Duality for Instantial Neighbourhood Logic via Coalgebra

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Abstract. Instantial Neighbourhood Logic (INL) has been introduced recently as a language for neighbourhood frames where existential information can be given about what kind of worlds occur in a neighbourhood of a current world. Apart from its semantics, its proof theory and bisimulation games have also been studied. However, conspicuously absent from the treatment of INL is the notion of *descriptive* frames.
This is the gap that we are closing in this paper. We introduce descriptive frames for INL and we prove that these are dual to boolean algebras with instantial operators (BAIOs), which give the algebraic semantics of INL. Our methods for establishing this duality make essential use of coalgebra: we observe that BAIOs are algebras for a functor on the category of boolean algebras and show that this functor is dual to the double Vietoris functor (i.e. the composition of the Vietoris functor with itself), thus obtaining a dual equivalence between double Vietoris coalgebras and BAIOs. The proof of our main result is then completed by showing that double Vietoris coalgebras correspond precisely to descriptive frames. As a corollary we obtain that every extension of INL is sound and complete with respect to descriptive frames, that descriptive frames enjoy the Hennessy-Milner property, and as a result, that finite neighbourhood frames enjoy the Hennessy-Milner property.

Keywords: Duality · Modal logic · Instantial Neighbourhood Logic · Descriptive frames · Coalgebra.

1 Introduction

The concept of *duality* arises in many areas of mathematics, logic and computer science. The first duality milestone in algebraic logic was the Stone representation theorem [31], which described the categorical duality between boolean algebras and so-called Stone spaces. Subsequently, many representation and duality theorems have been established, including a representation theorem for Riesz spaces [38], Priestley duality for distributive lattices [28], and Esakia duality for (bi-)Heyting algebras [12–14, 9].

The first representation theorem in the realm of modal logic was given by Jónsson and Tarski [20]. Although *op. cit.* does not mention modal logic explicitly, it introduces relational semantics and uses the representation theorem to relate algebraic and relational semantics. The full duality between boolean algebras with operators and descriptive general frames was established by Goldblatt [16], and many other dualities ensued [19, 8, 15, 25].

These dualities are useful because they provide two, algebraic and geometric, perspectives on the same object and allow one to translate results from the algebraic language to the geometric one via (descriptive) frames, and vice versa. For example, this was a key ingredient in Sambin and Vaccaro's simplified proof of the celebrated Sahlqvist completeness theorem in modal logic [30].

A novel system of modal logic for reasoning about neighbourhood frames, called *Instantial Neighbourhood Logic* (INL), was recently introduced in [3]. The modalities in INL give existential information about what kind of worlds occur in a neighbourhood of a current world. Motivations for investigating this logic range from topology to games, and from modelling notions of evidence to belief revision. Surprisingly however, a duality result is still absent from the theory of INL, despite an abundance of recent interest in the logic [32, 39, 2, 1, 40].

This is the gap we are filling in this paper. We introduce a suitable notion of descriptive frames for INL and prove a (categorical) duality between descriptive INL-frames and boolean algebras with instantial operators (BAIOs). The latter give the algebraic semantics for INL and play the same rôle for INL that boolean algebras with operators play for normal modal logic [7, 10] and boolean algebras with monotone operators for monotone modal logic [17, 18].

Our main technical tool for establishing this duality is *coalgebra*. Coalgebras arise as the dual notion of *algebras for a functor* and have found many applications in logic and computer science. For example, they are a natural setting for dealing with non-wellfounded data structures such as streams or infinite trees, and, triggered by Moss' paper on coalgebraic logic [26], they have become a widely-used framework for describing semantics for modal logics [29, 24, 22, 21].

