Vietoris bisimulations

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Abstract

Building on the fact that descriptive frames are coalgebras for the Vietoris functor on the category of Stone spaces, we introduce and study the concept of a Vietoris bisimulation between two descriptive modal models, together with the associated notion of bisimilarity. We prove that our notion of bisimilarity, which is defined in terms of relation lifting, coincides with Kripke bisimilarity (with respect to the underlying Kripke models), with behavioral equivalence, and with modal equivalence, but not with Aczel-Mendler bisimilarity. As a corollary we obtain that the Vietoris functor does not preserve weak pullbacks. Comparing Vietoris bisimulations between descriptive models to Kripke bisimulations on the underlying Kripke models, we prove that the closure of such a Kripke bisimulation is a Vietoris bisimulation. As a corollary we show that the collection of Vietoris bisimulations between two descriptive models forms a complete lattice. Finally, we provide a game-theoretic characterization of Vietoris bisimilarity.

Keywords Modal logic, bisimulation, descriptive frame, coalgebra, Vietoris construction.

1 Introduction

Bisimulations between Kripke models are of key importance in modal logic [8]. The starting observation is that the truth of modal formulas is invariant under bisimilarity, in the sense that any two states satisfy the same modal formulas if they are linked by some bisimulation. More fundamental connections between modal logic and bisimilarity are provided by results pointing in the converse direction. In particular, for some classes of models one may show that modal equivalence is *equivalent* to bisimilarity, i.e., the fact that two states are not bisimilar is witnessed by a separating modal formula. Such model classes are called *Hennessy-Milner* classes, after the authors of [15] who established this property for the class of image-finite models. As a second example, a celebrated theorem by van Benthem [7] identifies modal logic as the bisimulation invariant fragment of first-order logic: Any first order formula in one free variable that is invariant under bisimilarity is equivalent to (the standard translation of) a modal formula.

However, next to Kripke frames and models, other structures are worth studying. In fact, it follows from the work of Thomason [31], Blok [9], and others, that from a mathematical viewpoint Kripke frames do not provide an adequate semantics for the study of normal modal logics (where the word 'logic' is now taken in the technical meaning of a set of formulas containing certain axioms and being closed under certain derivation rules). To overcome this problem one may turn to the *algebraic* semantics of modal logic [32] which allows for a general completeness result: any axiomatic extension of the basic modal logic **K** is sound and complete with respect to some class of modal algebras. However, despite this mathematical advantage, many modal logicians still prefer frame-based semantics since they find relational structures more intuitive to work with. A good compromise is given by the *general frames* which are (almost) as intuitive as the familiar relational structures, but on the other hand, do support a general completeness result. In particular, the *descriptive frames* provide an interesting subclass of general frames, since there is a full categorical duality with the category of modal algebras [13]. Descriptive frames are formed by a Kripke frame nicely entangled with a Stone topology; a *descriptive model* arises by adding a valuation that interprets the proposition letters of the language as *clopens* of the topology. Such a valuation corresponds to an assignment of variables on the dual modal algebra.

Our paper addresses the question, that naturally arises in this context: What do bisimulations between *descriptive models* look like? We shall introduce and motivate the proper notion of a bisimulation between descriptive structures, together with the associated concept of bisimilarity. We shall then prove some of the intrinsic properties of these notions, and finally, compare them to the corresponding notions for Kripke structures.

Our approach is *coalgebraic*. The theory of universal coalgebra [29] aims at providing a general framework for the study of notions related to (possibly infinite) behavior such as invariance and observational indistinguishability. Coalgebras are simple but fundamental mathematical structures that capture the essence of dynamic or evolving systems [18]. Kripke frames and models provide key examples of set-based coalgebras, i.e., structures of the form $\langle S, \sigma : S \to \mathcal{T}S \rangle$, where S is some set, and $\mathcal{T}$ is some set functor denoting the *type* of the coalgebra. Many familiar notions and constructions, such as bisimulations and bounded morphisms, have analogues in other fields, and find their natural place at the abstraction level of coalgebra. Perhaps even more important is the realization that one may generalize the concept of modal logic itself, as a dynamic specification language for behavior, from Kripke frames to arbitrary coalgebras, see [32] and references therein. In fact, the link between (these generalizations of) modal logic and coalgebra is so tight, that one may even claim that just like equational logic is the natural logic for algebra, modal logic is that for coalgebras [22]. The point here is that not only Kripke frames and models, but also descriptive frames and models can be seen as coalgebras, not set-based but over the category Stone of Stone spaces [1, 21]. This provides a second motivation for this paper: to study bisimulations between coalgebras over a category different from Set.

In any case, once we have identified descriptive frames and models as coalgebras, the theory of universal coalgebra then suggests a natural definition of a bisimulation between descriptive frames and models, with an associated notion of bisimilarity. In fact, there are *two* natural candidates for such a definition. The *Vietoris bisimulations* are defined using

the so-called relation lifting associated with the Vietoris functor, while the notion of Aczel-Mendler bisimulation, named after the authors of [4], is defined in terms of a coalgebra structure imposed on the bisimulation relation itself.

We show that Vietoris bisimilarity coincides with modal equivalence (which in its turn is the same as the important coalgebraic notion of *behavioral equivalence*). Moreover, it turns out that there is a very simple characterization of this notion: a relation between two descriptive models is a Vietoris bisimulation if it is both a bisimulation between the two underlying Kripke models, and closed when seen as a subset in the product topology. In short:

<div align="center">Vietoris bisimulations are closed Kripke bisimulations.</div>

On the other hand, we show that Aczel-Mendler bisimilarity does not coincide with modal equivalence (and thus also differs from Vietoris bisimilarity and from behavioral equivalence), and that the poset of Aczel-Mendler bisimulations, ordered by inclusion, does not form a lattice. As a corollary of the first observation, we obtain the following contribution of this paper:

<div align="center">the Vietoris functor does not preserve weak pullbacks.</div>

This is all in contrast to the powerset functor, which *does* preserve weak pullbacks. Also, for coalgebras for the powerset functor, the notion of bisimilarity defined via relation lifting coincides with Aczel-Mendler bisimilarity, but, in general, is different from modal equivalence.

On the basis of the above-mentioned negative evidence, we conclude that for descriptive frames and models, Aczel-Mendler bisimilarity is not the right notion of bisimilarity, and we concentrate on Vietoris bisimulations. We turn to a comparison between Vietoris bisimulations and Kripke bisimulations, and we derive our main technical result, stating that

<div align="center">the closure of a Kripke bisimulation is a Vietoris bisimulation.</div>

As a corollary we prove that the collection of Vietoris bisimulations between two descriptive models forms a complete lattice, ordered by set inclusion.

We finish this introduction with an overview of the paper. After a brief presentation in Section 2 of descriptive frames and models from the coalgebraic point of view, in Section 3 we define the notion of Vietoris bisimulation, in terms of relation lifting. We prove that Vietoris bisimulations can be characterized as closed Kripke bisimulation and that Vietoris bisimilarity coincides with modal equivalence, and hence, with behavioral equivalence. In Section 4 we define Aczel-Mendler bisimilarity; we argue that in our setting this is not the right notion of bisimilarity by showing that it does not coincide with behavioral equivalence. We give an example showing why the Vietoris functor does not preserve weak pullbacks. In Section 5 we prove the main technical result of the paper that the closure of a Kripke bisimulation is a Vietoris bisimulation. As a corollary of this, we show that the collection of Vietoris bisimulations between two descriptive models forms a complete lattice, ordered by set inclusion. Section 6 provides a game-theoretic characterization of the notion of Vietoris bisimilarity. To finish off, we mention some conclusions and questions for further research.

2 Vietoris coalgebras

The aim of this section is to introduce the coalgebraic perspective on descriptive frames and models. This viewpoint essentially dates back to a talk given by Abramsky in 1988, of which the lecture notes remained unpublished until recently [1]. Abramsky's work had an early predecessor in Esakia [12], where the connection with the Vietoris functor was already made. Our presentation here is based on Kupke, Kurz & Venema [21], whereas the reader is referred to Venema [32] for some more context.

Notation First we need some auxiliary definitions. Given a binary relation $R \subseteq W \times W$, a state $u \in W$ and a set $U \subseteq W$, define

- $R(u) = \{y \in W : uRy\}$, $R(U) = \{y \in W : uRy$ for some $u \in U\}$,

- $\langle R \rangle U = \{y \in W : yRu$ for some $u \in U\}$,

- $[R]U = \{y \in W : u \in U$ for all u with $yRu\}$.

Observe that $\langle R \rangle U = R^{-1}(U)$, where R^{-1} is the converse relation of R, and that $[R]U = W \setminus \langle R \rangle (W \setminus U)$.

Kripke frames and models Recall that a *Kripke frame* is a pair $\mathbb{F} = (W, R)$, where W is a non-empty set and R is a binary relation on W. Throughout the paper we fix a set Φ of proposition letters. A *Kripke model* is a tuple $\mathbb{M} = (W, R, V)$, where (W, R) is a Kripke frame and $V : \Phi \to \mathcal{P}(W)$ sends proposition letters to the set of states where they are true. We assume familiarity with the notion of a bounded morphism from one Kripke frame or model to another [8]. Readers unfamiliar with this notion can derive its definition from Fact 2.3, or define bounded morphisms as maps of which the graph is a bisimulation.

Definition 2.1. *Let* $\mathbb{M} = (W, R, V)$ *and* $\mathbb{M}' = (W', R', V')$ *be two Kripke models. A relation* $B \subseteq W \times W'$ *is called a* (Kripke) bisimulation, *if for every* $x \in W$ *and* $x' \in W'$ *such that* xBx', *we have*

1. $x \in V(p)$ iff $x' \in V'(p)$, for all proposition letters p;

2. xRy implies that there exists $y' \in W'$ such that $x'R'y'$ and yBy';

3. $x'Ry'$ implies that there exists $y \in W$ such that xRy and yBy'.

4

Two points x and x' are called (Kripke) bisimilar*, notation:* $\mathbb{M}, x \leftrightarrow \mathbb{M}', x'$ *if they are linked by some Kripke bisimulation.*

We assume familiarity with the syntax and semantics of modal logic [8]. We use the symbol $\Vdash$ for the satisfaction relation, that is, we write '$\mathbb{M}, s \Vdash \phi$' to denote that formula ϕ is true, or satisfied, at state s in the Kripke model $\mathbb{M}$. Two points, s and s', in two models, $\mathbb{M}$ and $\mathbb{M}'$, respectively, are called *modally equivalent* if they satisfy exactly the same modal formulas. The relation of modal equivalence is denoted by $\leftrightsquigarrow$. One of the reasons why bisimilarity is such an important notion in modal logic, is that the truth of modal formulas is invariant under bisimilarity, or briefly: $\leftrightarrow \subseteq \leftrightsquigarrow$, while for some important classes of models the converse inclusion holds as well.

