quasi-state-morphism)**.** Let $\mathfrak{S}_1 = \langle \mathfrak{F}_1, l_1 \rangle, \mathfrak{S}_2 = \langle \mathfrak{F}_2, l_2 \rangle$ be Σ-quasi-states. Then a function f f is called a *quasi-state-morphism* from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ iff

- $f\colon \mathfrak{F}_1 \to \mathfrak{F}_2$ is a monotone function on the underlying frames,
- for all $x \in \mathfrak{F}_{1,\mathrm{w}}$ and $\Box_F \psi \in \Sigma$, $\Box_F \psi \in l_1(x)$ iff $\{\psi, \Box_F \psi\} \subseteq l_2(f(x))$, and
- for all $x \in \mathfrak{F}_{1,\mathrm{w}}$ and $\bigcirc \psi \in \Sigma$, $\bigcirc \psi \in l_1(x)$ iff $\psi \in l_2(f(x))$.

We now have all the ingredients to define a quasi-segment. Similar to how an stratified DFM is a sequence of Kripke models with functions between them, a quasi-segment is a sequence of quasi-states with quasi-state-morphisms in-between. For reasons that will become clear later, we also consider finite sequences.

Definition 8.8. A Σ-*quasi-segment* is a finite non-empty or ω-sequence of Σ-quasi-states $\mathfrak{S}_\_$ together with quasi-state-morphisms $f_i\colon \mathfrak{S}_i \to \mathfrak{S}_{i+1}$. It is called a $(\Sigma, \mathcal{F})$-quasi-segment iff each of the $\mathfrak{S}_\_$ is a $(\Sigma, \mathcal{F})$-quasi-state, and *infinite* iff the sequence is infinitely long.

Conversion. Each DFM induces an infinite quasi-segment.

Lemma 8.9. *Let $\langle \mathfrak{M}_\_, f_\_ \rangle$ be a stratified DFM. Define, for $i \in \omega$, $\mathfrak{F}_i := \mathfrak{M}_{i,\mathrm{fr}}$ and for $x \in \mathfrak{F}_{i,\mathrm{w}}$,*

$$l_i(x) := \{\varphi \in \Sigma \mid \langle \mathfrak{M}_\_, f_\_ \rangle, x \vDash \varphi\}.$$

Then $\langle \mathfrak{F}_\_, l_\_ \rangle$ with morphisms $f_\_$ forms a quasi-segment.

Proof. Let $i \in \omega$. There are three things to check:

- for each $x \in \mathfrak{F}_{i,\mathrm{w}}$, $l_i(x)$ is a Σ-type,
- $\langle \mathfrak{F}_i, l_i \rangle$ forms a quasi-state, i.e. eq. (8.6) holds, and
- f_i is a quasi-state-morphism from $\langle \mathfrak{F}_i, l_i \rangle$ to $\langle \mathfrak{F}_{i+1}, l_{i+1} \rangle$.

All are trivially checked. ∎

The quasi-segment constructed in this lemma is called the *induced* quasi-segment of $\langle \mathfrak{M}_\_, f_\_ \rangle$. By construction, a formula $\varphi \in \Sigma$ is satisfied in a point x of $\langle \mathfrak{M}_\_, f_\_ \rangle$ iff it is satisfied in the point x of the induced quasi-segment.

A converse construction however fails. The problem is that a point in a quasi-segment might satisfy $\neg\Box_F\varphi$ but each of the points reachable by following the quasi-state-morphisms satisfies φ. This is due to all conditions on the quasi-segment being *local* ones. We call an infinite quasi-segment a *quasi-model* iff it does not exhibit this problem.

Definition 8.10 (Quasi-model)**.** Let $\mathfrak{S}_\_ = \langle \mathfrak{F}_\_, l_\_ \rangle$ be an infinite Σ-quasi-segment with morphisms $f_\_$. We call it a Σ-*quasi-model* iff for all i, $\Box_F\psi \in \Sigma$ and every point x of $\mathfrak{S}_i = \langle \mathfrak{F}_i, l_i \rangle$ such that $\Box_F\psi \notin l_i$, there exists $j > i$ such that

$$\psi \notin \big(l_j \circ f_{j-1} \circ \ldots \circ f_i\big)(x).$$

When j is minimal we say $\mathfrak{S}_j$ *realises* the eventuality $\neg\Box_F\psi$ for x, and this eventuality is *realised in* the quasi-segment.[2]

Now we can move back and forth between stratified DFMs and quasi-models. First, the induced quasi-segment is actually a quasi-model, and we therefore call it the *induced quasi-model* from now on.

Proposition 8.11. *The induced quasi-segment of an stratified DFM is a quasi-model.*

Proof. Obvious. □

Proposition 8.12. *Let $\mathfrak{S}_\_ = \langle \mathfrak{F}_\_, l_\_ \rangle$ together with morphisms $f_\_$ be a quasi-model. Define for $i \in \omega$ and an atomic proposition $p \in \Sigma$,*

$$\mathfrak{V}_i(p) := \big\{ x \in \mathfrak{F}_{i,\mathrm{w}} \;\big|\; p \in l(x) \big\},$$

and set $\mathfrak{M}_i := \langle \mathfrak{F}_i, \mathfrak{V}_i \rangle$. Then $\langle \mathfrak{M}_\_, f_\_ \rangle$ is an stratified DFM such that for all points x of $\langle \mathfrak{S}_\_, f_\_ \rangle$ and $\varphi \in \Sigma$, x satisfies φ in $\langle \mathfrak{S}_\_, f_\_ \rangle$ iff it satisfies φ in $\langle \mathfrak{M}_\_, f_\_ \rangle$.

Proof. It is an stratified DFM by construction. The statement about satisfaction is proven by induction on $\varphi \in \Sigma$. The case for atomic propositions is by definition of the valuations $\mathfrak{V}_\_$. The cases for Boolean connectives and $\Box$ follow by our earlier observations about satisfaction in quasi-states.

For $\bigcirc$, assume $\bigcirc\varphi \in \Sigma$, and as induction hypothesis that for all points x, x satisfies φ in $\langle \mathfrak{S}_\_, f_\_ \rangle$ iff it satisfies φ in $\langle \mathfrak{M}_\_, f_\_ \rangle$. Let $i \in \omega$ and $x \in \mathfrak{F}_{i,\mathrm{w}}$. Then x satisfies $\bigcirc\varphi$ in $\langle \mathfrak{S}_\_, f_\_ \rangle$ iff $\bigcirc\varphi \in l_i(x)$. Since f_i is a quasi-state-morphism, this

[2]Note that the $\neg$ in 'the eventuality $\neg\Box_F\psi$' is just notation. There is no need for $\neg\Box_F\psi$ to be in Σ to speak about this eventuality.

is equivalent to $\varphi \in l_{i+1}(f_i(x))$. By the induction hypothesis this is equivalent to $\langle \mathfrak{M}_{\_}, f_{\_} \rangle, f_i(x) \vDash \varphi$, i.e. $\langle \mathfrak{M}_{\_}, f_{\_} \rangle, x \vDash \bigcirc(\varphi)$.

For $\Box_F$, assume $\Box_F \varphi \in \Sigma$, and as induction hypothesis that for all points x, x satisfies φ in $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$ iff it satisfies φ in $\langle \mathfrak{M}_{\_}, f_{\_} \rangle$. Let $i \in \omega$ and $x \in \mathfrak{F}_{i,\mathrm{w}}$. Suppose first that x satisfies $\bigcirc\varphi$ in $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$, i.e. $\Box_F \varphi \in l_i(x)$. Since each $f_{\_}$ is a quasi-state-morphism, one proves by induction on $n \in \omega \setminus \{0\}$ that $\{\varphi, \Box_F \varphi\} \subseteq (l_{i+n+1} \circ f_{i+n} \circ \ldots \circ f_i)(x)$. By the induction hypothesis it follows that

$$\langle \mathfrak{M}_{\_}, f_{\_} \rangle, (f_{i+n} \circ \ldots \circ f_i)(x) \vDash \varphi$$

for each $n \in \omega \setminus \{0\}$, hence $\langle \mathfrak{M}_{\_}, f_{\_} \rangle, x \vDash \Box_F \varphi$.

Conversely suppose $\Box_F \varphi \notin l_i(x)$. Since $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$ is a quasi-model there exists $j > i$ realising the eventuality $\Box_F \varphi$, i.e. $\varphi \notin (l_j \circ f_{j-1} \circ \ldots \circ f_i)(x)$. By the induction hypothesis

$$\langle \mathfrak{M}_{\_}, f_{\_} \rangle, (f_{j-1} \circ \ldots \circ f_i)(x) \nvDash \varphi$$

follows. Hence $\langle \mathfrak{M}_{\_}, f_{\_} \rangle, x \nvDash \Box\varphi$. □

This stratified DFM $\langle \mathfrak{M}_{\_}, f_{\_} \rangle$ is called the *induced DFM* of $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$.

Morphisms. Finally, we give some more notions of morphisms between quasi-states and quasi-segments. Recall from Definition 7.25 that a function is called label-monotone iff applying the function to a point makes its label increase in a given preorder. We call the function *label-preserving* iff it is label-monotone w.r.t. equality. Alternatively, a function $f \colon \langle \mathfrak{F}_1, l_1 \rangle \to \langle \mathfrak{F}_2, l_2 \rangle$ is label-preserving iff $l_2 \circ f = l_1$. It is called monotone or an embedding iff it is monotone respectively an embedding of the underlying frames, and an isomorphism iff it is both label-preserving and an isomorphism on the underlying frames. Quasi-states inherit all the terminology from labelled frames, as they are just particular labelled frames.