A key feature of the coalgebraic perspective on duality is that, in a sense, it isolates the essential part of the duality. In the case of modal logic, the traditional approach of Jónsson-Tarski duality considers modal algebras as boolean algebras with operators, and directly constructs the dual Kripke frames. From the point of view of coalgebra, this turns out to really be a duality between *functors*. Once we identify modal algebras as the algebras for a certain functor on the category of boolean algebras and recognise its dual functor as the Vietoris functor, the duality between algebras and coalgebras for these two functors, which piggy-backs Stone duality, is a trivial consequence. The same approach works well for other variations of modal logic such as monotone modal logic [18] or positive modal logic (which piggy-backs on Priestley duality) [27]. While these examples have been re-cast in this style with hindsight, we believe that this coalgebraic approach serves as a useful blueprint for establishing dualities for new logics. If a logical system and its algebras are given, and a natural candidate for dual "descriptive frames" is suggested, then proceed as follows:

1. Describe the algebras of the logic as algebras for a functor.
2. Find the dual functor T.
3. Show that T-coalgebras are equivalent to descriptive frames.

In the current paper we use this strategy to prove a duality theorem for INL. First, we observe that BAIOs are algebras for a functor on the category of boolean algebras and homomorphisms. Second, we recall that it was observed in [3] that the neighborhood semantics of INL corresponds to coalgebras for the *double covariant powerset functor*. Since Kripke frames are coalgebras for the covariant powerset functor, and modal algebras are dual to Vietoris coalgebras, this suggests that BAIOs should be dual to coalgebras for the *double Vietoris functor*. We verify that this is indeed the case. Lastly, we show that our descriptive frames are precisely the "double Vietoris coalgebras", in the sense that the categories are isomorphic. Hence, these descriptive frames are dual to BAIOs.

A rather trivial step in this proof is the representation of BAIOs as algebras for a functor I; the functor is simply read off from the defining equations of BAIOs. It was observed by Lutz Schröder (private communication) that the logic INL can be translated into the composition of standard modal logic with itself, and vice versa, in a semantics-preserving manner. This suggests that we could optionally have represented BAIOs as algebras for the functor $\mathsf{M} \circ \mathsf{M}$, where M is the functor whose algebras correspond to modal algebras. Since M is dual to the Vietoris functor, the dual equivalence of $\mathsf{M} \circ \mathsf{M}$-algebras with double Vietoris coalgebras would follow directly (in fact, as a special case of a more general result in [23]). While this alternative approach has a certain elegance to it, we have opted for a more direct representation of INL-algebras here. The main benefit is that our proof gives a quite standard canonical model construction for INL and its extensions, without a detour via a translation. This makes the duality better suited for dealing with issues like canonicity and Sahlqvist completeness, which we hope to address in future work. We obtain the equivalence of INL-algebras with algebras for $\mathsf{M} \circ \mathsf{M}$ as a corollary. Our proof also gives a representation of the double Vietoris functor where the topology is described in terms of the INL-modalities, rather than two layers of box and diamond modalities.

As a corollary of our main duality theorem we obtain that every extension of INL is sound and complete with respect to descriptive frames, that descriptive frames enjoy the Hennessy-Milner property, and as a result, that finite INL-frames enjoy the Hennessy-Milner property.

Outline. The paper is structured as follows: In Section 2 we recall the definitions of Instantial Neighbourhood Logic and its semantics, algebras and coalgebras, Stone duality, and the Vietoris functor. In Section 3 we define boolean algebras with instantial operators and descriptive INL-frames. The main results of the paper appear in Section 4, where we prove that the category of BAIOs is dually equivalent to the category of coalgebras for the double Vietoris functor, and in Section 5, where we identify coalgebra for the double Vietoris functor with descriptive INL-frames. Finally, we provide some applications of the developed theory in Section 6, and discuss possible direction for future work in Section 7.

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2 Preliminaries

2.1 Instantial neighbourhood logic

We briefly recall the language and semantics of instantial neighbourhood logic from [3]. The language $\mathcal{L}(\text{Prop})$ of instantial neighbourhood logic over some arbitrary but fixed set Prop of proposition letters is defined recursively by

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box(\varphi_1, \ldots, \varphi_n; \varphi),$$

where $p \in \text{Prop}$ and $n \in \omega$. Observe that we have a countably infinite number of modal operators: one for each $n \in \omega$. Formulas in $\mathcal{L}(\text{Prop})$ can be interpreted in neighbourhood frames, which we shall call *INL-frames*.