Descriptive frames and models   Roughly, a descriptive frame consists of a Kripke frame and a modal algebra nicely glued together in one structure. Part of the interest in descriptive frames stems from the fact that they provide a category which is dually equivalent to that of modal algebras [13].

Formally, a *field of sets* is a pair (W, A) with $A \subseteq \mathcal{P}(S)$ being closed under all Boolean set-theoretic operations. The elements of A are called the *admissible* subsets of S. A field of sets is called *differentiated* if for any two distinct points $s \neq t$ of X there is an admissible set $U \in A$ such that $s \in U$ and $t \notin U$, and *compact* if every subset of A with the finite intersection property has a non-empty intersection. A *general frame* is a triple (W, R, A) such that (W, R) is a Kripke frame and (W, A) is a field of sets such that A is closed under $\langle R \rangle$; that is, $U \in A$ implies $\langle R \rangle U \in A$. A binary relation R on a field of sets (W, A) is *tight* if

$$\forall x, y \in W \big(\neg Rxy \rightarrow \exists U \in A (y \in U \,\&\, x \notin \langle R \rangle U) \big). \tag{1}$$

The notions of differentiatedness, compactness and tightness apply to general frames in the obvious way. Finally, a *descriptive* frame is a general frame (W, R, A) that is differentiated, tight and compact. As *morphisms* between descriptive frames we take those bounded morphisms between the underlying Kripke frames such that the preimage of an admissible set is admissible.

A tuple $\mathbb{M} = (W, R, A, V)$ is called a *descriptive model* if (W, R, A) is a descriptive frame and V is an *admissible* valuation, that is, a map interpreting each proposition letter as an *admissible* set of states. The notion of truth in descriptive models is defined as for Kripke models; it is routine to verify that the interpretation of any formula, that is, the set of points where this formulas is true, is an admissible set.

Coalgebra   We also recall the definition of coalgebras [29].

Definition 2.2. *Let* C *be a category and* $\mathcal{T} : \mathsf{C} \rightarrow \mathsf{C}$ *an endofunctor. A* $\mathcal{T}$*-coalgebra is a pair* (X, σ)*, where* $\sigma : X \rightarrow \mathcal{T}X$ *is a morphism of* C*. In the case* C *is the category* Set *of sets with functions, we often speak of* systems *rather than coalgebras. A* morphism *between two coalgebras* (X, σ) *and* (X', σ') *is an arrow* f *(in* C*) such that the following diagram commutes:*

$$X \xrightarrow{\ f\ } X'$$
$$\downarrow{\scriptstyle\sigma} \qquad\qquad \downarrow{\scriptstyle\sigma'}$$
$$TX \xrightarrow[\ Tf\ ]{} TX'$$

Kripke frames can be seen as coalgebras for the powerset functor $\mathcal{P}$ on the category of sets. Recall that $\mathcal{P}$ maps a set W to its power set $\mathcal{P}(W)$, and a function $f : W \to W'$ to the direct image function $\mathcal{P}f$ given by $\mathcal{P}f(U) = f[U]$ (for all $U \subseteq W$). Coalgebraically, we see a Kripke frame (W, R) as a pair (W, ρ_R), where $\rho_R : W \to \mathcal{P}(W)$ maps a point x to the set $\rho_R(x)$ of its R-successors. In the sequel we will simply write R instead of ρ_R.

In order to represent a Kripke *model* (W, R, V) as a coalgebra, we represent the valuation as the function $\nu_V : W \to \mathcal{P}(\Phi)$ indicating for each world x which proposition letters are true at x. Formally, ν_V is defined by putting $\nu_V(x) := V^{-1}[x]$. The model (W, R, V) can then be represented as the pair (W, σ), with $\sigma : W \to \mathcal{P}(W) \times \mathcal{P}(\Phi)$ given by $\sigma(x) := (R(x), \nu_V(x))$. Kripke models over a set Φ of proposition letters are coalgebras for the *Kripke functor* $\mathcal{P}_\Phi$ which maps a set W to the set $\mathcal{P}(W) \times \mathcal{P}(\Phi)$, and maps a function $f : W \to W'$ to the function $\mathcal{P}f \times id$ given by $(\mathcal{P}f \times id)(U, Q) = (f[U], Q)$ (for all $U \subseteq W$ and all sets Q of proposition letters).

We do not know who first made this coalgebraic perspective on Kripke structures explicit. Aczel [3], discussing the operational semantics of process algebras in coalgebraic terms, is the most likely candidate. Other early references are Abramsky [1] and, independently, Rutten [27] and Barwise & Moss [6]. We summarize our findings formally as follows.

Fact 2.3. *The category of Kripke frames with bounded morphisms is isomorphic to the category of $\mathcal{P}$-coalgebras. Likewise, the category of Kripke models with bounded morphisms is isomorphic to the category of $\mathcal{P}_\Phi$-coalgebras.*

Stone spaces Descriptive frames (models) can be represented as coalgebras as well, but not for the functor $\mathcal{P}$ ($\mathcal{P}_\Phi$) or any other set functor. In fact, to represent descriptive structures as coalgebras we need to take the category of *Stone spaces* as our base category, and to define an analogue of the powerset functor on this category.

We first give some basic facts concerning Stone spaces. A *Stone space* is a compact Hausdorff space which is zero-dimensional, that is, has a basis of clopens. We let Stone denote the category of Stone spaces and continuous maps. Usually we will be sloppy concerning the distinction between a Stone space and its underlying set, writing for instance $x \in X$ where we mean that x is an element of the underlying set of X, etc. We assume some familiarity with notions and results pertaining to Stone spaces (or more generally, to compact Hausdorff spaces), such as the fact that the closed sets in a Stone space coincide with the compact ones. For future reference we explicitly mention the following facts, which can be found in any textbook on topology such as [11].

Fact 2.4. *Let X and X' be Stone spaces.*
(1) Given a subset S of X, the subspace induced by S is a Stone space iff S is closed.
(2) If $f : X \to X'$ is continuous, then the image $f[X]$ of X under f is a Stone space.

The earlier announced analog of the power set functor on the category of Stone spaces is given by the *Vietoris* construction, which may be defined as follows.

Definition 2.5. *For every Stone space X, its* Vietoris space $\mathcal{V}(X)$ *is defined as the set of all closed subsets of X, endowed with the topology generated by taking as a subbasis all sets of the form*

1. $[\ni]C = \{F \in \mathcal{V}(X) : F \subseteq C\}$

2. $\langle\ni\rangle C = \{F \in \mathcal{V}(X) : F \cap C \neq \emptyset\}$

where C ranges over all clopen subsets of X.

Note that usually, one builds up the Vietoris topology starting from basic elements of the form $[\ni]U$, $\langle\ni\rangle U$ with U an arbitrary *open* set rather than clopen. In the case of Stone spaces the two definitions are equivalent. It is well known that for every Stone space X, the space $\mathcal{V}(X)$ is also a Stone space, and the Vietoris construction $\mathcal{V}$ can in fact be extended to a *functor* $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$ which sends a map $f : X \to Y$ to the map $\mathcal{V}f$ given by $\mathcal{V}f(F) = f[F]$ (for all closed sets $F \subseteq X$).

Descriptive frames and models as Stone coalgebras   To explain how descriptive frames can be seen as Stone coalgebras, we first link the admissible sets of a descriptive frame to the clopen sets of a Stone space. For every field of sets (X, A) we can define a topology on X by declaring A as its basis. This topology is zero-dimensional by definition, and it is not hard to verify that it is a Stone space iff (X, A) is descriptive. Conversely, it is easy to see that the set $\mathsf{Clop}(X)$ of clopens of a Stone space X forms a field of sets that is differentiated and compact. In the sequel it will be convenient for us to identify Stone spaces with descriptive fields of sets.

Now consider a Stone space X and a binary relation R on (the underlying set of) X. We leave it for the reader to check the well-known fact that R satisfies the tightness condition (1) iff R is *point-closed*, that is, $R(x)$ is a closed set for all points $x \in X$. In other words, a relation R is tight on a Stone space X iff R, seen as a coalgebraic map, is a function from X to (the carrier set of) the Vietoris space $\mathcal{V}(X)$. However, this is not enough for making the structure $(X, R, \mathsf{Clop}(X))$ a descriptive frame: for $(X, R, \mathsf{Clop}(X))$ to be a *general frame*, $\mathsf{Clop}(X)$ must be closed under the operation $\langle R\rangle$. This however is taken care of by the *continuity* of R. For future reference we state this as follows, for any Stone space X:

$$R : X \to \mathcal{V}(X) \text{ is continuous iff } \mathsf{Clop}(X) \text{ is closed under } \langle R\rangle. \tag{2}$$

The above considerations show that descriptive frames can be identified with Stone coalgebras for the Vietoris functor $\mathcal{V}$.

In order to see descriptive *models* as Stone-based coalgebras, consider a clopen valuation on a descriptive frame (X, R), that is, a function $V : \Phi \to \mathcal{P}(X)$ mapping each $p \in \Phi$ to a *clopen* subset of X. In the coalgebraic representation of Kripke models, this valuation V was represented as the map $V^{-1} : X \to \mathcal{P}(\Phi)$. Now identify $\mathcal{P}(\Phi)$ with the set 2^Φ of *characteristic functions* on Φ, and observe that this set can be endowed with the product topology induced

by the discrete topology on the set $2 = \{0, 1\}$. The resulting topological space, denoted as 2^Φ, is a *Stone* space; for convenience, its elements will be identified with, and denoted as, subsets of Φ. It can be proved that a valuation $V : \Phi \to \mathcal{P}(X)$ is admissible iff V^{-1} is *continuous* when seen as a map between the spaces X and 2^Φ. Let $\mathcal{V}_\Phi : \mathsf{Stone} \to \mathsf{Stone}$ be the functor that maps a Stone space X to $\mathcal{V}(X) \times 2^\Phi$, and a function $f : X \to Y$ to the map $\mathcal{V}f \times id$ given by $(\mathcal{V}f \times id)(F, Q) = (f[F], Q)$ (for all $F \in \mathcal{V}(X)$ and all $Q \subseteq \Phi$).

The following fact, which in the sequel we may use without further notification, summarizes our findings. For a proof we refer to [21].

Fact 2.6. *The category of descriptive frames with continuous bounded morphisms is isomorphic to the category of $\mathcal{V}$-coalgebras. Likewise, the category of descriptive models with continuous bounded morphisms is isomorphic to the category of $\mathcal{V}_\Phi$-coalgebras.*

Convention 2.7. In the sequel we will make use of Fact 2.6 without explicit notification. For instance, we will usually present descriptive models coalgebraically as pairs $\mathbb{M} = (X, \sigma)$, or as triples $\mathbb{M} = (X, R, \nu)$. Here X is a Stone space and $\sigma : X \to \mathcal{V}_\Phi(X)$ is a Vietoris coalgebra map on X, which can also be presented as the pair $\langle R, \nu \rangle$ where $\nu : X \to 2^\Phi$ represents an admissible valuation, and one may think of R either as a tight/point-closed relation $R \subseteq X \times X'$, or as a continuous map $R : X \to \mathcal{V}(X)$.