We note the following.

Lemma 8.13. *Let $\mathfrak{S}_i$ be quasi-states for $i \in \{1, \ldots, 4\}$, $f \colon \mathfrak{S}_1 \to \mathfrak{S}_2$ and $h \colon \mathfrak{S}_3 \to \mathfrak{S}_4$ be label-preserving embeddings and $g \colon \mathfrak{S}_2 \to \mathfrak{S}_3$ a quasi-state-morphism. Then $h \circ g$ and $g \circ f$ are quasi-state-morphisms.*

Proof. All of f, g, h are monotone functions on the underlying frames, hence so are their compositions. Note that f and h preserve labels exactly, as they are embeddings. Therefore clearly composition with f or h preserves the requirements on labels for quasi-state-morphisms. □

For quasi-segments, a morphism is a sequence of functions mapping between the quasi-states pairwise, that admits the natural commutative diagram.

Definition 8.14 (Morphism of quasi-segments)**.** Let $\mathfrak{S}_\_$ together with morphisms $f_\_$ and $\mathfrak{S}'_\_$ together with morphisms $f'_\_$ be quasi-segments of the same length. Then a morphism from $\langle \mathfrak{S}_\_, f_\_ \rangle$ to $\langle \mathfrak{S}'_\_, f'_\_ \rangle$ is a sequence of functions $g_\_$ such that $g_i \colon \mathfrak{S}_i \to \mathfrak{S}'_i$ is a monotone function and $f'_i \circ g_i = g_{i+1} \circ f_i$, i.e. the following diagram commutes:

$$\begin{array}{ccccccc} \mathfrak{S}_0 & \xrightarrow{f_0} & \mathfrak{S}_1 & \xrightarrow{f_1} & \mathfrak{S}_2 & & \cdots \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\ \mathfrak{S}'_0 & \xrightarrow[f'_0]{} & \mathfrak{S}'_1 & \xrightarrow[f'_1]{} & \mathfrak{S}'_2 & & \cdots \end{array}$$

A morphism $g_\_$ of quasi-segments is called label-monotone, label-preserving, an embedding, or an isomorphism, respectively, iff each of the functions g_i is one of labelled frames.

8.4 Local Finiteness

Recall that we want to prove computable enumerability for certain DTL-logics through *semantic* means. As noted in Section 7.2, computing with objects requires coding them as natural numbers, which in turn requires those objects to be finite. This connection between finiteness and compatibility is exemplified by the famous result that a finitely axiomatisable modal logic with the fmp is decidable [7, Theorem 6.15]. In the setting of DTL, having a fmp is too much to ask, but instead we will consider a kind of 'local' version of finite frame property.

Definition 8.15 (Locally finite)**.** A stratified DFS $\langle \mathfrak{F}_\_, f_\_ \rangle$ or DFM $\langle \mathfrak{M}_\_, f_\_ \rangle$ is called *locally finite* iff each of the $\mathfrak{F}_i$ or $\mathfrak{M}_i$, respectively, is finite.

Definition 8.16 (Local fmp)**.** Let Λ be a DTL-logic. It is said to have the *local fmp* iff for every DTL-formula φ such that $\neg\varphi \notin \Lambda$, φ is satisfiable on an stratified locally finite DFS of Λ.

Two common techniques for establishing the fmp in unimodal logic are *filtration* and *selective filtration*. The former technique works by taking a certain quotient of the Kripke model, while in the latter technique, one takes a (p-morphic image of) a carefully selected submodel. We consider Fine's selective filtration via maximal points method [22, Section 4], as it admits a commutative square necessary to extend it to DFMs.

This method was introduced by Fine [22] in Section 4. It requires the Kripke model under consideration to have a certain maximal-point property. This method should not be confused with another selective filtration method introduced in the same paper in Section 6, which does not require this maximal-point property, but also lacks the same commutative square.

It is possible to use Fine's selective filtration via maximal points method on Kripke models and extend it to stratified DFMs from there, in order to prove the local fmp. However, we apply Fine's selective filtration via maximal points method to quasi-segments instead, and extend it to stratified DFMs by the two conversion propositions at the end of the previous section. This is a slightly easier approach, and we will need the selective filtration for quasi-segments in Section 8.5 too.

Selective filtration via maximal points. The idea of Fine's selective filtration via maximal points method is as follows. Given a Kripke model and a formula φ satisfiable on this model, we want to select a finite submodel which still satisfies φ. One starts with a point x where φ is satisfied. Preserving the truth of φ might require some other points, called *witnesses*, to be in the submodel as well. For example, if $\varphi = \Diamond\psi$ then preserving this formula requires one to include a successor y of x satisfying ψ in the submodel. This point y might then also need witnesses in order to preserve the truth of ψ. Recursively adding witnesses until no point needs new witnesses anymore gives the desired finite submodel.

The problem is how to make this procedure guaranteed to terminate. This is where Fine's selective filtration via maximal points method gets its name from. Whenever we select a witness, we take a maximal one, i.e. a witness such that no strict successor could also serve as a witness. Using transitivity of the frame, no successor of the witness y would again require a witness satisfying ψ. This makes the procedure terminate. It does however require such maximal witnesses to exists, which leads us to the maximal-point property.

Definition 8.17 (Maximal-point property)**.** A general frame $\mathfrak{f} = \langle X, R, A\rangle$ is said to have the *maximal-point property* iff for every $a \in A$ and every $x \in a$, $R^*(x) \cap a$ has a maximal point. A Kripke model $\mathfrak{M}$ is said to have the *maximal-point property* iff the induced general frame $\mathfrak{M}_{\mathrm{g}}$ has the maximal-point property.

Now, instead of Kripke models, we perform this same technique on quasi-states. The maximal-point property has a simple analogue in this setting.

Definition 8.18 (Maximal-point property for quasi-states)**.** A Σ-quasi-state $\mathfrak{S} = \langle X, R, l\rangle$ is said to have the *maximal-point property* iff for every $\varphi \in \Sigma$ and every $x \in [\![\varphi]\!]_{\mathfrak{S}}$, $R^*(x) \cap [\![\varphi]\!]_{\mathfrak{S}}$ has a maximal point.

For the rest of this section, let Σ be a subformula closed set of DTL-formulas such that only finitely many, say $n \in \omega$, formulas in Σ have $\Box$ as their top-level symbol.

Lemma 8.19. *There exists a constant $C_n \in \omega$, dependent on and computable in n, such that the following holds. Let $\mathfrak{S} = \langle \mathfrak{F}, l\rangle$ be a Σ-quasi-state with the maximal-point property and $X_0 \subseteq \mathfrak{F}_{\mathrm{w}}$. Then there exists a set $X \subseteq \mathfrak{F}_{\mathrm{w}}$ such that $X_0 \subseteq X$, $|X| \leq C_n \cdot |X_0|$ and $\mathfrak{S}$ restricted to X forms a Σ-quasi-state.*

Proof. To make the restriction of $\mathfrak{S}$ to X a quasi-state means precisely that whenever $x \in X$ and $\Box\psi \in \Sigma$ are such that $\Box\psi \notin l(x)$, then we need to include a witness y in X such that $\psi \notin l(y)$.

We give an algorithm to find X by constructing a finite sequence X_{-}. Take X_0 as given, and define for each $x \in X_0$,

$$\Psi_0^x := \{\psi \mid \Box\psi \in \Sigma, \Box\psi \notin l(x)\}.$$

The set of formulas Ψ_{-}^x represents the formulas for which x still requires witnesses. Note that by weak-transitivity of $\mathfrak{F}$, every successor of x can only ever require witnesses for formulas in Ψ_0^x.

Assume X_i and Ψ_i^{-} have been defined before, and

$$\Psi_i^x \subseteq \{\psi \mid \Box\psi \in \Sigma, \Box\psi \notin l(x)\}.$$

Let $x \in X_i$ and $\psi \in \Psi_i^x$. Then there exists a witness y above x such that $\psi \notin l(y)$, and since $\mathfrak{S}$ has the maximal-point property, a maximal such point. Pick such maximal witness and call it y_ψ^x. Now define X_{i+1} by adding to X_i all these y_{-}^{-}, i.e.

$$X_{i+1} := X_i \cup \left\{ y_\psi^x \;\middle|\; x \in X_i, \psi \in \Psi_i^x \right\}.$$

Now we need to define new sets of formulas Ψ_{i+1}^{-}. Since any point $x \in X_i$ now has all the witnesses it requires, we set $\Psi_{i+1,x} := \varnothing$ for such x. For $y := y_\psi^x$, note that since y was a maximal ψ-witness, y nor any of its successors need a ψ-witness: whenever such point needs a ψ-witness, it sees y. We also know as an induction invariant that any successor of x can only ever require witnesses for formulas in Ψ_i^x. Hence, for Ψ_{i+1}^y we can restrict ourselves to $\Psi_i^x \setminus \{\psi\}$. Therefore, we define

$$\Psi_i^y \subseteq \{\chi \in \Psi_i^x \setminus \{\psi\} \mid \Box\chi \notin l(y)\},$$

and note that by weak-transitivity of $\mathfrak{F}$ and the argument above, every successor of y can only ever require witnesses for formulas in Ψ_{i+1}^y.