Definition 2.1. An *INL-frame* is a pair (X, N) comprised of a set X and a neighbourhood functor $N : X \to \mathsf{PP}X$, where P is the (covariant) powerset functor. A *neighbourhood model* is a tuple (X, N, V) where (X, N) is a neighbourhood frame and $V : \text{Prop} \to \mathsf{P}X$ is a valuation of the proposition letters.

An *INL-morphism* from (X, N) to (X', N') is a map $f : X \to X'$ such that

$$N'(f(x)) = \{f[a] \mid a \in N(x)\},$$

for all $x \in X$. Here $f[a]$ denotes the direct image of a under f. An *INL-morphisms* $(X, N, V) \to (X', N', V')$ between neighbourhood models is an INL-morphism f between the underlying frames which additionally satisfies

$$x \in V(p) \quad \text{iff} \quad f(x) \in V'(p)$$

for every $p \in \text{Prop}$.

Write $\mathbf{INL}$ and $\mathbf{INL}^{\mathbf{M}}$ for the categories of neighbourhood frames and neighbourhood models, respectively, with their corresponding notion of morphism.

The interpretation of INL formulas in a neighbourhood model (X, N, V) is defined recursively, where the classical connectives are treated in the standard manner. For the modalities, let $x \Vdash \Box(\varphi_1, \ldots, \varphi_n; \psi)$ if there is a neighbourhood $w \in N(x)$ of x such that $y \Vdash \psi$ for all $y \in w$, and for each φ_i there is $y_i \in w$ such that $y_i \Vdash \varphi_i$. We write $[\![\varphi]\!] = \{x \in X \mid x \Vdash \varphi\}$ and say that x *satisfies* φ if $x \in [\![\varphi]\!]$. Two states are called *logically equivalent* if they satisfy precisely the same formulas.

The interpretation of the modalities can be conveniently reformulated using the following notion of *witnesses*.

Definition 2.2. Let X be a set and $w, a_1, \ldots, a_n, b$ subsets of X. We say that w *witnesses* $(a_1, \ldots, a_n; b)$ if and only if $w \cap a_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$ and $w \subseteq b$. We say that w *co-witnesses* $(a_1, \ldots, a_n; b)$ if and only if $w \subseteq a_i$ for some $i \in \{1, \ldots, n\}$ or $w \cap b \neq \emptyset$.

It is straightforward to see that in a neighbourhood model (X, N, V) a state x satisfies $\Box(\varphi_1, \dots, \varphi_n; \psi)$ if there is a $w \in N(x)$ witnessing $([\![\varphi_1]\!], \dots, [\![\varphi_n]\!]; [\![\psi]\!])$.

Definition 2.3. Let (X, N) be a neighbourhood frame. Define the map $m_{\Box,n} : (\mathsf{P}X)^{n+1} \to \mathsf{P}X$ by

$$m_{\Box,n}(a_1, \dots, a_n; b) = \{x \in X \mid \exists w \in N(x) \text{ which witnesses } (a_1, \dots, a_n; b)\}.$$

When there is no danger of confusion, we suppress the subscript n from $m_{\Box,n}$.

Yet another way to view the interpretation of the modalities in a neighbourhood model is via the equality $[\![\Box(\varphi_1, \dots, \varphi_n; \psi)]\!] = m_{\Box}([\![\varphi_1]\!], \dots, [\![\varphi_n]\!], [\![\psi]\!])$.

2.2 Stone Duality

Write **BA** for the category of boolean algebras and homomorphisms, and **Stone** for the category of Stone spaces and continuous functions.

The contravariant functor $\mathsf{uf} : \mathbf{BA} \to \mathbf{Stone}$ takes a boolean algebra B to the collection $\mathsf{uf}B$ of ultrafilters of B topologized by the base $\widetilde{B} = \{(\!| b |\!) \mid b \in B\}$, where $(\!| b |\!) = \{u \in \mathsf{uf}B \mid b \in u\}$. The action of uf on a homomorphism $h : B \to B'$ in **BA** is defined by $(\mathsf{uf}h)(u') = h^{-1}(u')$. In the converse direction, the contravariant functor $\mathsf{clp} : \mathbf{Stone} \to \mathbf{BA}$ takes a Stone space to its boolean algebra of clopens and a continuous function to its inverse. The functors uf and clp constitute a dual equivalence between **BA** and **Stone**.