Finality and behavioral equivalence To finish this introductory section we briefly discuss two key coalgebraic concepts in the context of descriptive models.

To start with the first, a coalgebra (Z, ζ) is *final* in the category of $\mathcal{T}$-coalgebras if for every $\mathcal{T}$-coalgebra C there is a unique coalgebra homomorphism $!_C : C \to (Z, \zeta)$. The categories of descriptive frames and models both have final objects: the canonical (general) frame in the case of descriptive frames, and the canonical model in the case of the descriptive models. From the perspective of the duality with modal algebras this should not come as a surprise, since the canonical general frame is the dual of the Lindenbaum-Tarski algebra, which is known to be a free object in the category of modal algebras (or an initial object, depending on the exact status of the proposition letters in Φ).

Recall that the *canonical model* for a set Φ of proposition letters is the structure $\mathbb{M}^c := (X^c, R^c, \nu^c)$. Here X^c is the set of maximal consistent sets of modal formulas (using proposition letters in Φ); the admissible/clopen sets are of the form $\widehat{\phi} := \{u \in X^c \mid \phi \in u\}$ for any formula ϕ in the modal language (with proposition letters from Φ); the canonical accessibility relation $R^c \subseteq X^c \times X^c$ is given by putting uR^cv if $\{\Diamond\phi \mid \phi \in v\} \subseteq u$; and the canonical valuation ν^c is defined by $\nu^c(u) := u \cap \Phi$. To say that the structure $\mathbb{M}^c$ is the final object in the category of descriptive models amounts to the following result, which is essentially due to Abramsky [1].

Fact 2.8. *Let $\mathbb{M}$ be a descriptive model. Then the theory map, sending a point x in $\mathbb{M}$ to the set of formulas that are true at x, is the unique coalgebra morphism from $\mathbb{M}$ to $\mathbb{M}^c$.*

In the case of set-based coalgebras there is often a natural interpretation of the states of a final algebra as the various distinct *behaviours* that a pointed coalgebra may display. This explains why we call a state s in a system C *behaviorally equivalent* to a state s' in a

system C' if $!_C(s) = !_{C'}(s')$. In the absence of a final coalgebra, we call two states s and s' in two systems C and C' *behaviorally equivalent* if s and s' can be identified by *some* pair of coalgebra morphisms, that is, if there is a coalgebra D and two coalgebra homomorphisms $f : C \to D$ and $f' : C' \to D$ such that $f(s) = f'(s')$.

As a consequence of Fact 2.8, we can show that behavioral equivalence coincides with modal equivalence (where both notions are defined in the same way as for systems). While the following theorem is to our knowledge new, it is a straightforward consequence of Abramsky's result, Fact 2.8, and work by Kurz & Pattinson [24] on the coalgebraic role of the canonical model.

Theorem 2.9. *Let $\mathbb{M}$ and $\mathbb{M}'$ be descriptive models, and let x and x' be points in $\mathbb{M}$ and $\mathbb{M}'$, respectively. Then x and x' are behaviorally equivalent iff $\mathbb{M}, x \leftrightsquigarrow \mathbb{M}', x'$.*

Bisimulations can be seen as a way to get hold of the notion of behavioral equivalence, without having to move outside of the two coalgebras involved. In the next two sections we will discuss two ways of defining the notion of a bisimulation for descriptive models.

3 Vietoris Bisimulations

The first characterization of bisimulations between systems (that is, set-based coalgebras) that we will generalize to the setting of Stone coalgebras is the one using the notion of *relation lifting*. As far as we know, this characterization is due to Rutten [28], who generalized earlier results by Hermida & Jacobs [16] on set coalgebras for so-called polynomial set functors. See also [5, 30].

We first consider the case of the powerset functor. Let W and W' be sets and let $B \subseteq W \times W'$ be a binary relation. We define the *relation lifting* $\widetilde{\mathcal{P}}(B) \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$ of B by

$$\widetilde{\mathcal{P}}(B) := \{(U, U') : \forall x \in U \ \exists x' \in U' \ xBx', \text{ and } \forall x' \in U' \ \exists x \in U \ xBx'\}. \tag{3}$$

Similarly, for the functor $\mathcal{P}_\Phi$, we define the *relation lifting* $\widetilde{\mathcal{P}}_\Phi(B) \subseteq \mathcal{P}_\Phi(W) \times \mathcal{P}_\Phi(W')$ of B by

$$\widetilde{\mathcal{P}}_\Phi(B) := \{((U, Q), (U', Q')) : (U, U') \in \widetilde{\mathcal{P}}(B) \text{ and } Q = Q'\}. \tag{4}$$

Fact 3.1. *Let (W, σ) and (W', σ') be Kripke models. A relation $B \subseteq W \times W'$ is a Kripke bisimulation iff $(\sigma(w), \sigma'(w'))$ belongs to $\widetilde{\mathcal{P}}_\Phi(B)$, for all $(w, w') \in B$.*

The notion of relation lifting can be defined for an arbitrary set functor, as follows. Given a relation $B \subseteq S \times S'$, consider the following diagram, where π and π' denote the projection maps.

$$S \xleftarrow{\quad \pi \quad} B \xrightarrow{\quad \pi' \quad} S'$$

If we apply the functor $\mathcal{T}$ to this diagram, it follows from the category-theoretic properties of the product $\mathcal{T}S \times \mathcal{T}S'$ (with projection maps p and p') that there is a unique map $\rho = \langle \mathcal{T}\pi, \mathcal{T}\pi' \rangle$ from $\mathcal{T}R$ to $\mathcal{T}S \times \mathcal{T}S'$ such that $p \circ \rho = \mathcal{T}\pi$ and $p' \circ \rho = \mathcal{T}\pi'$:

9

The *relation lifting* of B is defined as the image of $\mathcal{T}B$ under ρ, that is, as the relation

$$\widetilde{\mathcal{T}}B := \{((\mathcal{T}\pi)(u), (\mathcal{T}\pi')(u)) \mid u \in \mathcal{T}B\}. \tag{5}$$

It is an easy exercise to check that this definition, in the case of the Kripke functor $\mathcal{P}_\Phi$, coincides with (4).

This suggests the following definition for relation lifting on Stone spaces. Given two Stone spaces X and X', consider a subspace S of the product space $X \times X'$, with the projections $\pi : S \to X$ and $\pi' : S \to X'$. Observe that the projections are continuous, and hence indeed are arrows in Stone. Now apply the Vietoris functor to the diagram:

$$X \xleftarrow{\pi} S \xrightarrow{\pi'} X'$$

obtaining

$$\mathcal{V}X \xleftarrow{\mathcal{V}\pi} \mathcal{V}S \xrightarrow{\mathcal{V}\pi'} \mathcal{V}X'$$

Since the category Stone has products, there is a unique arrow $\langle \mathcal{V}\pi, \mathcal{V}\pi' \rangle$ from $\mathcal{V}S$ to $\mathcal{V}X \times \mathcal{V}X'$. We will define $\widetilde{\mathcal{V}}(S)$ as the subspace of $\mathcal{V}X \times \mathcal{V}X'$ induced by the image of $\mathcal{V}S$ under this map — this subspace is a Stone space by Fact 2.4(2).

This definition easily generalizes to an arbitrary endofunctor $\mathcal{T}$ on Stone.

Definition 3.2. *Let $\mathcal{T}$ be an endofunctor on the category Stone, and let X and X' be two Stone spaces. Given a subspace S of $X \times X'$, with projections π and π', let the $\mathcal{T}$-lifting $\widetilde{\mathcal{T}}(S)$ of S denote the subspace $\langle \mathcal{T}\pi, \mathcal{T}\pi' \rangle[\mathcal{T}S]$ of $\mathcal{T}X \times \mathcal{T}X'$. Given the identification of subspaces of a product with binary relations between the respective spaces that are closed in the product topology (Fact 2.4(1)), this definition induces a notion of* relation lifting *of closed binary relations, that will be denoted by $\widetilde{\mathcal{T}}$ as well.*

Observe that for any closed relation $B \subseteq X \times X'$, the relation $\widetilde{\mathcal{T}}(B) \subseteq \mathcal{T}X \times \mathcal{T}X'$ is closed as well (in the product topology of $\mathcal{T}X$ and $\mathcal{T}X'$). See Remark 3.7 on how to extend this definition of relation lifting to relations that are not closed.

On the basis of Definition 3.2 we define our notion of Vietoris bisimulation.

Definition 3.3. *Let (X, σ) and (X', σ') be two descriptive models, and suppose that $B \subseteq X \times X'$ is a closed set in the product topology. Then we say that B is a* Vietoris bisimulation *if $(\sigma(x), \sigma(x')) \in \widetilde{\mathcal{V}}_\Phi(B)$ for all $(x, x') \in B$.*

Here is a more direct characterization of the Vietoris relation lifting.

Proposition 3.4. *Given a closed relation B between two Stone spaces X and X', we may characterize the relation liftings $\widetilde{\mathcal{V}}(B)$ and $\widetilde{\mathcal{V}}_\Phi(B)$ as follows:*

$$\widetilde{\mathcal{V}}(B) = \{(F, F') : \forall x \in F\ \exists x' \in F'\ xBx', \text{ and } \forall x' \in F'\ \exists x \in F\ xBx'\}. \tag{6}$$

and
$$\widetilde{\mathcal{V}}_\Phi(B) = \{((F,Q),(F',Q')) : (F,F') \in \widetilde{\mathcal{V}}(B) \ and \ Q = Q'\} \tag{7}$$

In other words, when we restrict attention to closed subsets of X and X', the relations $\widetilde{\mathcal{P}}(B)$ and $\widetilde{\mathcal{V}}B$ coincide, and likewise for $\widetilde{\mathcal{P}}_\Phi(B)$ and $\widetilde{\mathcal{V}}_\Phi(B)$. That is, another way of formulating (6) and (7) is as follows:

$$\begin{aligned} \widetilde{\mathcal{V}}(B) &= \widetilde{\mathcal{P}}(B) \cap \big(\mathcal{V}X \times \mathcal{V}X'\big). \\ \widetilde{\mathcal{V}}_\Phi(B) &= \widetilde{\mathcal{P}}_\Phi(B) \cap \big(\widetilde{\mathcal{V}}_\Phi X \times \widetilde{\mathcal{V}}_\Phi X'\big). \end{aligned}$$

Proof. We only prove the first characterization, the second one being an immediate consequence of the first. For the inclusion '$\subseteq$' of (6), take an element in $\mathcal{V}(B)$, i.e., a closed subset $G \subseteq B$. Then the sets $\pi[G]$ and $\pi'[G]$ are closed subsets of X and X', respectively, and by definition this pair of sets satisfies the property that $\forall x \in \pi[G] \ \exists x' \in \pi'[G] \ xGx'$, and vice versa. But then by G being a subset of B, the pair $(\pi[G], \pi'[G])$ belongs to the set on the right hand side of (6). For the opposite inclusion, take a pair $(F,F') \in \widetilde{\mathcal{P}}(B) \cap \big(\mathcal{V}X \times \mathcal{V}X'\big)$. Then the set $G := (F \times F') \cap B$ is a closed subset of B, and so it belongs to the set $\mathcal{V}(B)$. But since (F,F') belongs to $\widetilde{\mathcal{P}}(B)$, it is straightforward to check that $\pi[G] = F$ and $\pi'[G] = F'$. From this it follows that $(F,F') = \langle \mathcal{V}\pi, \mathcal{V}\pi' \rangle(G)$ and so by definition, $(F,F') \in \widetilde{\mathcal{V}}(B)$. This proves (6). $\qquad \square$

Remark 3.5. In passing we note the following connection between $\widetilde{\mathcal{P}}(B)$ and $\widetilde{\mathcal{V}}(\overline{B})$. Let (X,R) and (X',R') be descriptive frames. Then for any relation $B \subseteq X \times X'$:

$$\text{if } (U,U') \in \widetilde{\mathcal{P}}(B), \text{ then } (\overline{U}, \overline{U'}) \in \widetilde{\mathcal{V}}(\overline{B}). \tag{8}$$

The fairly direct proof of (8) is left to the reader.