Clearly the maximal size Ψ_i^x strictly decreases as a function of i, so the construction terminates. To be more precise, $|\Psi_0^x| \leq n$ for $x \in X_0$, and hence $\Psi_n^x = \varnothing$ for all $x \in X_n$. Then every $x \in X_n$ has all the witnesses it needs, so we can take $X := X_n$.

To determine the constant C_n, note that at every step we introduce, for every point of X_i at most n 'new' points to X_{i+1}. Since there are n steps, we can take $C_n := (n+1)^n$ as a very rough upper bound. Counting more carefully, one can note that every point in X_0 introduces at most n witnesses, each of which require at most $n-1$ witnesses, etc., which gives the better constant

$$C_n := \sum_{i=1}^{n-1} \frac{n!}{i!} \leq (n+1)^n.$$

□

Recall that the more categorical version of $\mathfrak{F}$ being a subframe of $\mathfrak{G}$ is the existence of an embedding from $\mathfrak{F}$ to $\mathfrak{G}$. To be more precise, $\mathfrak{F}$ embeds into $\mathfrak{G}$ iff $\mathfrak{F}$ is *isomorphic* to a subframe of $\mathfrak{G}$. For Kripke models, the analogue holds, where embeddings are required to preserve atomic propositions. For quasi-states, the analogue is a label-preserving embedding. Now the commutative square that we have been referring to, is the following lemma.

Lemma 8.20. *Let $\mathfrak{S}_1, \mathfrak{S}'_1, \mathfrak{S}_2$ be Σ-quasi-states, $f_1\colon \mathfrak{S}'_1 \to \mathfrak{S}_1$ a label-preserving embedding and $g\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ a quasi-state-morphism. Then there exist a Σ-quasi-state $\mathfrak{S}'_2$, a label-preserving embedding $f_2\colon \mathfrak{S}'_2 \to \mathfrak{S}_2$ and a quasi-state-morphism $g'\colon \mathfrak{S}'_1 \to \mathfrak{S}'_2$ such that $|\mathfrak{S}'_2| \leq C_n \cdot |\mathfrak{S}'_1|$ and $f_2 \circ g' = g \circ f_1$, i.e. the following square commutes:*

$$\begin{array}{ccc} \mathfrak{S}'_1 & \overset{g'}{\dashrightarrow} & \mathfrak{S}'_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathfrak{S}_1 & \xrightarrow{\;g\;} & \mathfrak{S}_2 \end{array}$$

Proof. Let X_0 be the image of $g \circ f_1$. By Lemma 8.19 there exists X extending X_0 such that the restriction of $\mathfrak{S}_2$ to X is a Σ-quasi-state and $|X| \leq C_n \cdot |X_0|$. Define $\mathfrak{S}'_2$ to be this restriction. Now the cardinality requirement is satisfied, and for f_2 we can take the restricted identity from $\mathfrak{S}'_2$ to $\mathfrak{S}_2$, which is clearly an embedding.

Finally define $g' := g \circ f_1$. Note that this is well-defined since the image of $g \circ f_1$ is contained in $\mathfrak{S}'_2$. By Lemma 8.13, g' is a quasi-state-morphism as required. Clearly, this g' makes the diagram commute. □

Selective filtration for quasi-segments. Repeatedly applying the previous lemma gives selective filtration theorem for quasi-segments. Note that the quasi-states that we select are not just finite, but we have an explicit upper bound on their sizes.

Definition 8.21 (Sized quasi-segment)**.** We say a finite or ω-sequence of Σ-quasi-states $\mathfrak{S}_\_$ is *sized* iff for all i, $|\mathfrak{S}_i| \leq C_n^{i+1}$. By extension, a Σ-quasi-segment is *sized* iff its sequence of quasi-states is sized.

Lemma 8.22. *Let $\varphi \in \Sigma$ and let $\mathfrak{S}_\_$ with morphisms $f_\_$ be a Σ-quasi-segment satisfying φ in $\mathfrak{S}_0$. Then there exist a sized Σ-quasi-segment of the same length $\mathfrak{S}'_\_$ with morphisms $f'_\_$ satisfying φ in $\mathfrak{S}'_0$, and a label-preserving embedding $g_\_$ of quasi-segments from $\langle \mathfrak{S}'_\_, f'_\_ \rangle$ to $\langle \mathfrak{S}_\_, f_\_ \rangle$. If the initial segment of $\mathfrak{S}_\_$ of length n is sized, we can take $\mathfrak{S}'_i := \mathfrak{S}_i$ for all $i < n$.*

Proof. Find a point x_0 in $\mathfrak{S}_0$ which has φ in its label. By Lemma 8.19 there exists a subset X of the domain of $\mathfrak{S}_0$ with $x_0 \in X$ and $|X| \leq C_n$ such that the restriction of $\mathfrak{S}_0$ to X is a quasi-state. Define $\mathfrak{S}'_0$ to be this restriction and set g_0 to be the restricted identity. Now, repeatedly applying the previous lemma produces the required commutative diagram. □

Note that label-preserving embeddings reflect the realisation of eventualities. Hence if the original quasi-segment $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$ was a quasi-model, then so is $\langle \mathfrak{S}'_{\_}, f'_{\_} \rangle$.

The local fmp. Using the conversion propositions from the previous section, we get an analogous statement for stratified DFMs. We use this to prove the local fmp for the DTL-logics that we are interested in, assuming they are complete w.r.t. their DFMs with the maximal-point property.

Definition 8.23. [Maximal-point property for DFMs] A DFM $\langle X, R, f, \mathfrak{V} \rangle$ is said to have the *maximal-point property* iff for every DTL-formula φ and every $x \in [\![\varphi]\!]_{\langle \mathfrak{F}, f, \mathfrak{V} \rangle}$, $R^*(x) \cap [\![\varphi]\!]_{\langle \mathfrak{F}, f, \mathfrak{V} \rangle}$ has a maximal point.

Theorem 8.24. *Let $\mathcal{F}$ be a class of Kripke frames closed under taking disjoint unions and subframes. Suppose* DTL($\mathcal{F}$) *is complete w.r.t. DFMs, over frames in $\mathcal{F}$, that have the maximal-point property for DFMs. Then* DTL($\mathcal{F}$) *is complete w.r.t. stratified locally finite DFSs on frames in $\mathcal{F}$. In particular, it has the local fmp.*

Proof. Let φ be a DTL-formula such that $\neg\varphi \notin \Lambda$. By the completeness assumption in the theorem statement, there exists a DFM $\langle \mathfrak{F}, f, \mathfrak{V} \rangle$ with the maximal-point property, satisfying φ in some point, such that $\mathfrak{F} \in \mathcal{F}$. By Remark 8.5 we can turn this into an stratified DFM $\langle \mathfrak{M}_{\_}, f_{\_} \rangle$. Note that each $\mathfrak{M}_{i,\mathrm{fr}} \in \mathcal{F}$. It is easy to check that this stratified DFM inherits the maximal-point property.

Define $\Sigma \coloneqq \mathrm{Sub}(\varphi)$. This set is finite, so in particular only finitely many formulas in Σ have $\Box$ as their top-level symbol.

By Proposition 8.11, $\langle \mathfrak{M}_{\_}, f_{\_} \rangle$ induces a $(\Sigma, \mathcal{F})$-quasi-model $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$ which satisfies φ in $\mathfrak{S}_0$. It is easy to check that each $\mathfrak{S}_{\_}$ inherits the maximal-point property from $\langle \mathfrak{M}_{\_}, f_{\_} \rangle$.

By Lemma 8.22, there exists a sized Σ-quasi-segment $\langle \mathfrak{S}'_{\_}, f'_{\_} \rangle$ still satisfying φ in $\mathfrak{S}'_0$ and a label-preserving embedding from $\langle \mathfrak{S}'_{\_}, f'_{\_} \rangle$ to $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$. Since $\mathcal{F}$ is closed under taking subframes, the frames underlying the $\mathfrak{S}'_{\_}$ are (isomorphic to) frames in $\mathcal{F}$. Since $\langle \mathfrak{S}_{\_}, f_{\_} \rangle$ is a quasi-model, so is $\langle \mathfrak{S}'_{\_}, f'_{\_} \rangle$.

Finally, by Proposition 8.12 this quasi-model induces an stratified DFM $\langle \mathfrak{M}'_{\_}, f'_{\_} \rangle$ which still satisfies φ in $\mathfrak{M}'_0$. Clearly it is locally finite. Since $\mathfrak{M}'_{i,\mathrm{fr}} \in \mathcal{F}$ for all i, and $\mathcal{F}$ is closed under disjoint union, the underlying frame is a frame in $\mathcal{F}$. Hence $\langle \mathfrak{M}'_{\_}, f'_{\_} \rangle$ is a DFM of DTL($\mathcal{F}$). □

8.5 Reduced Quasi-Segments and Computable Enumerability

In this section we will apply the selective filtration for quasi-segments developed in the previous section a second time. This time, we combine it with Kruskal's tree theorem and make good use of the effective bound on the sizes of quasi-states, to prove a co-computable enumerability result for the formulas that are satisfiable on quasi-models.