2.3 Algebra, Coalgebra, and the Vietoris functor

We recall the definitions of algebras, coalgebras, and the Vietoris functor.

Definition 2.4. Let F be an endofunctor on a category $\mathbf{C}$. An F-*coalgebra* is a pair (c, γ) such that $\gamma : c \to \mathsf{F}c$ is a morphism in $\mathbf{C}$. An F-*coalgebra morphism* $(c, \gamma) \to (c', \gamma')$ is a morphism $f : c \to c'$ in $\mathbf{C}$ satisfying $\gamma' \circ f = \mathsf{F}f \circ \gamma$. The collection of F-coalgebras and F-coalgebra morphisms constitutes the category $\mathbf{Coalg}(\mathsf{F})$.

The dual notion of a coalgebra is that of an algebra: an F-algebra is a morphism $\gamma : \mathsf{F}c \to c$ in $\mathbf{C}$, an F-algebra morphism $(c, \gamma) \to (c', \gamma')$ is a morphism $f : c \to c'$ in $\mathbf{C}$ such that $\gamma' \circ \mathsf{F}f = f \circ \gamma$, and they form the category $\mathbf{Alg}(\mathsf{F})$.

Coalgebras for an endofunctor on **Set** are used to describe *systems* [29], but can also be used to characterise the frame semantics for a wide variety of modal logics [24]. In particular, as noted in [3, Section 7.5], we have:

Proposition 2.5. *The category* **INL** *of INL-frames and INL-morphisms is isomorphic to* $\mathbf{Coalg}(\mathsf{PP})$.

In the realm of modal logic a well-known example of a category of algebras is the category **MA** of modal algebras: we have $\mathbf{MA} \cong \mathbf{Alg}(\mathsf{M})$, where M is an endofunctor on the category **BA** of boolean algebras and homomorphisms defined as follows:

Definition 2.6. For a boolean algebra B let $\mathsf{M}B$ be the free boolean algebra generated by the set $\{\Box a \mid a \in B\}$, modulo the relations $\Box a \wedge \Box b = \Box(a \wedge b)$ and $\Box\top = \top$. This assignment extends to an endofunctor on **BA** by defining the action of M on a homomorphism $h : B \to B'$ via $\mathsf{M}h(\Box a) = \Box h(a)$ and extending this (uniquely) to $\mathsf{M}B$.

A prime example of a category of coalgebras for an endofunctor on a category different from **Set** is the category **DGF** of descriptive general frames and appropriate morphisms. This is isomorphic to the category of coalgebras for the *Vietoris functor* V on **Stone**, the category of Stone spaces and continuous functions [23]. We recall the definition of the Vietoris functor on **Top** (originally introduced by Leopold Vietoris for compact Hausdorff spaces [37], see also [19, Section III.4] for a localic perspective), which restricts to the Vietoris functor V on **Stone**.

Definition 2.7. For a topological space $\mathbb{X}$ let $\mathsf{V}'\mathbb{X}$ be the set of compact subsets of $\mathbb{X}$ topologized by

$$\boxdot a = \{b \in \mathsf{V}'\mathbb{X} \mid b \subseteq a\}, \quad \diamondot a = \{b \in \mathsf{V}'\mathbb{X} \mid b \cap a \neq \emptyset\},$$

where a ranges over the open sets of $\mathbb{X}$. This is called the *Vietoris topology*. For a continuous function $f : \mathbb{X} \to \mathbb{X}'$ define $\mathsf{V}'f : \mathsf{V}'\mathbb{X} \to \mathsf{V}'\mathbb{X}' : c \mapsto f[c]$, i.e. $\mathsf{V}'f$ is the direct image of f. Then V' defines a functor on **Top** called the *Vietoris functor*.