As an easy but important corollary of Proposition 3.4 we obtain the following characterization of Vietoris bisimulations. We list it as a theorem because of its conceptual, not its technical importance.

Theorem 3.6. *Let (X,σ) and (X',σ') be two descriptive models, and let B be a relation $B \subseteq X \times X'$. Then B is a Vietoris bisimulation iff B is a closed Kripke bisimulation.*

Proof. Let B be a relation $B \subseteq X \times X'$. Then B is a Vietoris bisimulation iff B is closed and $(\sigma(x), \sigma'(x')) \in \widetilde{\mathcal{V}}_\Phi(B)$ for each pair $(x,x') \in B$. But by Proposition 3.4 this is equivalent to requiring that B is a closed Kripke bisimulation. $\qquad \square$

Remark 3.7. Given the fact that (6) and (7) characterize the Vietoris relation lifting of a closed binary relation, we may take these identities as the *definition* of a notion of relation lifting for arbitrary (i.e., not necessarily closed) relations between X and X'. That is, for any relation $B \subseteq X \times X'$, define

$$\widetilde{B} := \{(F,F') \in \mathcal{V}X \times \mathcal{V}X' \mid \forall x \in F \ \exists x' \in F' \ xBx', \text{ and } \forall x' \in F' \ \exists x \in F \ xBx'\}.$$

Interestingly, we may prove that

$$B \text{ is closed (in } X \times X') \text{ iff } \widetilde{B} \text{ is closed (in } \mathcal{V}X \times \mathcal{V}X'). \tag{9}$$

The implication '⇒' follows from (6): If B is closed then $\widetilde{\mathcal{V}}B$ is a closed subset in $\mathcal{V}X \times \mathcal{V}X'$, simply by Definition 3.2. But by (6), $\widetilde{\mathcal{V}}B = \widetilde{B}$, and so the latter is indeed closed.

For the implication in the opposite direction, suppose that $\widetilde{B}$ is closed in $\mathcal{V}X \times \mathcal{V}X'$. In order to prove that B is closed, we will show its complement to be open. For this purpose, take $(x, x') \notin B$. It suffices to find an open neighborhood of (x, x') that does not intersect B. Since every singleton of a Stone space is closed, we have that $(\{x\}, \{x'\}) \in \mathcal{V}X \times \mathcal{V}X'$. Moreover, $(\{x\}, \{x'\}) \notin \widetilde{B}$. Therefore, there exist basic open sets $U \subseteq \mathcal{V}X$ and $U' \subseteq \mathcal{V}X'$ such that $\{x\} \in U$, $\{x'\} \in U'$ and $(U \times U') \cap \widetilde{B} = \emptyset$. By definition of the Vietoris topology, there are clopens C_i, C_i' with $U = \langle \ni \rangle C_1 \cap \cdots \cap \langle \ni \rangle C_k \cap [\ni] C_{k+1} \cap \cdots \cap [\ni] C_n$ and $U' = \langle \ni \rangle C_1' \cap \cdots \cap \langle \ni \rangle C_{k'}' \cap [\ni] C_{k'+1}' \cap \cdots \cap [\ni] C_{n'}'$. Then $x \in \bigcap_{i=1}^n C_i$ and $x' \in \bigcap_{i=1}^{n'} C_i'$. Let $C = \bigcap_{i=1}^n C_i$ and $C' = \bigcap_{i=1}^{n'} C_i'$. Then $(x, x') \in C \times C'$. Now we show that $(C \times C') \cap B = \emptyset$. Suppose there exists $(s, s') \in B$ such that $s \in C$ and $s' \in C'$. Then $(\{s\}, \{s'\}) \in \widetilde{B}$ and $(\{s\}, \{s'\}) \in U \times U'$. This is a contradiction since the intersection of $U \times U'$ with $\widetilde{B}$ is empty. Thus, $C \times C'$ is a neighborhood of (x, x') that does not intersect B. This finishes the proof of (9).

Our main reason for believing that Definition 3.3 provides the right notion of a Vietoris bisimulation is that, unlike the Aczel-Mendler bisimilarity of the following section, Vietoris bisimilarity exactly captures the notion of *behavioral equivalence* between two Stone spaces. As a matter of fact, we can show that the relation of behavioral equivalence is itself a Vietoris bisimulation.

Theorem 3.8. *Let $\mathbb{M} = (X, R, \nu)$ and $\mathbb{M}' = (X', R', \nu')$ be descriptive models. Then the relation of behavioral equivalence between $\mathbb{M}$ and $\mathbb{M}'$ is a Vietoris bisimulation.*

Proof. By the Theorems 2.9 and 3.6 it suffices to prove that the relation $\leftrightsquigarrow$ of modal equivalence is a closed Kripke bisimulation between $\mathbb{M}$ and $\mathbb{M}'$.

We first prove that $\leftrightsquigarrow$ is a Kripke bisimulation. Let $x \leftrightsquigarrow x'$. Then obviously x and x' satisfy the same propositional variables. Now assume xRy and suppose there is no $y' \in R'(x')$ such that y and y' are modally equivalent. This means that for every $y' \in R'(x')$ there exists a formula $\varphi_{y'}$ such that $y \not\models \varphi_{y'}$ and $y' \models \varphi_{y'}$. Therefore, $R'(x') \subseteq \bigcup_{y' \in R'(x')} \llbracket \varphi_{y'} \rrbracket^{\mathbb{M}'}$, where $\llbracket \varphi_{y'} \rrbracket^{\mathbb{M}'}$ denotes the set of points in $\mathbb{M}'$ where $\varphi_{y'}$ holds. Since $\mathbb{M}$ is a descriptive model, for each $\varphi_{y'}$, the set $\llbracket \varphi_{y'} \rrbracket^{\mathbb{M}'}$ is clopen. (It is routine to verify that in a general frame, the interpretation of any modal formula is a clopen set.) Moreover, $R'(x')$, being a closed set of a compact space, is also compact. Thus, there exist $\varphi_1, \ldots, \varphi_n$ such that $R'(x') \subseteq \bigcup_{i=1}^n \llbracket \varphi_i \rrbracket^{\mathbb{M}'}$ and for each $i \leq n$ we have $y \not\models \varphi_i$. This implies that $x' \models \Box(\varphi_1 \vee \cdots \vee \varphi_n)$ and $y \not\models \varphi_1 \vee \cdots \vee \varphi_n$. Finally, since x and x' are modally equivalent, we obtain that $x \models \Box(\varphi_1 \vee \cdots \vee \varphi_n)$, which together with xRy gives us a contradiction. The proof of the forth condition is similar.

It remains to show that $\leftrightsquigarrow$ is closed. Suppose x and x' are not modally equivalent. Then there exists a formula φ such that $\mathbb{M}, x \models \varphi$ and $\mathbb{M}', x' \not\models \varphi$. Let U be the basic open set $\llbracket \varphi \rrbracket^{\mathbb{M}} \times \llbracket \neg \varphi \rrbracket^{\mathbb{M}'}$. Then $(x, x') \in U$, while it is obvious that $\leftrightsquigarrow \cap U = \emptyset$. Thus, we found an open neighborhood of (x, x') contained in the complement of $\leftrightsquigarrow$. This means that $\leftrightsquigarrow$ is closed. $\square$

Remark 3.9. Theorem 3.8 is closely related to results in the literature on so-called m-saturated models. A modal model is called *m-saturated* if, for every state $w \in W$, every set Σ

of formulas which is finitely satisfiable in the set of successors of w, is itself satisfiable in the set of successors of w. Then a result, originally due to A. Visser (unpublished, see [8, section 2.5] for a proof), states that the class of m-saturated models has the so-called Hennessy-Milner property. That is, between any two m-saturated models, the relation of modal equivalence is a (Kripke) bisimulation. The connection with our approach here is that one may easily show that descriptive models are in fact m-saturated, so that Theorem 3.8 can be seen as a corollary of Visser's result. For more work in this direction, see Goldblatt [14] or Hollenberg [17].

As a fairly direct corollary of the previous theorem we find that the notions of behavioral equivalence, modal equivalence, Vietoris bisimilarity and Kripke bisimilarity all *coincide*.

Corollary 3.10. *Let $\mathbb{M}$ and $\mathbb{M}'$ be descriptive models, and let x and x' be two points in $\mathbb{M}$ and $\mathbb{M}'$, respectively. Then the following are equivalent:*

1. *x and x' are behaviorally equivalent*

2. *x and x' are modally equivalent*

3. *x and x' are Vietoris bisimilar*

4. *x and x' are Kripke bisimilar.*

Proof. The equivalence '1 ⇔ 2' forms the content of Theorem 2.9. The implication '1 ⇒ 3' is immediate by the previous theorem, and the implication '3 ⇒ 4' follows from Theorem 3.6. The remaining implication '4 ⇒ 2' follows from the invariance of the truth of modal formulas under Kripke bisimilarity. $\square$

Note that the equivalence '1 ⇔ 4' in Corollary 3.10 is a special case of a more general result by Kupke, Kurz & Pattinson [20, Proposition 26] which compares the notion of behavioral equivalence between certain Stone coalgebras to that of their associated set coalgebras.

4   Aczel-Mendler bisimulations

In this section we discuss an earlier notion of bisimulation, due to Aczel & Mendler [4]. Here the basic idea is that the bisimulation relation is itself the carrier of a coalgebraic structure. Aczel & Mendler's definition, which applied to set coalgebras only, easily generalizes to the following definition for any category in which finite products exist.