Let $\mathcal{F}$ be a class of Kripke frames. Let us say that a DTL-formula φ is *$\mathcal{F}$-quasi-satisfiable* iff it is satisfiable on a $(\mathrm{Sub}(\varphi), \mathcal{F})$-quasi-model. Propositions 8.11 and 8.12 show a strong connection with satisfiability on stratified DFMs. If $\mathcal{F}$ is closed under taking disjoint unions, then together with Remark 8.5 this shows that φ is *$\mathcal{F}$-quasi-satisfiable* iff it is satisfiable on a DFS of $\mathrm{DTL}(\mathcal{F})$. Hence we would like to prove that the set of *$\mathcal{F}$-quasi-satisfiable* is co-computably enumerable, so that $\mathrm{DTL}(\mathcal{F})$ is computably enumerable.

Let us call a $(\mathrm{Sub}(\varphi), \mathcal{F})$-quasi-segment a $(\varphi, \mathcal{F})$-quasi-segment iff is φ is satisfiable in the first quasi-state, and analogously for quasi-models. Then *$\mathcal{F}$-quasi-satisfiable* iff there exists a $(\varphi, \mathcal{F})$-quasi-segment.

In this section we restrict our attention to the case where $\mathcal{F}$ consists of only finite forest-like frames, and is closed under taking subframes. The finite and forest-like conditions make Kruskal's tree theorem, in particular the version stated in Theorem 8.3, apply to quasi-states. The closedness under taking subframes makes the selective filtration technique from the last section apply. Since a quasi-state on a finite frame has the maximal-point property, the latter implies that if a DTL-formula φ is satisfiable on a $(\Sigma, \mathcal{F})$-quasi-segment respectively a $(\Sigma, \mathcal{F})$-quasi-model, then it is satisfiable on a sized one.

For a given length $k \in \omega$, there are only finitely many sized $\mathrm{Sub}(\varphi)$-quasi-segments of length k up to isomorphism. Assuming $\mathcal{F}$, up to isomorphism, is decidable, one can check for each one whether it is a $(\varphi, \mathcal{F})$-quasi-segment. Hence, the set of DTL-formulas φ such that there exists, for each finite length $k \in \omega$, a sized $(\varphi, \mathcal{F})$-quasi-segment of length k, is co-computably enumerable.

Using Kőnig's lemma, one can show that if for each $k \in \omega$, such sized $(\varphi, \mathcal{F})$-quasi-segment of length k exists, then there exists an infinite $(\varphi, \mathcal{F})$-quasi-segment. The converse direction follows trivially after applying selective filtration to make the quasi-segment sized. Hence, the set of formulas φ such that there exists an infinite $(\varphi, \mathcal{F})$-quasi-segment, is co-computably enumerable.

However, not every infinite quasi-segment is a quasi-model, so this does not establish co-computable enumerability of *$\mathcal{F}$-quasi-satisfiability.* In order to solve this issue, we introduce reduced quasi-segments. The idea will be that the existence of reduced $(\varphi, \mathcal{F})$-quasi-segments for all lengths $k \in \omega$ does imply the existence of a $(\varphi, \mathcal{F})$-quasi-model. The proof of this claim will make use of Kruskal's tree theorem.

Let us fix a DTL-formula φ for the rest of the section. We will from now on leave the $(\varphi, \mathcal{F})$ for quasi-segments and quasi-models implicit.

Lemma 8.25. *Let $\mathfrak{S}_\_$ be a $(\varphi, \mathcal{F})$-quasi-segment with morphisms $f_\_$. Suppose there exist i, j such that $i < j$, and an label-preserving embedding $g \colon \mathfrak{S}_i \to \mathfrak{S}_j$. Then*

$$\mathfrak{S}_0 \overset{f_0}{\to} \cdots \overset{f_{i-1}}{\to} \mathfrak{S}_i \overset{f_j \circ g}{\to} \mathfrak{S}_{j+1} \overset{f_{j+1}}{\to} \mathfrak{S}_{j+2} \cdots \tag{8.1}$$

is a $(\varphi, \mathcal{F})$*-quasi-segment.*

Proof. By Lemma 8.13. □

Note that in the previous lemma, if some eventuality was realised in the original quasi-segment, then either it is still so in the shorter one, or it was realised in some $\mathfrak{S}_k$ for $k \in \{i+1, \dots, j\}$. This leads to the following definition.

Definition 8.26 (Progress)**.** Let $\mathfrak{S}\_$ be a quasi-segment with morphisms $f\_$. It is said to *progress* at k iff $\mathfrak{S}_k$ realises an eventuality for some point of $\mathfrak{S}_j$, where j is an index such that for all $i < j$, all eventualities for points in $\mathfrak{S}_i$ are realised *before* $\mathfrak{S}_k$, i.e. in $\mathfrak{S}_0, \dots \mathfrak{S}_{k-1}$.

Definition 8.27 (Reduced quasi-segment)**.** A quasi-segment is called *n-reduced* iff either

- for every pair $\langle i, j \rangle$ such that $i < j$, $i < n$ and $\mathfrak{S}_i$ label-preservingly embeds into $\mathfrak{S}_j$, there exists $k \in \{i+1, \dots, j\}$ such that the quasi-segment progresses at k, or
- every eventuality in it is realised before $\mathfrak{S}_n$.

It is called *reduced* iff it is n-reduced for all n.

We derive two lemmata about this reducedness. First we show that a reduced and infinite quasi-segment is a quasi-model. Conversely, we show that any quasi-model can be turned into a sized and n-reduced one.

For the first lemma we will use Kruskal's tree theorem. Note that this requires a well-quasi-order on the labels, i.e. on ty_Σ. Since we take $\Sigma \coloneqq \mathrm{Sub}(\varphi)$, Σ is finite, and hence so is ty_Σ. Therefore ty_Σ forms a well-quasi-order under *equality*. Then a function is label-preserving iff it is label-monotone.

Lemma 8.28.

(i) A reduced infinite $(\varphi, \mathcal{F})$*-quasi-segment progresses infinitely often.*

(ii) A $(\varphi, \mathcal{F})$*-quasi-segment that progresses infinitely often is a* φ*-quasi-model.*

Proof.

(i) Let $\mathfrak{S}\_$ together with some morphisms be a reduced infinite $(\varphi, \mathcal{F})$-quasi-segment. By Kruskal's tree theorem, in particular Theorem 8.3, there exist $i, j \in \omega$ such that $i < j$ and $\mathfrak{S}_i$ label-preservingly embeds into $\mathfrak{S}_j$. Since the quasi-segment is reduced, there is a $k \in \{i+1, \dots, j\}$ such that the quasi-segment progresses at k. By applying Kruskal's tree theorem again on the sequence from $\mathfrak{S}\_$ from k onwards, we find another $k' > k$ where the quasi-segment progresses, etc.

(ii) Let $n \in \omega$. We show that all eventualities in the first n quasi-states get realised. The first n quasi-states of the quasi-segment only have finitely many eventualities that need to be realised. Suppose the quasi-segment progresses at $\mathfrak{S}_k$. Then an eventuality of $\mathfrak{S}_j$ gets realised that was not realised before, where $\mathfrak{S}_j$ is the first quasi-state with not-yet-realised eventualities. In particular, either $j < n$, or all eventualities of the first n quasi-states are already realised, in which case we are done. But in the former case, the number of eventualities in the first n quasi-states that are not-yet-realised decreases, which can happen only finitely often.

□

Lemma 8.29. *Let $\mathfrak{S}_\_$ together with some morphisms be a $(\varphi, \mathcal{F})$-quasi-model. Then for every $n \in \omega$ there exists a sized and n-reduced $(\varphi, \mathcal{F})$-quasi-model.*

Proof. By induction on n. For the base case, note that there exists by assumption (iv) a quasi-model, and we can turn this into a sized one by Lemma 8.22. Obviously it is 0-reduced.

For the inductive step, assume we have a sized and n-reduced quasi-model. Suppose it is not $(n+1)$-reduced. Then there exists $j > n$ such that $\mathfrak{S}_n$ label-preservingly embeds into $\mathfrak{S}_j$ and the quasi-segment does not progress at indices $n+1, \dots, j$.

First assume no maximal such j exists. Then the quasi-model never progresses after $\mathfrak{S}_n$. Since we have a quasi-model, every eventuality gets realised. Hence this must happen before $\mathfrak{S}_n$, so the quasi-model is by definition already reduced.

Second, assume a maximal such j does exists. Then by Lemma 8.25 we find a quasi-model on

$$\mathfrak{S}_0, \dots, \mathfrak{S}_n, \mathfrak{S}_{j+1}, \dots$$

which is $(n+1)$-reduced.

Note that the initial segment of length $n+1$ is sized. Applying Lemma 8.22 we can turn this into a sized quasi-model without changing the first $n+1$ quasi-states:

$$\mathfrak{S}_0, \dots, \mathfrak{S}_n, \mathfrak{S}'_{j+1}, \dots$$

Now this new quasi-model is still $(n+1)$-reduced, because any label-preserving embedding $\mathfrak{S}_i$ to $\mathfrak{S}'_k$ (for some $i \in \{0, \dots, n\}$ and $k > j$) composes with one of the label-preserving embeddings from Lemma 8.22 to a label-preserving embedding $\mathfrak{S}_i$ to $\mathfrak{S}_k$ in the original sequence. □

We are now in the position to prove the main equivalence result of this section.

Proposition 8.30. *For any formula φ, the following are equivalent:*

(i) For every $n \in \omega$ there exists a sized and reduced $(\varphi, \mathcal{F})$-quasi-segment of length at least n.

(ii) There exists an infinite sized and reduced $(\varphi, \mathcal{F})$-quasi-segment.