It is well known that V' restricts to an endofunctor on **Stone**, which we denote by V. Moreover, if $\mathbb{X}$ is a Stone space then the topology on $\mathsf{V}\mathbb{X}$ is generated by $\boxdot a, \diamondot a$, where a ranges over the *clopen* subsets of $\mathbb{X}$. The fact that V is the (Stone) dual of M then implies

$$\mathbf{MA} \cong \mathbf{Alg}(\mathsf{M}) \equiv^{\mathrm{op}} \mathbf{Coalg}(\mathsf{V}) \cong \mathbf{DGF},$$

where we use $\cong$ to indicate an isomorphism of categories and $\equiv^{\mathrm{op}}$ for a dual equivalence. For details we refer to [23].

3 BAIOs and general frames

In this section we define boolean algebras with instantial operators (BAIOs) and general INL-frames.

A boolean algebra with instantial operators comprises of a boolean algebra B and an ω-indexed family of functions, reflecting the infinite number of modal operators in INL, and provide algebraic semantics for instantial neighbourhood logic. They play the same rôle for INL that modal algebras play for normal modal logic [10, 7] and boolean algebras with monotone operators for monotone modal logic [17, 18].

A general INL-frame is an INL-frame together with a collection of "admissible subsets" which is closed under certain operations. As usual, these admissible subsets form a BAIO. In the converse direction, we will show that every BAIO gives rise to a general INL-frame.

Definition 3.1. A *boolean algebra with instantial operators* is a pair $(B, (f_n)_{n\in\omega})$ consisting of a boolean algebra B and an ω-indexed set of functions $f_n : B^{n+1} \to B$ satisfying the following equations for all $n \in \omega$:

(B_1) $f_n(a_1, \dots, a_{n-1}, \bot; b) = \bot$;
(B_2) $f_n(a_1, \dots, a_i, a_{i+1}, \dots, a_n; b) = f_n(a_1, \dots, a_{i+1}, a_i, \dots, a_n; b)$;
(B_3) $f_n(a_1, \dots, a_n; b) \leq f_n(a_1, \dots, a_n \vee a'_n; b \vee b')$;
(B_4) $f_n(a_1, \dots, a_n; b) \leq f_n(a_1, \dots, a_n \wedge b; b)$;
(B_5) $f_n(a_1, \dots, a_n; b) \leq f_{n+1}(a_1, \dots, a_n, c; b) \vee f_n(a_1, \dots, a_n; b \wedge \neg c)$;
(B_6) $f_{n+1}(a_1, \dots, a_{n+1}; b) \leq f_n(a_1, \dots, a_n; b)$;
(B_7) $f_n(a_1, \dots, a_n; b) \leq f_{n+1}(a_1, \dots, a_n, a_n; b)$.

A morphism between BAIOs $(B, (f_n)_{n\in\omega})$ and $(B', (f'_n)_{n\in\omega})$ is a boolean algebra homomorphism $h : B \to B'$ which satisfies

$$h(f_n(a_1, \dots, a_n; b)) = f'_n(ha_1, \dots, ha_n; hb)$$

for all $a_i, b \in B$ and $n \in \omega$. The collection of BAIOs and BAIO morphisms forms a category (a variety of algebras, in fact) denoted by **BAIO**.

Every INL-frame (X, N) gives rise to a BAIO, namely its complex algebra.

Example 3.2. Let (X, N) be an INL-frame. Let $\mathsf{P}X$ be the powerset of X viewed as a boolean algebra and define $f_n(a_1, \dots, a_n; b) = m_\Box(a_1, \dots, a_n; b)$. Then it is easy to verify that $(\mathsf{P}X, (f_n)_{n\in\omega})$ is a BAIO. This is called the *complex algebra* of (X, N).