Definition 4.1. *Let $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ be an endofunctor on a category C, which we assume to have binary products. Consider two $\mathcal{T}$-coalgebras (S, σ) and (S', σ'), and let B be a subobject of $S \times S'$, with associated projections $\pi : B \to S$ and $\pi' : B \to S'$. The triple (B, π, π') is called a $\mathcal{T}$-bisimulation if there exists a morphism $\beta : B \to \mathcal{T}B$ such that the following diagram commutes:*

$$
\begin{array}{ccccc}
S & \xleftarrow{\ \pi\ } & B & \xrightarrow{\ \pi'\ } & S' \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\sigma'} \\
\mathcal{T}S & \xleftarrow[\mathcal{T}\pi]{} & \mathcal{T}B & \xrightarrow[\mathcal{T}\pi']{} & \mathcal{T}S'
\end{array}
$$

13

Remark 4.2. If the base category C does not have binary products, then bisimulations can be defined as so-called *mono spans* (B, π, π'); see, e.g., Kurz [23, Section 1.1.3].

Note that the above definition, when applied to Kripke structures, admits bisimulations between two Kripke frames that are not concretely based on actual binary relations between the carriers of the two coalgebras. If we restrict our attention to concrete relations, as we will do in this paper, then the two notions coincide.

Fact 4.3. *Let* $\mathbb{M} = (W, R, \nu)$ *and* $\mathbb{M}' = (W', R', \nu')$ *be Kripke models and therefore coalgebras for the Kripke functor* $\mathcal{P}_\Phi$. *Let* $B \subseteq W \times W'$ *be a binary relation between* W *and* W' *with associated projections* $\pi : (x, x') \mapsto x$ *and* $\pi' : (x, x') \mapsto x'$. *Then* B *is a Kripke bisimulation iff* (B, π, π') *is a* $\mathcal{P}_\Phi$-*bisimulation.*

Now we are ready to define bisimulations of descriptive models. Given Fact 4.3, it seems natural to take the following coalgebraic perspective, analogous to Definition 4.1.

Definition 4.4. *Let* (X, σ) *and* (X', σ') *be two descriptive models, and let* $B \subseteq X \times X'$ *be a binary relation with associated projections* $\pi : (x, x') \mapsto x$ *and* $\pi' : (x, x') \mapsto x'$. *Then we call* B *an* Aczel-Mendler bisimulation, *if, seen as a subspace of the product topology* $X \times X'$, *it is a Stone space which can be endowed with a Vietoris coalgebra structure* β *that makes the diagram below commute.*

$$
\begin{array}{ccccc}
X & \xleftarrow{\ \ \pi\ \ } & B & \xrightarrow{\ \ \pi'\ \ } & X' \\
\sigma \downarrow & & \beta \downarrow & & \downarrow \sigma' \\
\mathcal{V}_\Phi X & \xleftarrow[\mathcal{V}_\Phi \pi]{} & \mathcal{V}_\Phi B & \xrightarrow[\mathcal{V}_\Phi \pi']{} & \mathcal{V}_\Phi X'
\end{array}
$$

Two points x *and* x' *are called* Aczel-Mendler bisimilar *if they are linked by some Aczel-Mendler bisimulation.*

Note that the diagram of Definition 4.4 is a diagram in the category of *Stone spaces*. In particular, the arrow β must be a *continuous* map from B to $\mathcal{V}_\Phi B$.

Proposition 4.5. *Every Aczel-Mendler bisimulation between two descriptive models is also a Kripke bisimulation. As a corollary, Aczel-Mendler bisimilarity implies Kripke bisimilarity, Vietoris bisimilarity, behavioral equivalence, and modal equivalence.*

Proof. That every Aczel-Mendler bisimulation is a Kripke bisimulation, follows directly from the commutativity of the diagram of Definition 4.4. (For details: the argument of [32, Example 168], which applies to the powerset functor, goes through for the Vietoris functor as well.) From this the second statement of the proposition is immediate by Corollary 3.10. $\qquad\square$

The following example shows that the converse of Proposition 4.5 does not hold in general: Aczel-Mendler bisimilarity is *not* the same as Vietoris bisimilarity, and hence, it differs from modal equivalence and behavioral equivalence as well.
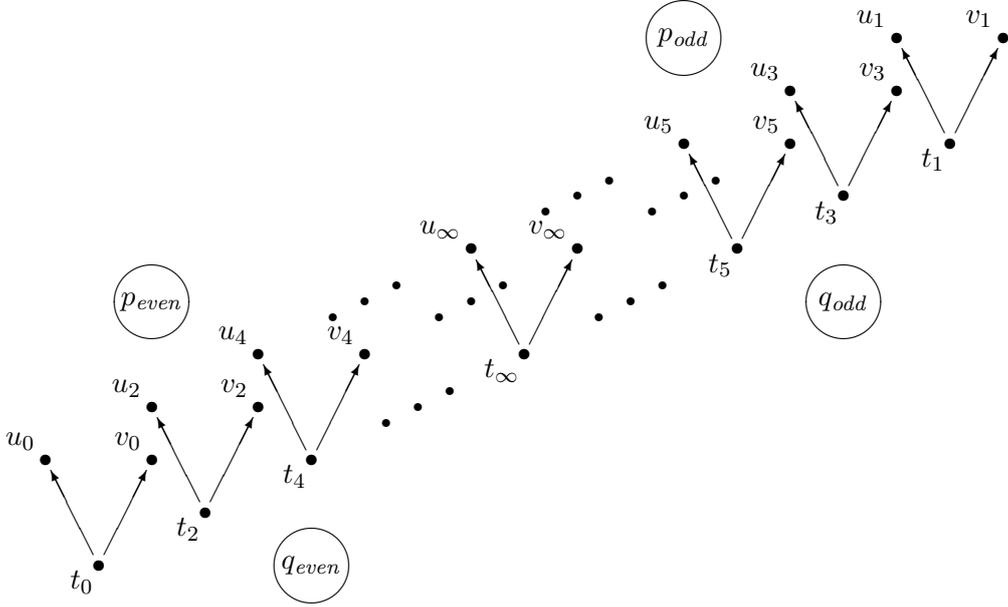
Figure 1: The descriptive model $\mathbb{X}$ of Example 4.6

Example 4.6. We will give an example of two descriptive models and two states in these models that are modally equivalent, but not Aczel-Mendler bisimilar. Let $T_0 = \{t_i \mid i \in \omega\}$, $U_0 = \{u_i \mid i \in \omega\}$ and $V_0 = \{v_i \mid i \in \omega\}$ be countable sets, endowed with the discrete topology. Let $T = T_0 \cup \{t_\infty\}$, $U = U_0 \cup \{u_\infty\}$ and $V = V_0 \cup \{v_\infty\}$ be the Alexandroff compactifications of T_0, U_0, and V_0, respectively (for the definition of the Alexandroff compactification see for instance [11, 3.5.11]). In other words, the clopens of T are the finite subsets of T_0, together with the cofinite subsets of T containing the point t_∞, and likewise for U and V. We let X denote the topological sum $T \oplus U \oplus V$, and define a relation R on X by

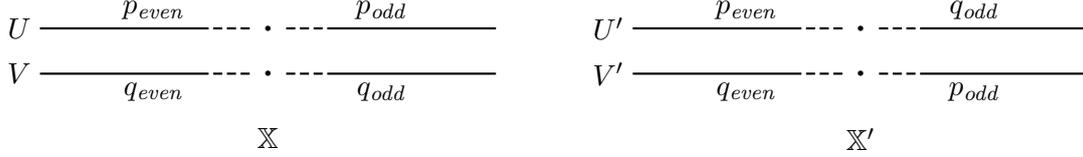$$R := \{(t_i, u_i), (t_i, v_i) \mid i \in \omega \cup \{\infty\}\}.$$

See Figure **??** for a picture of (X, R). Let (X', R') be an isomorphic copy of (X, R), then it is routine to check that (X, R) and (X', R') are descriptive frames. For the set $\{p_i, q_i \mid i \in \omega\}$ of proposition letters, define valuations ν and ν' on X and X' by

- $\nu(u_i) = \nu'(u_i') = \{p_i\}$ for all even $i \in \omega$,

- $\nu(u_i) = \nu'(v_i') = \{p_i\}$ for all odd $i \in \omega$,

- $\nu(v_i) = \nu'(v_i') = \{q_i\}$ for all even $i \in \omega$,

- $\nu(v_i) = \nu'(u_i') = \{q_i\}$ for all odd $i \in \omega$,

- $\nu(u_\infty) = \nu(v_\infty) = \nu'(u_\infty') = \nu'(v_\infty') = \emptyset$,

15

- $\nu(t_i) = \nu'(t'_i) = \emptyset$ for all $i \in \omega \cup \{\infty\}$.

It is easy to check that ν and ν' are admissible valuations and therefore $\mathbb{X} = (X, R, \nu)$ and $\mathbb{X}' = (X', R', \nu')$ are descriptive models.

To explain the valuations, we have indicated in Figure **??** where the various proposition letters are satisfied in the descriptive model $\mathbb{X}$. The picture below, which only depicts the sets U, V, U' and V', should clarify the difference between the two models.

We first focus on the relation $B := \leftrightsquigarrow$ of modal equivalence. It is easy to see that

$$
\begin{aligned}
B \;=\; & \{(t_i, t'_i) \mid i \in \omega \cup \{\infty\}\} \,\cup \\
& \{(u_i, u'_i), (v_i, v'_i) \mid i \text{ is even }\} \cup \{(u_i, v'_i), (v_i, u'_i) \mid i \text{ is odd }\} \cup (\{u_\infty, v_\infty\} \times \{u'_\infty, v'_\infty\})
\end{aligned}
$$

Claim 1. B is a Vietoris bisimulation, but not an Aczel-Mendler bisimulation.

Proof of Claim In the previous section we already proved that the relation of modal equivalence between two descriptive models is a Vietoris bisimulation. This takes care of the first part of the claim, so it is left to prove that B is not an Aczel-Mendler bisimulation. Suppose for contradiction that $\beta_\Phi : B \to \mathcal{V}_\Phi(B)$ is a continuous map making the diagram of Definition 4.4 commute. Let $\beta : B \to \mathcal{V}(B)$ be the composition of $\beta_\Phi : B \to \mathcal{V}(B) \times 2^\Phi$ with the first projection of $\mathcal{V}(B) \times 2^\Phi$ to $\mathcal{V}(B)$. Obviously, β_Φ is continuous iff β is continuous. Therefore, it suffices to show that $\beta : B \to \mathcal{V}(B)$ is not continuous.