(iii) There exists an sized and reduced $(\varphi, \mathcal{F})$-quasi-model.

(iv) There exists a $(\varphi, \mathcal{F})$-quasi-model.

(v) For every $n \in \omega$ there exists a sized and n-reduced $(\varphi, \mathcal{F})$-quasi-model.

Proof. (i) ⇒ (ii): Let us say that a quasi-segment $\langle \mathfrak{S}_{-}, f_{-} \rangle$ *is a successor of* a quasi-segment $\langle \mathfrak{S}'_{-}, f'_{-} \rangle$ iff $\langle \mathfrak{S}'_{-}, f'_{-} \rangle$ is an initial segment of $\langle \mathfrak{S}_{-}, f_{-} \rangle$. Then the set of sized and reduced $(\varphi, \mathcal{F})$-quasi-segments up to isomorphism forms a tree T under this successor relation.

Note that there are only finitely many sized $(\varphi, \mathcal{F})$-quasi-segments of length n up to isomorphism. Hence T is finitely branching. Since there are sized and reduced $(\varphi, \mathcal{F})$-quasi-segments of arbitrary lengths, T is infinite. Hence, by Kőnig's lemma there exists an infinite path in T. Obviously this gives rise to a quasi-segment whose initial segments are the points in this path. Since all these initial segments are sized and reduced, so is the full quasi-segment.

(ii) ⇒ (iii): By Lemma 8.28.

(iii) ⇒ (iv): Trivial.

(iv) ⇒ (v): By Lemma 8.29.

(v) ⇒ (i): Take the initial segment of length n of the sized and n-reduced $(\varphi, \mathcal{F})$-quasi-model. □

As an immediate consequence we find the theorem that we were after: the set of formulas that are $\mathcal{F}$-quasi-satisfiable is co-computably enumerable.

Theorem 8.31. *Let $\mathcal{F}$ be a class of finite forest-like Kripke frames closed under taking subframes, such that membership of $\mathcal{F}$ (up to equivalence) is decidable. Then the set of DTL-formulas φ that are $\mathcal{F}$-quasi-satisfiable is co-computably enumerable.*

Proof. By the previous proposition having a $(\varphi, \mathcal{F})$-quasi-model is equivalent to having sized and reduced $(\varphi, \mathcal{F})$-quasi-segments for all finite lengths. For a given formula φ and length n we can decide whether there is a sized and reduced $(\varphi, \mathcal{F})$-quasi-segment of length n, by checking all finitely many (up to isomorphism) candidates. Hence universally quantifying over $n \in \omega$ gives a set that is co-computably enumerable. □

8.6 Computable Enumerability over CWF Frames

In this section we apply the main theorem from the previous section to prove computable enumerability of DTL($\mathcal{F}$) for a certain frame class $\mathcal{F}$. There are however two requirements on Theorem 8.31 that we need to alleviate. First, the requirement of finiteness of the frames in $\mathcal{F}$, which is alleviated using the local fmp result from

Section 8.4. Second, Theorem 8.31 applies only to forest-like frames. This is satisfied by applying a kind of tree unravelling.

The general approach is as follows. We start with a class of frames $\mathcal{F}$ which is closed under taking disjoint unions and subframes and satisfies some addition requirements to make the subsequent steps work. We will impose a condition on $\mathcal{F}$ that implies the maximal-point property for all stratified DFMs on a frame in $\mathcal{F}$. Then, by Theorem 8.24, DTL($\mathcal{F}$) has the local fmp. Hence a formula φ satisfiable on a DFS of DTL($\mathcal{F}$), is satisfiable on locally finite DFM on a frame $\mathfrak{F} \in \mathcal{F}$.

Next, we assume $\mathcal{F}$ is closed under a kind of tree unravelling. Then, by performing this tree unravelling, we can assume that $\mathfrak{F}$ is a forest-like frame. Using Proposition 8.11 this stratified DFM induces a quasi-model. Since $\mathfrak{F}$ is forest-like and the DFM was locally finite, this is a $(\varphi, \mathcal{F}')$-quasi-model, where $\mathcal{F}'$ the class of finite forest-like frames of $\mathcal{F}$. Since $\mathcal{F}$ is closed under disjoint union, using Proposition 8.12 such $(\varphi, \mathcal{F}')$-quasi-model also induces a DFM on a frame in $\mathcal{F}$, satisfying φ in some point.

We conclude that satisfiability on a DFS on a frame in $\mathcal{F}$ is equivalent to $\mathcal{F}'$-quasi-satisfiability. Since $\mathcal{F}'$ satisfies the requirements of Theorem 8.31, the set of satisfiable formulas is co-computably enumerable. Hence DTL($\mathcal{F}$) is computably enumerable.

The maximal-point property. As explained, we need the maximal-point property for all stratified DFMs on a frame in $\mathcal{F}$, in order to apply Theorem 8.24. In fact, a unimodal property that guarantees this is converse pre-well-foundedness, which we already studied in the context of degrees in Chapter 4.

Lemma 8.32. *Let $\mathfrak{F}$ be a weakly-transitive* CWF *Kripke frame. Then any general frame $\mathfrak{f} = \langle \mathfrak{F}, A \rangle$ or Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on it has the maximal-point property.*

Proof. Obviously it suffices to prove that an arbitrary non-empty subset $Y \subseteq \mathfrak{F}_{\mathrm{w}}$ has a maximal point. Recall the definitions of depth and converse pre-well-foundedness from Section 2.10. Since $\mathfrak{F}$ is CWF, $\mathfrak{F} = \mathfrak{F}^{\mathrm{upper}}$. Therefore, as Y is non-empty and the ordinals are well-ordered, there exists some $y_0 \in Y$ with minimal depth.

We claim this y_0 is maximal for Y. For suppose $y \in Y$ such that y_0 sees y. Then by definition of depth, $\mathrm{depth}(y) \leq \mathrm{depth}(y_0)$. Since y_0 had minimal depth, $\mathrm{depth}(y) = \mathrm{depth}(y_0)$. But by construction of depth then y also sees y_0. ☐

Clearly, this implies the maximal-point property for DFMs on a CWF frame.

Lemma 8.33. *Let $\mathcal{F}$ be a class of* CWF *Kripke frames. Then* DTL($\mathcal{F}$) *is complete w.r.t. DFMs, over frames in $\mathcal{F}$, with the maximal-point property.*

Proof. By definition DTL($\mathcal{F}$) is complete w.r.t. DFMs $\langle \mathfrak{F}, f, \mathfrak{V} \rangle$ such that $\mathfrak{F} \in \mathcal{F}$. The set

$$\left\{ [\![\varphi]\!]_{\langle \mathfrak{F}, f, \mathfrak{V} \rangle} \;\middle|\; \varphi \in \mathrm{DTL} \right\}$$

generates a general frame on $\mathfrak{F}$, which has the maximal-point property by the previous lemma. It clearly follows that $\langle \mathfrak{F}, f, \mathfrak{V} \rangle$ has the maximal-point property. □

By Theorem 8.24 the local fmp follows.

Unravelling. Next, we look at the tree unravelling. To be used as in the proof outline at the start of this section, we need an unravelling that preserves the local finiteness of DFSs. Fortunately, in Section 2.12 we developed an unravelling preserving finiteness, the *finite tree-like unravelling*. Using the functoriality it enjoys, extending it to locally finite DFSs is easy.

Definition 8.34 (p-Morphism of stratified DFSs)**.** Let $\langle \mathfrak{F}_\_, f_\_ \rangle$ and $\langle \mathfrak{G}_\_, g_\_ \rangle$ be stratified DFSs. Then a p-morphism from $\langle \mathfrak{G}_\_, g_\_ \rangle$ to $\langle \mathfrak{F}_\_, f_\_ \rangle$ is a sequence of functions $h_\_$ such that for all $i \in \omega$, $h_i \colon \mathfrak{G}_i \to \mathfrak{F}_i$ is a p-morphism and $f_i \circ h_i = h_{i+1} \circ g_i$, i.e. the following diagram commutes:

$$\begin{array}{ccccccc} \mathfrak{G}_0 & \xrightarrow{g_0} & \mathfrak{G}_1 & \xrightarrow{g_1} & \mathfrak{G}_2 & & \cdots \\ {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \\ \mathfrak{F}_0 & \xrightarrow[f_0]{} & \mathfrak{F}_1 & \xrightarrow{f_1} & \mathfrak{F}_2 & & \cdots \end{array}$$

It is called surjective iff each h_i is surjective, in which case $\langle \mathfrak{F}_\_, f_\_ \rangle$ is called a p-morphic image of $\langle \mathfrak{G}_\_, g_\_ \rangle$.

Lemma 8.35. *Let $\langle \mathfrak{F}_\_, f_\_ \rangle$ be an stratified locally finite DFS. Then there exists an stratified locally finite DFS $\langle \mathfrak{G}_\_, g_\_ \rangle$ such that for each $i \in \omega$, $\mathfrak{G}_i$ is the finite tree-like unravelling of $\mathfrak{F}_i$ and $\langle \mathfrak{F}_\_, f_\_ \rangle$ is a p-morphic image of $\langle \mathfrak{G}_\_, g_\_ \rangle$.*

Proof. Let for each i, $\mathfrak{G}_i$ be the finite tree-like unravelling of $\mathfrak{F}_i$, $h_i \colon \mathfrak{G}_i \to \mathfrak{F}_i$ the surjective p-morphism from Lemma 2.106, and $g_i \colon \mathfrak{G}_i \to \mathfrak{G}_{i+1}$ the monotone function induced by f as in Lemma 2.107. □

p-Morphism for DFSs preserves valid DTL-formulas and reflects satisfiability, analogous to p-morphism for Kripke frames w.r.t. unimodal formulas.