Example 3.3. Recall that Prop is an arbitrary but fixed set of proposition letters and let $\mathcal{L} = \mathcal{L}(\text{Prop})$ be the collection of instantial formulas as defined in Subsection 2.1. Write $\varphi \equiv \psi$ if two formulas are provably equivalent in the axiomatization given in [3, Section 4] and write $[\varphi]$ for the equivalence class of φ under $\equiv$. Then $\mathcal{L}/\!\equiv$ is a BAIO, where $f_n([\varphi_1], \dots, [\varphi_n]; [\psi])$ is defined to be $[\Box(\varphi_1, \dots, \varphi_n; \psi)]$. This is of course the free BAIO generated by Prop, and is known as the Lindenbaum-Tarski algebra.

Towards a duality theorem for BAIOs, we define general INL-frames. These are INL-frames together with a subalgebra of their complex algebra.

Definition 3.4. A *general INL-frame* is a triple (X, N, A) such that (X, N) is an INL-frame and $A \subseteq \mathsf{P}X$ is a collection of admissible sets that is closed under boolean operations and the operation $m_\Box : (\mathsf{P}X)^{n+1} \to \mathsf{P}X$ (see Definition 2.3).

A *general INL-morphism* from (X, N, A) to $(X', N'A')$ is an INL-morphism $f : (X, N) \to (X', N')$ satisfying $f^{-1}(a') \in A$ for all $a' \in A'$. Write **G-INL** for the category of general INL-frames and general INL-morphisms.

Since every algebra is a subalgebra of itself, every INL-frame can be seen as a general INL-frame:

Example 3.5. If (X, N) is an INL-frame, then setting $A = \mathsf{P}X$ yields a general INL-frame.

Example 3.6. For a BAIO $(B, (f_n)_{n\in\omega})$, let $\mathsf{uf}B$ be the collection of ultrafilters of B and let $\widetilde{B} = \{(\!| a |\!) \mid a \in B\}$, where $(\!| a |\!) = \{u \in \mathsf{uf}B \mid a \in u\}$. Define a neighbourhood function N for $\mathsf{uf}B$ via

$$N(u) = \{d \subseteq \mathsf{uf}B \mid f_n(a_1, \dots, a_n; b) \in u \text{ whenever } d \text{ witnesses } ((\!| a_1 |\!), \dots, (\!| a_n |\!); (\!| b |\!))\}.$$

Then $(\mathsf{uf}B, N, \widetilde{B})$ is a general INL-frame.

In the converse direction of Example 3.6 we have the functor $\mathsf{F} : \mathbf{G\text{-}INL} \to \mathbf{BAIO}$ which sends a general INL-frame (X, N, A) to $(A, (m_{\Box,n})_{n\in\omega})$.

One can now ask whether **G-INL** can be restricted to a category of *descriptive* INL-frames such that the restriction of F to these descriptive frames gives rise to a dual equivalence with **BAIO**. This turns out to be the case for the following definition of descriptive INL-frames:

Definition 3.7. A general INL-frame (X, N, A) is called:

- *differentiated* if for any two distinct points $x, y \in X$ there is $a \in A$ such that $x \in a$ and $y \notin a$;
- *compact* if $\bigcap A' \neq \emptyset$ for any subset A' of A with the finite intersection property;
- *crowded* if for all $x \in X$ and $d \subseteq X$ such that $d \notin N(x)$ we can find $a_1, \dots, a_n, b$ such that d witnesses $(a_1, \dots, a_n; b)$ while no $d' \in N(x)$ witnesses $(a_1, \dots, a_n; b)$.

A *descriptive INL-frame* is a general INL-frame that is differentiated, compact and crowded. Denote by **D-INL** the full subcategory of **G-INL** whose objects are descriptive INL-frames.

The notion of crowdedness is the INL analogue of the tightness condition for normal modal logic [7, Definition 5.65]. Intuitively, it states that $N(x)$ is, in a sense, determined by the admissible subsets. In passing, we make the following observation, the proof of which is straightforward.