Chasing the diagram of Definition 4.4, it is not hard to show that

$$\text{if } i \in \omega, \text{ then } \beta(t_i, t'_i) = (R(t_i) \times R'(t'_i)) \cap B, \tag{10}$$

and

$$\beta(t_\infty, t'_\infty) \cap (\{u_\infty, v_\infty\} \times \{u'_\infty, v'_\infty\}) \neq \emptyset. \tag{11}$$

To see why (11) leads to a contradiction, suppose that $(u_\infty, u'_\infty) \in \beta(t_\infty, t'_\infty)$. (A symmetric argument will apply to any other element of $\{u_\infty, v_\infty\} \times \{u'_\infty, v'_\infty\}$ belonging to $\beta(t_\infty, t'_\infty)$.) Consider the set $C = (U \times U') \cap B$. Obviously, $C = \{(u_i, u'_i) \mid i \text{ is even}\} \cup \{(u_\infty, u'_\infty)\}$. Since U and U' are clopen in X and X', respectively, the set $U \times U'$ is clopen in $X \times X'$. Therefore, C is clopen in B, and so by definition of $\mathcal{V}(B)$, $\langle\ni\rangle C$ is open in $\mathcal{V}(B)$. Then by the supposed continuity of β, the set $\beta^{-1}(\langle\ni\rangle C) = \{(x, x') \mid \beta(x, x') \cap C \neq \emptyset\}$ is open in B. However, using (10) and (11), it is not hard to show that

$$\beta^{-1}(\langle\ni\rangle C) = \{(t_i, t'_i) \mid i = \infty \text{ or } i \text{ is even}\},$$

and this set is clearly not open in B. (For instance, one may show that the pair $(t_\infty, t'_\infty) \in \beta^{-1}(\langle\ni\rangle C)$ is a limit point of the complement $B \setminus \beta^{-1}(\langle\ni\rangle C)$ of $\beta^{-1}(\langle\ni\rangle C)$). This gives us the desired contradiction, and thus proves the claim. $\triangleleft$

Note that, although our example shows that the modal equivalence relation B itself is not always an Aczel-Mendler bisimulation, it does not show that modal equivalence differs from Aczel-Mendler *bisimilarity*. On the contrary, one may easily check that for any two modally equivalent points $x \in X$ and $x' \in X'$, the relation $\{(x,x')\} \cup ((R(x) \times R'(x')) \cap B)$ is an Aczel-Mendler bisimulation, making x and x' Aczel-Mendler bisimilar.

In order to show that the relation of modal equivalence *does not coincide* with Aczel-Mendler bisimilarity in general, we need to modify our example a bit. Let Y be the Stone space we obtain from X by adding one single point r, i.e., we let Y be the topological sum $X \oplus \{r\}$. We also let $S = R \cup \{(u_i,r),(v_i,r),(r,t_i) \mid i \in \omega \cup \{\infty\}\}$. It is easy to check that (Y,S) is a descriptive frame. Let (Y',S') be an isomorphic copy of (Y,S). We define valuations μ and μ' on Y and Y', respectively, by

- $\mu(x) = \nu(x)$ and $\mu'(x') = \nu'(x')$ for each $x \in X$ and $x' \in X'$,

- $\mu(r) = \mu'(r') = \{p\}$.

It is easy to check that μ and μ' are descriptive valuations. Thus, $\mathbb{Y} = (Y,S,\mu)$ and $\mathbb{Y}' = (Y',S',\mu')$ are descriptive models.

Claim 2. The points r and r' are Vietoris bisimilar, but not Aczel-Mendler bisimilar.

Proof of Claim It is easy to verify that r and r' are modally equivalent, and so by Corollary 3.10 they are Vietoris bisimilar. Now for contradiction assume that r and r' are also Aczel-Mendler bisimilar, that is, there is an Aczel-Mendler bisimulation B' linking r and r'. By Proposition 4.5 we obtain that B' is a Kripke bisimulation, and so from $rB'r'$ it is not hard to derive that

$$
\begin{aligned}
B' &= \{(r,r')\} \cup \{(t_i,t_i') \mid i \in \omega \cup \{\infty\}\} \cup \\
&\quad \{(u_i,u_i'),(v_i,v_i') \mid i \text{ is even }\} \cup \{(u_i,v_i'),(v_i,u_i') \mid i \text{ is odd }\} \cup B_{uv}
\end{aligned}
$$

for some set $B_{uv} \subseteq \{u_\infty,v_\infty\} \times \{u_\infty',v_\infty'\}$ satisfying $(\{u_\infty,v_\infty\},\{u_\infty',v_\infty'\}) \in \widetilde{\mathcal{P}}(B_{uv})$. In particular, B_{uv} is non-empty, and so the same kind of argument as above shows that B' is not an Aczel-Mendler bisimulation. ◁

Remark 4.7. In fact we can avoid the use of propositional variables in Example 4.6. We let T_n denote the tree obtained from the disjoint union of all the chains of length k for $k \le n$, by identifying their roots. We let (Z,S) be the descriptive frame obtained by substituting each u_i and v_i in (X,R), for $i \in \omega$, by T_{2i+1} and T_{2i}, respectively, and let (Z',S') be an isomorphic copy of (Z,S). Then an argument similar to the one used in Example 4.6 shows that the modal equivalence between the descriptive *frames* (Z,S) and (Z',S') is not an Aczel-Mendler bisimulation. We skip the details.

As a corollary to Example 4.6, we obtain that the Vietoris functor does not preserve weak pullbacks. (For the definition of weak pullbacks and weak pullback preserving functors we refer to [32].) In fact, the structures of Example 4.6 themselves provide an explicit counterexample.

Corollary 4.8. *The Vietoris functor does not preserve weak pullbacks.*
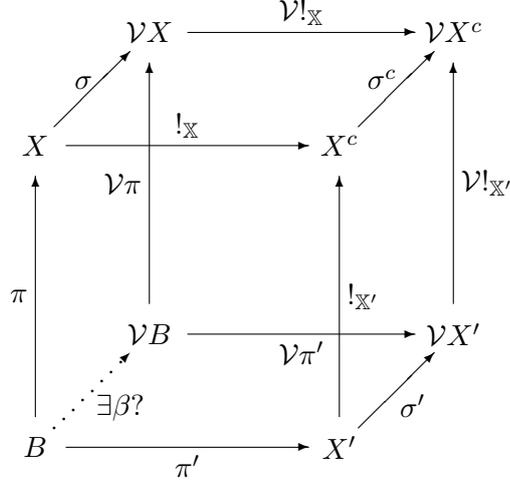
Figure 2: An explicit counterexample to $\mathcal{V}$ preserving weak pullbacks

Proof. The Corollary itself is immediate on the basis of Example 4.6, but it might be useful to see the concrete weak pullback that is not preserved. Let X and X be the Stone spaces of Example 4.6. Also let $!_{\mathbb{X}} : X \to X^c$ and $!_{\mathbb{X}'} : X' \to X^c$ be the theory maps discussed in Section 2, where X^c is the underlying Stone space of the canonical model $\mathbb{M}^c = (X^c, R^c, \nu^c)$. Then clearly, $!_{\mathbb{X}} : X \to X^c$ and $!_{\mathbb{X}'} : X' \to X^c$ are coalgebra homomorphisms from the descriptive frames (X, R) and (X', R'), respectively, to (X^c, R^c). It is easy to check that the modal equivalence $B \subseteq X \times X'$ described in Example 4.6 is equal to $\{(x, x') \in X \times X' \mid !_{\mathbb{X}}(x) = !_{\mathbb{X}'}(x')\}$. Therefore, B is the pullback of $!_{\mathbb{X}}$ and $!_{\mathbb{X}'}$. So if $\mathcal{V}$ preserves weak pullbacks, then $\mathcal{V}B$ must be a weak pullback of $\mathcal{V}(!_{\mathbb{X}})$ and $\mathcal{V}(!_{\mathbb{X}'})$.

We show that this gives a contradiction. Consider the maps $\sigma \circ \pi : B \to \mathcal{V}X$ and $\sigma' \circ \pi' : B \to \mathcal{V}X'$. First we claim that

$$\mathcal{V}(!_{\mathbb{X}}) \circ \sigma \circ \pi = \mathcal{V}(!_{\mathbb{X}'}) \circ \sigma' \circ \pi'. \tag{12}$$

This follows from the facts that $!_{\mathbb{X}}$ is a coalgebra morphism ($\mathcal{V}(!_{\mathbb{X}}) \circ \sigma = \sigma^c \circ !_{\mathbb{X}}$), that B is a pullback ($!_{\mathbb{X}} \circ \pi = !_{\mathbb{X}'} \circ \pi'$) and that $!_{\mathbb{X}'}$ is a coalgebra morphism ($\sigma^c \circ !_{\mathbb{X}'} = \mathcal{V}(!_{\mathbb{X}'}) \circ \sigma'$), see the diagram in Figure 1.

Now if $\mathcal{V}B$ is a weak pullback of $!_{\mathbb{X}}$ and $!_{\mathbb{X}'}$, there must exist a map $\beta : B \to \mathcal{V}B$ such that $\mathcal{V}\pi \circ \beta = \sigma \circ \pi$ and $\mathcal{V}\pi' \circ \beta = \sigma' \circ \pi'$. But this means that B is an Aczel-Mendler bisimulation, contradicting Claim 1 in Example 4.6. □

We finish this section by showing that the set of Aczel-Mendler bisimulations between two descriptive models do not form a lattice.

Corollary 4.9. *There exist pairs of descriptive models for which the poset of Aczel-Mendler bisimulations is not a lattice.*

18

Proof. Let $\mathbb{X}$ and $\mathbb{X}'$ be the models described in Example 4.6. Let

$$B_1 := \{(u_i, u_i'), (v_i, v_i'), (t_i, t_i') \mid i \text{ is odd or } i = \infty\}$$

and

$$B_2 := \{(u_i, v_i'), (v_i, u_i'), (t_i, t_i') \mid i \text{ is even or } i = \infty\}.$$

We show that B_1 and B_2 are Aczel-Mendler bisimulations. Without loss of generality we consider the case of B_1. It is easy to see that B_1 is closed, therefore it is a Stone space. As in Example 4.6, instead of $\mathcal{V}_\Phi(B_1)$, we work with $\mathcal{V}(B_1)$. We define a map $\beta_1 : B_1 \to \mathcal{V}(B_1)$ by

$$\beta_1(x, x') = (R(x) \times R'(x')) \cap B_1.$$

Since R and R' are closed, β_1 is well defined. The standard argument also shows that the diagram of Definition 4.4 commutes. Finally, let C be clopen in B_1. Then $C = \bigcup_{i=1}^n (C_i \times C_i') \cap B_1$, where C_i and C_i' are clopens in X and X', respectively. It is now routine to check that for each clopen A and A' in X and X', respectively, we have $\beta_1^{-1}(\langle \ni \rangle (A \times A') \cap B_1) = (\langle R \rangle A \times \langle R' \rangle A') \cap B_1$ and $\beta_1^{-1}([\ni](A \times A') \cap B_1) = ([R]A \times [R']A') \cap B_1$. Thus, β_1 is continuous and B_1 is an Aczel-Mendler bisimulation.