Lemma 8.36. *Let $\langle \mathfrak{G}_\_, g_\_ \rangle$ be an stratified DFS and $\langle \mathfrak{F}_\_, f_\_ \rangle$ a p-morphic image of it. Let φ be a DTL-formula satisfiable on $\langle \mathfrak{F}_\_, f_\_ \rangle$. Then it is satisfiable on $\langle \mathfrak{G}_\_, g_\_ \rangle$.*

Proof. Let $h_\_$ be the surjective p-morphism from $\langle \mathfrak{G}_\_, g_\_ \rangle$ to $\langle \mathfrak{F}_\_, f_\_ \rangle$. Find valuations $\mathfrak{V}_i$ such that φ is satisfied in a point of $\langle \mathfrak{M}_\_, f_\_ \rangle$, where $\mathfrak{M}_i := \langle \mathfrak{F}_i, \mathfrak{V}_i \rangle$. Define $\mathfrak{N}_i := \langle \mathfrak{G}_i, h_i^{-1} \circ \mathfrak{V}_i \rangle$.

We show by induction on formulas $\psi \in \mathrm{Sub}(\varphi)$ that ψ is satisfied in a point x of $\langle \mathfrak{N}_\_, g_\_ \rangle$ iff it is satisfied in $h(x)$ of $\langle \mathfrak{M}_\_, f_\_ \rangle$. For the atomic propositions, Boolean connectives and modal operator $\Box$ the inductive steps are standard and follow from the fact that h is a p-morphism.

For $\bigcirc$, assume as induction hypothesis that for all $y \in \mathfrak{G}_{\mathrm{w}}$, $\mathfrak{N}, y \vDash \psi$ iff $\mathfrak{M}, h(y) \vDash \psi$. Then for a point $x \in \mathfrak{G}_{i,\mathrm{w}}$,

$$\mathfrak{N}, x \vDash \bigcirc\psi \iff \mathfrak{N}, g_i(x) \vDash \psi \iff \mathfrak{M}, h_{i+1}(g_i(x)) \vDash \psi$$
$$\iff \mathfrak{M}, f_i(h_i(x)) \vDash \psi \iff \mathfrak{M}, h_i(x) \vDash \bigcirc\psi.$$

That the case for $\Box_F$ follows from that for $\bigcirc$ since satisfying $\Box_F\psi$ is equivalent to simultaneously satisfying all of $\bigcirc^n\psi$ for $n \in \omega \setminus \{0\}$. (The proof is similar to that of conjunction, but with 'infinitely many conjunctions'.) □

Together, these two lemmata tell us that if $\mathcal{F}$ is closed under finite tree-like unravelling then any formula satisfiable on an stratified locally finite DFS on a frame in $\mathcal{F}$ is satisfiable on an stratified locally finite DFS on a forest-like frame in $\mathcal{F}$. In particular, when combined with Theorem 8.24 we get the following.

Proposition 8.37. *Let $\mathcal{F}$ be a class of Kripke frames closed under taking subframes, disjoint union and finite tree-like unravelling. Suppose* DTL($\mathcal{F}$) *is complete w.r.t. DFMs, over frames in $\mathcal{F}$, that have the maximal-point property for DFMs. Then* DTL($\mathcal{F}$) *is complete w.r.t. stratified locally finite DFSs on forest-like frames in $\mathcal{F}$.*

Proof. Let φ be a formula such that $\neg\varphi \notin \mathrm{DTL}(\mathcal{F})$. By Theorem 8.24, φ is satisfiable on an stratified locally finite DFS $\langle \mathfrak{F}_\_, f_\_\rangle$. By Lemma 8.35 $\langle \mathfrak{F}_\_, f_\_\rangle$ is a p-morphic image of its finite tree-like unravelling $\langle \mathfrak{G}_\_, g_\_\rangle$. Since $\mathcal{F}$ is closed under finite tree-like unravelling, the disjoint union of the frames $\mathfrak{G}_\_$ is a frame of $\mathcal{F}$. Hence, since finite tree-like unravelling preserves finiteness, $\langle \mathfrak{G}_\_, g_\_\rangle$ is an stratified locally finite DFS on forest-like frame in $\mathcal{F}$. By Lemma 8.36 φ is satisfiable on $\langle \mathfrak{G}_\_, g_\_\rangle$. □

Conclusion. Combining this last result with Lemma 8.33 and Theorem 8.31 gives us our desired computable enumerability result. Since now all hypotheses are unimodal in nature, the theorem is easily applied.

Theorem 8.38. *Let $\mathcal{F}$ be a class of* CWF *Kripke frames closed under taking disjoint unions, subframes and finite tree-like unravelling, such that for finite forest-like frames membership of $\mathcal{F}$ (up to isomorphism) is decidable. Then* DTL($\mathcal{F}$) *is computably enumerable.*

Proof. Define $\mathcal{F}'$ to be the class of finite forest-like elements of $\mathcal{F}$. By Theorem 8.31, the set of DTL-formulas that are $\mathcal{F}'$-quasi-satisfiable is co-computably enumerable. We will show that these are precisely the formulas φ such that $\neg\varphi \notin \mathrm{DTL}(\mathcal{F})$, formulas whose negationis a non-theorem of DTL($\mathcal{F}$). Hence the non-theorems of DTL($\mathcal{F}$) form a co-computably enumerable set, so DTL($\mathcal{F}$) itself is computably enumerable.

Let φ be a DTL-formula. First, suppose φ is $\mathcal{F}'$-quasi-satisfiable. Then there exists a $(\varphi, \mathcal{F}')$-quasi-model, and by Proposition 8.12 it induces an stratified DFM satisfying φ in a point. The underlying frame is a disjoint union of frames in $\mathcal{F}' \subseteq \mathcal{F}$, hence an element of $\mathcal{F}$. By definition of $\mathrm{DTL}(\mathcal{F})$, $\neg\varphi \notin \mathrm{DTL}(\mathcal{F})$.

Conversely suppose $\neg\varphi \notin \mathrm{DTL}(\mathcal{F})$. By Lemma 8.33 $\mathrm{DTL}(\mathcal{F})$ is complete w.r.t. DFMs over $\mathcal{F}$ with the maximal-point property. Then it follows by Proposition 8.37 that $\mathrm{DTL}(\mathcal{F})$ is complete w.r.t. stratified locally finite DFSs on forest-like frames in $\mathcal{F}$. Therefore φ is satisfied such a DFS. By Proposition 8.11 it is satisfied on the induced $\mathrm{Sub}(\varphi)$-quasi-model. Since the DFS was locally finite and on a forest-like frames in $\mathcal{F}$, this is a $(\mathrm{Sub}(\varphi), \mathcal{F}')$-quasi-model. Say φ is satisfied in k-th quasi-state. We obtain a $(\varphi, \mathcal{F}')$-quasi-model by dropping the first $k-1$ quasi-states. Hence φ is $\mathcal{F}'$-quasi-satisfiable. □

In particular, when Λ is a unimodal subframe logic whose frames are closed under finite tree-like unravelling, then $\mathrm{DTL}(\mathcal{F})$ is computably enumerable for

- $\mathcal{F}$ the class of all disjoint unions of finite Λ-frames, and
- $\mathcal{F}$ the class of all CWF Λ-frames.

Applying the theorem shows that for example the following DTL-logics are computably enumerable.

Corollary 8.39. *Let $\mathcal{F}$ be any of the following frame classes:*

(i) disjoint unions of the finite frames of **wK4**, **K4** *or* **S4**,

(ii) the CWF frames of **wK4**, **K4** *or* **S4**,

(iii) the frames of **GL** *or* **Grz**,

(iv) the frames of $\mathbf{K4} \oplus \Box^n \bot$ *for any* $n \in \omega$, *or*

(v) linear frames of any of the previously mentioned frame classes, i.e. disjoint unions of the finite or CWF frames of **wK4.3**, **K4.3** *or* **S4.3** *or the frames of* **GL.3**, **Grz.3**, *or* $\mathbf{K4.3} \oplus \Box^n \bot$ *for any* $n \in \omega$.

Then $\mathrm{DTL}(\mathcal{F})$ *is computably enumerable.*

Remark 8.40. For the classes of linear frames with bounded depth in the corollary, i.e. the frames of $\mathbf{K4.3} \oplus \Box^n \bot$ for some $n \in \omega$, the DTL-logic is not just computably enumerable but even decidable. To see this, suppose a DTL-formula φ is satisfiable on a φ-quasi-model on a frame of $\mathbf{K4.3} \oplus \Box^n \bot$. Using a selection argument it is easy to see that one can make each quasi-state in the quasi-model rooted, giving say quasi-states $\mathfrak{S}_\_$ with morphisms $f_\_$. Then each quasi-state is based on a chain of at most n elements, and there are only finitely many such $\mathrm{Sub}(\varphi)$-quasi-states (up to isomorphism). Hence there exist $i, j \in \omega$ such that $\mathfrak{S}_i = \mathfrak{S}_j$, and *any quasi-state that occurs infinitely often in the quasi-model does occur between indices i and j.*

Now using this property of i and j, we can construct a finite DFM on which φ is still satisfied. First construct a new quasi-segment by first taking states $\mathfrak{S}_0, \dots, \mathfrak{S}_{i-1}$ and then repeating $\mathfrak{S}_i, \dots, \mathfrak{S}_{j-1}$ infinitely often (all with the respective morphisms from the quasi-model):

$$\mathfrak{S}_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{j-1}} \mathfrak{S}_j = \mathfrak{S}_i \xrightarrow{f_i} \dots \xrightarrow{f_{j-1}} \mathfrak{S}_j = \mathfrak{S}_i \xrightarrow{f_i} \dots$$

It is easy to check that this quasi-segment is a φ-quasi-model. By Proposition 8.12 it induces a stratified DFM. This can be turned into a finite DFM by identifying all the parts that are induced by one of the repetitions of $\mathfrak{S}_k$ in the quasi-model, for each $k \in \{i, \dots, j-1\}$.

Hence any DTL-formula φ that is satisfiable on a DFS on a frame of $\mathbf{K4.3} \oplus \Box^n \bot$, is satisfiable on a finite such DFS. It follows that the DTL-logic of these DFSs is co-computably enumerable. Since we already proved computable enumerability, it follows that it is decidable.

A similar proof works without the *lin* axiom, i.e. for $\mathbf{K4} \oplus \Box^n \bot$. However, now we need to work with quasi-states up to bisimilarity[3] instead of up to isomorphism. The crucial properties here are that there are only finitely many quasi-states on frames of $\mathbf{K4} \oplus \Box^n \bot$ up-to-bisimilarity, and that for bisimilar quasi-states there exist label-preserving p-morphisms in both directions between them. Note that a label-preserving p-morphisms compose 'nicely' with quasi-state-morphisms just as label-preserving embeddings do. ∎

This corollary gives a sense of the generality of Theorem 8.38, which is the main result of the chapter. However, also note how restricting the converse pre-well-foundedness condition is. *Many* important modal logics, e.g. **K4** or **S4**, have frames that are not CWF, even tough they are CWF-frame complete. Also recall that we based our proof on that of a result about DTL_1 of Konev et al. [28, Theorem 7], generalising some parts of the construction. However, due to this converse pre-well-foundedness condition, their result does not follow from ours.

In fact, the converse pre-well-foundedness condition can be lifted when restricting to the fragment of DTL, where, under a positive $\Box$, the future looking box $\Box_F$ is only allowed to occur positively. This fragment significantly extends DTL_1, so Theorem 7 of Konev et al. [28] would follow from this result. The proof requires a fair bit of duality theory though, and is therefore omitted due to size and time constraints on this thesis.

[3] A bisimulation between quasi-states is a bisimulation between the underlying Kripke frames which only relates points with the same label.

Chapter 9

Conclusions and Future Work

In this thesis we looked into various topics in modal logic related to completeness, as summarised in the introduction. For WF- and CWF-degrees, quasi-canonicity and canonical approximations, our investigations are essentially the first investigations into the topic. Therefore, our results should be seen as initial observations, founding the theory and study of these notions. We leave many questions open for future investigation. In this concluding chapter, we discuss our main contributions, important questions that we leave open and general possible directions for future research. As in the introduction, we discuss this topic by topic, for each of our four lines of study.

Degrees of completeness. We introduced a general notion of degrees of completeness, generalising and unifying the study of the degrees of incompleteness of Fine [19] and the 'degrees of fmp', which we call finite-frame degrees, of G. Bezhanishvili, N. Bezhanishvili and Moraschini [5]. We briefly considered WF-frame degrees, and showed WF-model degrees to be trivial in a sense, as every such degree over the extensions of **K4** is singleton. Our main contributions lie in the study of CWF-frame degrees, with Theorems 4.6 and 4.26. In particular, these results show the existence of singleton and continuum CWF-frame degrees over the extensions and Kripke complete extensions of **K4** and **S4**.

Considering Block's dichotomy theorem and the anti-dichotomy theorem, this immediately raises the question whether there are CWF-frame degrees of cardinality strictly in-between 1 and continuum. This question can be asked both for degrees over extensions of **K4** or **S4**, as well as degrees over *Kripke complete* such extensions. The former could be seen as a potential first step towards solving the same problem for degrees of incompleteness, which is considered a major open problem in the field [11, Problem 10.5]. The latter might be more interesting when studying converse pre-well-foundedness in its own right, as the effects of Kripke incompleteness are in a sense factored out.

There are two more specific questions that caught our attention, that we leave open. First, as noted in Proposition 2.82, finite-frame completeness and finite-model completeness coincide. In Section 3.6 we saw that the same does not hold for WF-frame and WF-model completeness. It is to our knowledge however still open, whether CWF-frame and CWF-model completeness coincide.

The second question concerns the structural theory of degrees. With Proposition 3.27 we showed that WF- and CWF-frame degrees and degrees of incompleteness are closed under finite non-empty intersections. For finite-frame degrees, G. Bezhanishvili, N. Bezhanishvili and Moraschini [5, Theorem 10.3] already showed a stronger property: these degrees are closed under arbitrary non-empty intersections. This raises the question whether the same holds for WF- and CWF-frame degrees. An affirmative answer would yield that the respective degrees form a complete lattice w.r.t. the subset order, improving on the corollary of Proposition 3.28 that they form lattices.

As a more general direction of possible future research, WF-frame degrees can be studied. Based on our preliminary investigations, these seem easier to work with than CWF-frame degrees; compare for example the proof of Theorem 3.32, that there exists an infinite WF-frame degree, with that of Proposition 4.22, that there exists an infinite (even continuum) CWF-frame degree. One can also note that from the proof of the latter, the existence of a continuum WF-degree can be easily derived. Despite this,[1] we mostly left WF-frame degrees unstudied, and focused on the CWF-frame degrees.

Quasi-canonicity. After studying degrees of completeness, we had a brief look at quasi-canonicity, as introduced by Takapui [42], in Chapter 5. We showed that quasi-canonicity is *strictly* in-between canonicity and Kripke completeness, improving on a result of Takapui [42]. Moreover, we proved, using frames and techniques from our study of CWF degrees, non-quasi-canonicity of **GL** and **Grz**. In fact, using a slightly different frame, it is possible to get the same result for **Grz.2**, but we did not pursue this further.

Currently the main motivation for the notion is a construction by Takapui [42], which would generalise if **GL** turned out to be quasi-canonical. We showed that **GL** is, however, not quasi-canonical. Therefore, it would be good to have further applications of this notion. In particular, since we showed that there are quasi-canonical logics that are not canonical, it would be interesting to check whether existing constructions using canonicity can be generalised to quasi-canonicity.

Canonical approximations. Our other line of research involving canonicity might be of even more interest though. We introduced a general notion of approximations of logics in complete lattices, of which specific instances were already studied in the literature [4], and in particular studied approximations for the lattice of canonical logics. Our main contributions are Proposition 6.13, where we showed that for

[1] Or even maybe because of this.

logics with the fmp, the canonical approximations from above an below coincide, and Theorems 6.19 and 6.28, where we compute the canonical approximations of **Grz.3** and **Grz.2**. Concerning the former, it can also be noted that there are logics (without the fmp) where the approximations from above and below differ, although we did not prove this in this thesis.

In the direction of the latter two theorems, several interesting questions remain open. First, we computed the canonical approximations of two extensions of **Grz**, but computing the canonical approximations of **Grz** itself remains open. Looking at the canonical approximations of **Grz.3** and **Grz.2**, the pattern looks like the **Grz** 'part' turns into **S4.1**, so one might hypothesise that **S4.1** is the canonical approximation of **Grz**. It is not difficult to see that **S4.1** is a lower bound for it, but whether it is also an upper bound remains open.

A second question is whether analogues of our results hold for extensions of **GL** instead of **Grz**. Third, but maybe most important, it would be good to have more general results for computing canonical approximations, instead of having to compute each one individually. For example, one could wonder whether the operation of taking canonical approximations preserves sums. Also, we computed the canonical approximations of **Grz.3** completely independently of those of **Grz.2**, while the former logic extends the latter logic. Thus, one might ask whether there is a single more general theorem concerning some class of extensions of **Grz.2** that yields both of our results as corollaries.

Computable enumerability of dynamic topological logics. Finally, we studied computable enumerability for dynamic topological logics in Chapter 8. As our main result, we proved, as Theorem 8.38, that the DTL-logics of certain classes of CWF frames are computably enumerable. Combining the groundwork laid down in the chapter with some duality theory, one can in fact lift the converse pre-well-foundedness condition, at the cost of restricting to the fragment of DTL, where, under a positive $\Box$, the future looking box $\Box_F$ is only allowed to occur positively. This result would significantly generalise a result due to Konev et al. [28, Theorem 7], on whose proof techniques our investigation in based.

Ideally though, the converse pre-well-foundedness condition could be lifted without restricting to a fragment of DTL at all. Our current techniques seem unable to achieve this. It would be interesting to check whether more recently developed techniques, based on 'non-deterministic' structures [1, 14], allow to generalise our results in this direction.

Index

Bibliography

[1] J. P. Aguilera et al. *Gödel-Dummett linear temporal logic.* June 2023. arXiv: `2306.15805 [cs.LO]`.

[2] P. S. Alexandrov. 'Sur les espaces discrets'. French. In: *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* (13th May 1935), pp. 1649–1651.

[3] P. S. Alexandrov. 'Diskrete Räume'. German. In: *Recueil Mathématique. Nouvelle Série* 2 (1937), pp. 501–518.