We observe that the modal equivalence relation B of Example 4.6 is equal to $B_1 \cup B_2$. Therefore, $B_1 \cup B_2$ is not an Aczel-Mendler bisimulation. Moreover, since $B_1 \cup B_2$ is modal equivalence, and by Proposition 4.5 Aczel-Mendler bisimilarity implies modal equivalence, there is no Aczel-Mendler bisimulation that contains $B_1 \cup B_2$. Thus, the Aczel-Mendler bisimulations B_1 and B_2 do not have an upper bound in the poset of all Aczel-Mendler bisimulations. $\square$

We see the results of this section as strong evidence that Aczel-Mendler bisimilarity is not the right notion of bisimilarity between descriptive models. Therefore, we prefer to work with the Vietoris bisimulations defined in the previous section.

5   Kripke versus Vietoris bisimulations

In Section 2 we already saw that the Vietoris bisimulations between two descriptive models can be identified with those Kripke bisimulations (between the underlying Kripke models) that are closed in the product space of the two Stone topologies. In this section we prove something rather stronger, namely that the closure of any Kripke bisimulation is a Vietoris bisimulation. Given Theorem 3.6, the point to prove is that the topological closure of a Kripke bisimulation is again a Kripke bisimulation. The proof of this result involves some topology theory related to the concept of a *net*. In order to keep the paper self-contained, we recall the necessary definitions and facts that are used in the proof. Readers who would like to have more background on nets are referred to [33].

Let X be a topological space. A *net* is a map from a directed set into the space X. (Recall that a directed set is a nonempty set I together with a preorder $\leq$ having the additional property that every pair of elements has an upper bound.) Our usual notation for a net will be $N = \{x_i \mid i \in I\}$, where I is the directed set corresponding to the net N and x_i is the

point assigned to $i \in I$. A net $P = \{y_j \mid j \in J\}$ is a *subnet* of a net $N = \{x_i \mid i \in I\}$ if there is an monotone cofinal map $f : J \to I$ such that $y_j = x_{f(j)}$, for all $j \in J$. (Recall that f is *monotone* if for all $j_0, j_1 \in J$, we have $j_0 \leq j_1$ implies that $f(j_0) \leq f(j_1)$, and *cofinal* if for every $i_0 \in I$, there is $j_0 \in J$ such that $f(j_0) \geq i_0$.)

Let V be a subset of X, and $N = \{x_i \mid i \in I\}$ a net. Then N *is in* V if $x_i \in V$ for all $i \in I$; N is *eventually* in V if there exists $i_0 \in I$ such that $x_i \in V$, for all $i \geq i_0$; and N is *cofinally* in V if for all $i_0 \in I$, there exists $i_1 \geq i_0$ such that $x_{i_1} \in V$. N *converges* to a point $x \in X$, or has *limit* x, if for every open neighborhood V of x, N is eventually in V. A point $x \in X$ is a *cluster point* of a net N if for each open neighborhood V of x, N is cofinally in V.

Fact 5.1 ([33]). *Let X and X' be topological spaces and let V be a subset of X.*
(1) A point x is in the closure of V iff there exists a net N in V converging to x.
(2) A net N converges to a point $x \in X$ iff every subnet of N converges to x.
(3) A point x is a cluster point of a net N iff there is a subnet of N converging to x.
(4) If X is Hausdorff, then every net has at most one limit point.
(5) If X is compact, then every net has a cluster point.

The following result is one of the main technical contributions of the paper.

Theorem 5.2. *Let (X, R, ν) and (X', R', ν') be descriptive models. If $B \subseteq X \times X'$ is a bisimulation between the underlying Kripke models, then $\overline{B}$ is a Vietoris bisimulation.*

Proof. Given a Kripke bisimulation B, we will show that $\overline{B}$ is a Vietoris bisimulation.

First, suppose for contradiction that $\nu(x) \neq \nu'(x')$ for some pair (x, x') in $\overline{B}$. Then without loss of generality we may assume that $p \in \nu(x) \setminus \nu'(x')$, for some $p \in \Phi$. Consider the set $\Omega = \nu^{-1}(p) \times (X' \setminus \nu'^{-1}(p))$. It is not hard to see that Ω is a clopen neighbourhood of (x, x'). Since (x, x') belongs to $\overline{B}$, this means that $\Omega \cap B$ is nonempty. But any $(u, u') \in \Omega$ will have the property that $u \Vdash p$ while $u' \nVdash p$, so the nonemptiness of $\Omega \cap B$ contradicts the fact that B is a bisimulation. Hence we may conclude that $\nu(x) = \nu'(x')$.

It remains to check that $\overline{B}$ satisfies the back and forth condition. We only consider the forth condition, leaving the back condition to be proved by a symmetric argument. Consider again a pair (x, x') in $\overline{B}$, and suppose that xRy for some $y \in X$. We need to come up with a y' such that $x'R'y'$ and $y\overline{B}y'$. Roughly, the idea of the proof is the following. Since (x, x') is in $\overline{B}$, there is a net $M = \{(x_i, x_i') \mid i \in I\}$ converging to (x, x') and such that $M \subseteq B$. We let M_X be the net $\{x_i \mid i \in I\}$ and we let $M_{X'}$ be the net $\{x_i' \mid i \in I\}$. First we define a subnet $N_X = \{u_j \mid j \in J\}$ of M_X and a net $P_X = \{v_j \mid j \in J\}$ converging to y, such that $u_j R v_j$, for all $j \in J$. Then we consider the subnet $N_{X'} = \{u_j' \mid j \in J\}$ of $M_{X'}$ corresponding to N_X. This means that (u_j, u_j') belongs to B, for all $j \in J$, so using the fact that B is a bisimulation, we construct a net $\{v_j' \mid j \in J\}$ in X' such that $(v_j, v_j') \in B$ and $u_j' R' v_j'$, for all $j \in J$. We take y' to be an arbitrary cluster point of $P_{X'}$ and we show that $x'R'y'$ and $(y, y') \in \overline{B}$.

Turning to the details, take a non-empty directed set I and a net $M = \{(x_i, x_i') \mid i \in I\}$ converging to (x, x') and such that $M \subseteq B$. Without loss of generality, we may assume that I is unbounded. Let M_X be the net $\{x_i \mid i \in I\}$ and let $M_{X'}$ be the net $\{x_i' \mid i \in I\}$. It is routine to check that M_X converges to x and $M_{X'}$ converges to x'.

Claim 1. There is a subnet $N_X = \{u_j \mid j \in J\}$ of M_X, witnessed by a monotone, cofinal map $h : J \to I$, and a net $P_X = \{v_j \mid j \in J\}$ converging to y, satisfying $u_j R v_j$, for all $j \in J$.

Proof of Claim Let K be the directed set $\{(z,i) \mid i \in I, x_i R z\}$ where the order is defined by

$$(z_1, i_1) \leq (z_2, i_2) \text{ iff } \big(i_1 < i_2 \text{ or } (z_1, i_1) = (z_2, i_2)\big),$$

for all $i_1, i_2 \in I$ and $z_1 \in R(x_{i_1}), z_2 \in R(x_{i_2})$. Let P be the net $\{y_k \mid k \in K\}$ such that $y_k = z$, if $k = (z,i)$.

We show that y is a cluster point of P. Let V be a clopen neighborhood of y and let $(z_0, i_0) \in K$. We need to find a $k \geq (z_0, i_0)$ such that y_k belongs to V. Remark first that $\langle R \rangle V$ is a clopen neighborhood of x. Since M_X converges to x, there exists $i_1 \in I$ such that $x_i \in \langle R \rangle V$, for all $i \geq i_1$. Since I is directed and unbounded, we can choose an element i_2 of I such that $i_2 > i_0$ and $i_2 \geq i_1$. Then by $i_2 \geq i_1$ we have that x_{i_2} belongs to $\langle R \rangle V$. That is, there exists $z_2 \in V$ such that $x_{i_2} R z_2$. Now let k be the pair (z_2, i_2), then it is immediate that $k \geq (z_0, i_0)$ and that $y_k = z_2$ belongs to V. This finishes the proof that P is cofinally in V, and since V was arbitrary, it shows that y is a cluster point of P.

It follows that there is a subnet $P_X = \{v_j \mid j \in J\}$ of P converging to y, witnessed by a monotone cofinal map $g : J \to K$ such that $v_j = y_{g(j)}$. If we recall $y_k = z$ iff $k = (z,i)$ for some $i \in I$, this means that for any $j \in J$ we may write $g(j) = (v_j, h(j))$, and so by definition of K we have $v_j \in R(x_{h(j)})$. Then it is routine to verify that $h : J \to I$ is monotone and cofinal. Hence, if we let u_j be the element $x_{h(j)}$, then $N_X = \{u_j \mid j \in J\}$ is a subnet of M such that $u_j R v_j$, for all $j \in J$. This finishes the proof of the Claim. ◁

Next, we define the net $N_{X'}$ as the net $\{u'_j \mid j \in J\}$, where $u'_j = x'_{h(j)}$. It is not hard to show that $N_{X'}$ is a subnet of $M_{X'}$, and that $(u_j, u'_j) \in B$, for all $j \in J$. Putting this together with $u_j R v_j$ and the fact that B is a bisimulation, we get that for all $j \in J$, there exists $v'_j \in X'$ such that $(v_j, v'_j) \in B$ and $u'_j R v'_j$. Let $P_{X'}$ be the net $\{v'_j \mid j \in J\}$.

Claim 2. Let y' be a cluster point of $P_{X'}$. Then $(y, y') \in \overline{B}$ and $x' R' y'$.

Proof of Claim First we prove that (y, y') belongs to $\overline{B}$. Since y' is a cluster point of $P_{X'}$, there is a subnet $S_{X'} = \{z'_l \mid l \in L\}$ of $P_{X'}$ converging to y'. It follows that there exists a subnet $S_X = \{z_l \mid l \in L\}$ of P_X satisfying $(z_l, z'_l) \in B$, for all $l \in L$. Now it is easy to check that $\{(z_l, z'_l) \mid l \in L\}$ is a net converging to (y, y'). Since $\{(z_l, z'_l) \mid l \in L\}$ is a subset of B, we obtain that (y, y') belongs to $\overline{B}$.

Turning to the second part of the claim, we first prove that $x \in \langle R' \rangle V'$ for any clopen neighborhood V' of y'. To see this, observe that the net $P_{X'}$ is cofinally in V' since y' is a cluster point of $P_{X'}$. In other words, for every $j \in J$, there is $j_1 \geq j$ such that v'_{j_1} belongs to V. It follows that for every $j \in J$, there is $j_1 \geq j$ such that u'_{j_1} belongs to $\langle R' \rangle V'$ (since $u'_{j_1} R' v'_{j_1}$). That is, the net $N_{X'}$ is cofinally in $\langle R' \rangle V'$. Now suppose for contradiction that $x' \notin \langle R' \rangle V'$. Then there is a clopen neighborhood U' of x' such that $U' \cap \langle R' \rangle V' = \emptyset$. Since $N_{X'}$ converges to x', there is $j_0 \in J$ such that $u'_j \in U'$, for all $j \geq j_0$. Putting this together with $U' \cap \langle R' \rangle V' = \emptyset$, we have that for all $j \geq j_0$, u'_j does not belong to $\langle R' \rangle V'$. This contradicts the fact that $N_{X'}$ is cofinally in $\langle R' \rangle V'$ and so finishes the proof that x' belongs to $\langle R' \rangle V'$.