[4] G. Bezhanishvili, N. Bezhanishvili and J. Ilin. 'Subframization and Stabilization for Superintuitionistic Logics'. In: *Journal of Logic and Computation* 29.1 (2019), pp. 1–35. DOI: `10.1093/logcom/exy035`.

[5] G. Bezhanishvili, N. Bezhanishvili and T. Moraschini. *Degrees of the finite model property: the antidichotomy theorem.* July 2023. arXiv: `2307.07209 [math.LO]`.

[6] N. Bezhanishvili. 'Lattices of Intermediate and Cylindric Modal Logics'. PhD thesis. Universiteit van Amsterdam, 26th Jan. 2006.

[7] P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic.* Vol. 53. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001. ISBN: 978 0 521 52714 9.

[8] W. J. Blok. *On the degree of incompleteness of modal logics and the covering relation in the lattice of modal logics.* Tech. rep. 78-07. Department of Mathematics, University of Amsterdam, 1978.

[9] G. Boole. *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities.* Project Gutenberg. [1854] 19th July 2017. URL: `https://www.gutenberg.org/files/15114/15114-pdf.pdf`.

[10] R. A. Bull. 'That All Normal Extensions of S4.3 Have the Finite Model Property'. In: *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 12 (1966), pp. 341–344. DOI: `10.1002/malq.19660120129`.

[11] A. Chagrov and M. Zakharyaschev. *Modal Logic.* Vol. 35. Oxford Logic Guides. Oxford University Press, 1997. ISBN: 0 19 853779 4.

[12] A. Church. 'An Unsolvable Problem of Elementary Number Theory'. In: *American Journal of Mathematics* 58.2 (Apr. 1936), pp. 345–363. DOI: 10.2307/2371045. JSTOR: 2371045.

[13] W. Craig. 'On Axiomatizability Within a System'. In: *The Journal of Symbolic Logic* 18.1 (Mar. 1953), pp. 30–32. ISSN: 0022-4812. DOI: 10.2307/2266324.

[14] D. Fernández-Duque. 'Non-deterministic semantics for dynamic topological logic'. In: *Annals of Pure and Applied Logic* 157.2 (2009). Kurt Gödel Centenary Research Prize Fellowships, pp. 110–121. ISSN: 0168-0072. DOI: 10.1016/j.apal.2008.09.015.

[15] D. Fernández-Duque. 'A sound and complete axiomatization for Dynamic Topological Logic'. In: *The Journal of Symbolic Logic* 77.3 (Sept. 2012), pp. 947–969. DOI: 10.2178/jsl/1344862169.

[16] D. Fernández-Duque. 'Non-finite Axiomatizability of Dynamic Topological Logic'. In: *Advances in Modal Logic* (Copenhagen, Denmark, Aug. 2012). Ed. by T. Bolander et al. Vol. 9. London: College Publications, 2012, pp. 200–216. ISBN: 978-1-84890-068-4.

[17] K. Fine. 'The Logics Containing S4.3'. In: *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 17 (1971), pp. 371–376. DOI: 10.1002/malq.19710170141.

[18] K. Fine. 'An ascending chain of S4 logics'. In: *Theoria* 40.2 (1974), pp. 110–116. DOI: 10.1111/j.1755-2567.1974.tb00081.x.

[19] K. Fine. 'An incomplete logic containing S4'. In: *Theoria* 40.1 (1974). DOI: 10.1111/j.1755-2567.1974.tb00076.x.

[20] K. Fine. 'Logics containing K4. part I'. In: *The Journal of Symbolic Logic* 39.1 (Mar. 1974). DOI: 10.2307/2272340.

[21] K. Fine. 'Some Connections Between Elementary and Modal Logic'. In: *Proceedings of the Third Scandinavian Logic Symposium* (Uppsala, 1973). Ed. by S. Kanger. Vol. 82. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1975, pp. 15–31. DOI: 10.1016/S0049-237X(08)70723-7.

[22] K. Fine. 'Logics containing K4. part II'. In: *The Journal of Symbolic Logic* 50.3 (Sept. 1985). DOI: 10.2307/2274318.

[23] D. Gabelaia et al. 'Non-primitive recursive decidability of products of modal logics with expanding domains'. In: *Annals of Pure and Applied Logic* 142.1-3 (Oct. 2006), pp. 245–268. ISSN: 0168-0072. DOI: 10.1016/j.apal.2006.01.001.

[24] J. Ilin. 'Filtration Revisited: Lattices of Stable Non-Classical Logics'. PhD thesis. Universiteit van Amsterdam, May 2018.

[25] V. A. Jankov. 'Constructing a sequence of strongly independent superintuitionistic propositional calculi'. Trans. from the Russian by A. Yablonski. In: *Soviet Mathematics* 9.4 (1968), pp. 806–807.

[26] S. C. Kleene. 'λ-Definability and Recursiveness'. In: *Duke Mathematical Journal* 2.2 (June 1936), pp. 340–353. DOI: 10.1215/S0012-7094-36-00227-2.

[27] S. C. Kleene. 'General recursive functions of natural numbers'. In: *Mathematische Annalen* 112 (1st Dec. 1936), pp. 727–742. ISSN: 1432-1807. DOI: 10.1007/BF01565439.

[28] B. Konev et al. 'Dynamic Topological Logics Over Spaces with Continuous Functions'. In: *Advances in Modal Logic* (Noosa, Queensland, Australia, Sept. 2006). Ed. by G. Governatori, I. Hodkinson and Y. Venema. Vol. 6. London: College Publications, 2006, pp. 299–318. ISBN: 1-904987-20-6.

[29] D. Kőnig. 'Sur les correspondances multivoques des ensembles'. French. In: *Fundamenta Mathematicae* 8.1 (1926), pp. 114–134. ISSN: 0016-2736. URL: https://eudml.org/doc/214882.

[30] D. Kőnig. 'Über eine Schlussweise aus dem Endlichen ins Unendliche'. German. In: *Acta Scientiarum Mathematicarum* 3 (1927), pp. 121–130. ISSN: 0324-5462. URL: https://acta.bibl.u-szeged.hu/13338/.

[31] P. Kremer and G. Mints. 'Dynamic topological logic'. In: *Annals of Pure and Applied Logic* 131.1 (2005), pp. 133–158. ISSN: 0168-0072. DOI: 10.1016/j.apal.2004.06.004.

[32] J. B. Kruskal. 'Well-quasi-ordering, the Tree Theorem, and Vazsonyi's Conjecture'. In: *Transactions of the American Mathematical Society* 95.2 (May 1960), pp. 210–225. DOI: 10.1090/s0002-9947-1960-0111704-1.

[33] T. Litak. 'A Continuum of Incomplete Intermediate Logics'. In: *Reports on Mathematical Logic* (), pp. 131–141. arXiv: 1808.06284 [cs.LO].

[34] J. C. C. McKinsey and A. Tarski. 'Some Theorems About the Sentential Calculi of Lewis and Heyting'. In: *The Journal of Symbolic Logic* 13.1 (Mar. 1948), pp. 1–15. DOI: 10.2307/2268135. JSTOR: 2268135.

[35] C. S. J. A. Nash-Williams. 'On well-quasi-ordering finite trees'. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 59.4 (Oct. 1963), pp. 833–835. DOI: 10.1017/S0305004100003844.

[36] C. A. Naturman. 'Interior Algebras and Topology'. PhD thesis. University of Cape Town, Nov. 1990. HDL: 11427/18244.

[37] P. G. Odifreddi. *Classical Recursion Theory. The Theory of Functions and Sets of Natural Numbers.* Ed. by J. Barwise et al. 1st ed. Studies in Logic and The Foundations of Mathematics 125. Elsevier Science Publishers B.V., 4th Feb. 1992. ISBN: 0-444-87295-7.

[38] W. Rautenberg. 'Splitting lattices of logics'. In: *Archiv für mathematische Logik und Grundlagenforschung* 20 (Sept. 1980), pp. 155–159. DOI: 10.1007/BF02021134.

[39] H. Rogers Jr. *Theory of Recursive Functions and Effective Computability.* Paperback edition. The MIT Press, [1967] 1987. ISBN: 0-262-68052-1.

[40] V. B. Shehtman. 'On Incomplete Propositional Logics'. Trans. from the Russian by B. F. Wells. In: *Soviet Mathematics* 18.4 (1977), pp. 985–989.

[41] G. Takapui. 'General topological semantics for modal provability logics'. Master's Thesis Seminar (Radboud University Nijmegen). 11th Mar. 2024.

[42] G. Takapui. 'General topological semantics for modal provability logics'. MA thesis. Radboud University Nijmegen. forthcomming.

[43] A. M. Turing. 'Computability and λ-Definability'. In: *The Journal of Symbolic Logic* 2.4 (Dec. 1937), pp. 153–163. DOI: 10.2307/2268280. JSTOR: 2268280.

[44] A. M. Turing. 'On Computable Numbers, with an Application to the Entscheidungsproblem'. In: *Proceedings of the London Mathematical Society* s2-42.1 (Jan. 1937), pp. 230–265. ISSN: 0024-6115. DOI: 10.1112/plms/s2-42.1.230.

[45] J. F. A. K. van Benthem. 'Some kinds of modal completeness'. In: *Studia Logica* 39.2/3 (June 1980), pp. 125–141. ISSN: 1572-8730. DOI: 10.1007/BF00370316.