Thus, the set $R'[x'] \cap V'$ is non-empty. It follows that the family $\{R'[x'] \cap V' \mid V' \text{ clopen}, y \in V'\}$ of closed sets has the finite intersection property. As the space X' is compact, we may

conclude that this family has a non-empty intersection. Since X' is a Hausdorff space, we also have that $\bigcap\{V' \mid V' \text{ clopen}, y \in V'\} = \{y'\}$. Putting everything together, we have $R'[x'] \cap \{y'\} \neq \emptyset$. That is, $x'R'y'$, which finishes the proof of Claim 2. ◁

Now we can finish the proof by observing that by compactness of X', $P_{X'}$ has a cluster point y', and that this y' has the desired properties by Claim 2. □

Remark 5.3. In section 2 we proved that between two descriptive models, the notions of Kripke bisimilarity and Vietoris bisimilarity coincide (Corollary 3.10). We could also have derived this result from Theorem 5.2. For the nontrivial direction, any two points that are linked by some Kripke bisimulation B, are also linked by the Vietoris bisimulation $\overline{B}$.

An interesting corollary of Theorem 5.2 is the following.

Corollary 5.4. *The collection of Vietoris bisimulations between two descriptive models* $\mathbb{M}$ *and* $\mathbb{M}'$ *forms a complete lattice, with joins given by*

$$\bigvee_{i \in I} B_i \; := \; \overline{\bigcup_{i \in I} B_i}. \tag{13}$$

Proof. It suffices to prove, with $\{B_i \mid i \in I\}$ a collection of Vietoris bisimulations between $\mathbb{M}$ and $\mathbb{M}'$, that $\overline{\bigcup_I B_i}$ is the smallest Vietoris bisimulation containing each B_i.

It follows from Theorem 3.6 that each B_i is a Kripke bisimulation. Since the collection of Kripke bisimulations between two models is closed under taking unions, this implies that $\bigcup_I B_i$ is a Kripke bisimulation between (the underlying Kripke models of) $\mathbb{M}$ and $\mathbb{M}'$. But then by Theorem 5.2, $\overline{\bigcup_I B_i}$ is a Vietoris bisimulation between $\mathbb{M}$ and $\mathbb{M}'$. It is then easy to see that it is in fact the smallest closed bisimulation containing every B_i. □

Remark 5.5. While (13) provides a fairly direct description of the joins in a lattice of Vietoris bisimulations, it seems to be harder to find an intuitive characterization of the meets. In particular, meets are generally not given by intersections: while the intersection of a family of Vietoris bisimulations is closed, the (simple) example below shows that in general it is not a (Kripke) bisimulation.

Let $\mathbb{M}$ and $\mathbb{M}'$ be two isomorphic copies of the 2-fork model; that is, $\mathbb{M} = (X, R, \nu)$ and $\mathbb{M}' = (X', R', \nu')$, where X is the discrete space on the set $\{w_0, w_1, w_2\}$, X' likewise on $\{w_0', w_1', w_2'\}$, and $R = \{(w_0, w_1), (w_0, w_2)\}$, $R' = \{(w_0', w_1'), (w_0', w_2')\}$, and $\nu(w_i) = \nu'(w_j')$ for all i and j. Then it is easy to check that $B_1 = \{(w_0, w_0'), (w_1, w_1'), (w_2, w_2')\}$ and $B_2 = \{(w_0, w_0'), (w_1, w_2'), (w_2, w_1')\}$ are bisimulations. However, $B_1 \cap B_2 = \{(w_0, w_0')\}$ is not a bisimulation.

6 Bisimilarity Games

Both the notion of a Kripke bisimulation and that of Kripke bisimilarity can be studied in a game-theoretic framework. For instance, the relation $\leftrightarrow$ of Kripke bisimilarity between two Kripke models can be characterized as the *largest bisimulation* between the two models. Then from Fact 3.1 we may infer that $\leftrightarrow$ is the greatest postfixpoint of some monotone operation

Position	Player	Admissible moves
(x, x')	$\exists$	$\{Z \in \mathcal{V}(X \times X') : (R(x), R'(x')) \in \widetilde{\mathcal{V}}(Z)\}$
$Z \in \mathcal{V}(X \times X')$	$\forall$	Z

Table 1: Bisimilarity game for descriptive models.

on the complete lattice of binary relations between the universes of the two models, and from this it is standard [2] to derive a game-theoretic characterization. Clearly, by Corollary 3.10, the very same games would also work to characterize the notion of Vietoris bisimilarity. But we can do better than this: in this section we briefly discuss games for Vietoris bisimilarity between two descriptive models which only refers to elements and subobjects of the product space, that is: *closed* relations.

Let $\mathbb{M} = (X, R, \nu)$ and $\mathbb{M}' = (X', R', \nu')$ be two descriptive models. A match of the *bisimilarity game* $\mathcal{G}$ between $\mathbb{M}$ and $\mathbb{M}'$ is played by two players, $\exists$ (Éloise) and $\forall$ (Abélard). These players move a token around from one *position* of the game to the next one. In the game there are two kinds of positions: pairs of the form $(s, s') \in X \times X'$ are called *basic positions* and belong to $\exists$. The other positions are closed relations between X and X' and belong to $\forall$.

The idea of the game is that at a position (s, s'), $\exists$ claims that s and s' are Vietoris bisimilar, and to substantiate this claim she proposes a 'local bisimulation' Z, i.e., a closed relation $Z \subseteq X \times X'$ such that $R(x)$ and $R'(x')$ are related by $\widetilde{\mathcal{V}}(Z)$. $\forall$ then challenges her by picking a pair $(t, t') \in Z$ as the next basic position. These rules are summarized in Table 1.

A *match* of the games thus consists of a sequence of positions. $\exists$ *loses* at the instant such a match arrives at a position (s, s') where s and s' do not satisfy the same proposition letters. On the other hand, $\forall$ loses if $\exists$ could (legitimately) choose the empty relation as a candidate bisimulation. In the case that neither player loses after finitely many steps, we are dealing with an infinite match, and the agreement is that all infinite matches are won by $\exists$. Basically then, $\exists$ wins a match if she manages not to get stuck. A position p of this game is called *winning* for $\exists$ if she has a strategy in the instantiation of the game $\mathcal{G}$ initialized at position p.

The difference with the standard (Kripke) bisimilarity game is that here, $\exists$ can only choose binary relations that are *closed* in the product topology. This would make it in principle harder for her to win the game, but nevertheless we can prove the following characterization.

Theorem 6.1. *Let* $\mathbb{M} = (X, R, \nu)$ *and* $\mathbb{M}' = (X', R', \nu')$ *be descriptive models. Then for all* $x \in X$ *and* $x' \in X'$, x *and* x' *are Vietoris bisimilar iff* (x, x') *is a winning position for* $\exists$ *in the bisimilarity game.*

Proof. For the direction from right to left, suppose that B is a Vietoris bisimulation linking x and x'. We need to provide $\exists$ with a winning strategy for the game $\mathcal{G}$ starting at (x, x'). Suppose that $\exists$, starting from position (x, x'), always chooses the relation B. Using the definition of a Vietoris bisimulation, it is straightforward to verify that this is a legitimate, winning strategy for her.

For the converse direction, it clearly suffices to prove that the set $W \subseteq X \times X'$ of winning positions for $\exists$ is itself a Vietoris bisimulation. We leave it for the reader to verify that W

is a Kripke bisimulation. Thus it remains to show that W is closed. Since W is a Kripke bisimulation, it follows from Theorem 5.2 that $\overline{W}$ is a Vietoris bisimulation. As we saw already in the first part of the proof, this implies that every pair (s, s') in $\overline{W}$ is a winning position for $\exists$. But then from the fact that W was defined as the set of *all* winning positions, we may conclude that $\overline{W}$ is a subset of W. It follows that $W = \overline{W}$; in other words, W is closed. $\square$

7 Conclusions & Further Work

In this paper we sketched some rudiments of the coalgebraic theory of descriptive models, focussing on the notion of a bisimulation. We saw that this theory is reasonably well-behaved, and intimately related to the theory of bisimulations between Kripke models. In particular, we found that the notions of Vietoris bisimilarity, Kripke bisimilarity, behavioral equivalence, modal equivalence, all coincide. These four notions do not coincide with Aczel-Mendler bisimilarity, however: this is due to the fact that the Vietoris functor does not preserve weak pullbacks, something we also established in this paper. Most of our positive results could be obtained through relatively easy proofs, with the possible exception of our main technical observation, viz., that the topological closure of a Kripke bisimulation is a Vietoris bisimulation.

Our work can be extended in various directions. Clearly, to start with, one may investigate the Vietoris coalgebras over more general categories of topological spaces, such as compact Hausdorff spaces.

Orthogonal to this, one may generalize the *functor* rather than the base category. In particular, it would be interesting to understand the relation between bisimulations for *other functor pairs* than Vietoris and power set. For instance, using the fact that Stone is the Pro-completion of the category of finite sets [19], one may associate with any set functor $\mathcal{T}$ that sends finite sets to finite sets, a 'Stone companion' $\widehat{\mathcal{T}}$: Stone $\to$ Stone. The relation between $\mathcal{T}$ and $\widehat{\mathcal{T}}$ generalizes that between $\mathcal{P}$ and $\mathcal{V}$ in the sense that $\mathcal{V} = \widehat{\mathcal{P}}$. An interesting question is then whether the analog of Theorem 5.2 holds in this generality, i.e., whether the topological closure of a $\mathcal{T}$-bisimulation between $\widehat{\mathcal{T}}$-coalgebras is always a $\widehat{\mathcal{T}}$-bisimulation. For some related work, see [20, 25].

Finally, descriptive models are concrete coalgebras, that is, coalgebras for an endofunctor on a *concrete* category. Roughly speaking, these are categories in which the objects are sets with some additional structure, and all morphisms are functions between those sets. For such coalgebras, notions like 'state', 'behaviour' and 'bisimulation' make sense. As a different example, Desharnais, Edalat & Panangaden [10] study bisimulations between so-called Labelled Markov processes. These are probabilistic transition systems and can formally be modelled as coalgebras over a base category where the carriers are certain measurable spaces, and the morphisms are measurable functions. We think it would be interesting to study the notion of bisimulation for concrete coalgebras for other examples, and in more generality.

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