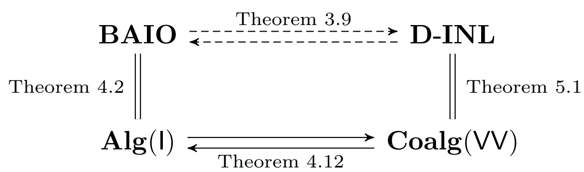
Proposition 3.8. *Let (X, N) be an INL-frame and suppose X is finite. Then $A = \mathsf{P}X$ is the unique set of admissible sets making (X, N, A) a descriptive INL-frame.*

We aim to prove the following duality result:

Theorem 3.9. *We have a dual equivalence*

$$\mathbf{D\text{-}INL} \equiv^{\mathrm{op}} \mathbf{BAIO}.$$

In order to prove this, we adhere to the strategy suggested in the introduction. First we identify **BAIO** with the category of algebras for some functor I on **BA**, then we determine the (Stone) dual of I, and finally, we show that descriptive INL-frames are precisely coalgebras for this dual functor. In a diagram:



(1)

Concretely, the dual equivalence from Theorem 3.9 will be given by the construction from Example 3.6 and the subsequent paragraph, see Remark 5.7 below.

4 Duality

We show that BAIOs are algebras for the functor $\mathsf{I} : \mathbf{BA} \to \mathbf{BA}$, and that I is the (Stone) dual of the double Vietoris functor VV on **Stone**. As a consequence we obtain an algebra/coalgebra duality in Theorem 4.12.

Definition 4.1. Let B be a boolean algebra. Abbreviate $(\boldsymbol{a}; b) = (a_1, \dots, a_n; b)$ for an $(n+1)$-tuple of elements of B. Let $\mathsf{I}B$ be the boolean algebra generated by $\Box(a_1, \dots, a_n; b)$, where $n \in \omega$ and $a_i, b \in B$, subject to the relations

- (I_1) $\Box(\boldsymbol{a}, \bot; b) = \bot$;
- (I_2) $\Box(a_1, \dots, a_i, a_{i+1}, \dots, a_n; b) = \Box(a_1, \dots, a_{i+1}, a_i, \dots, a_n; b)$;
- (I_3) $\Box(\boldsymbol{a}, c; b) \leq \Box(\boldsymbol{a}, c \vee c', b \vee b')$;
- (I_4) $\Box(\boldsymbol{a}, c; b) \leq \Box(\boldsymbol{a}, c \wedge b; b)$;
- (I_5) $\Box(\boldsymbol{a}; b) \leq \Box(\boldsymbol{a}, c; b) \vee \Box(\boldsymbol{a}; b \wedge \neg c)$;
- (I_6) $\Box(\boldsymbol{a}, c; b) \leq \Box(\boldsymbol{a}; b)$;
- (I_7) $\Box(a_1, \dots, a_n; b) \leq \Box(a_1, \dots, a_n, a_n; b)$.

For a homomorphism $f : B \to B'$ define $\mathsf{I}f : \mathsf{I}B \to \mathsf{I}B'$ on generators by

$$(\mathsf{I}f)(\Box(a_1, \dots, a_n; b)) = \Box(f(a_1), \dots, f(a_n); f(b)).$$

The assignment I determines an endofunctor on **BA**, called the *instantial functor*.

Item I_2 of Definition 4.1 allows us to put the a_i in any desired order. In I_4 equality holds because of I_3 and in I_7 equality hold because of I_6. The proof of the following theorem is standard.

Theorem 4.2. **BAIO** $=$ **Alg**(I).

As stated in the introduction, it seems reasonable to expect that the double Vietoris functor VV is the dual of I under the dual equivalence between boolean algebras and Stone spaces. We prove that this is indeed the case. More concretely, we give a natural isomorphism

$$\mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp} \to \mathsf{VV}. \tag{2}$$

We first work towards an isomorphism $\mathsf{uf} \circ \mathsf{l} \circ \mathsf{clp}\mathbb{X} \cong \mathsf{VV}\mathbb{X}$, where $\mathbb{X}$ is a Stone space. Lemma 4.11 then states that the collection of these isomorphisms is in fact natural. This ultimately proves the duality between (the categories of) l-algebras and VV-coalgebras (Theorem 4.12).

We commence by giving an alternative subbase for the double Vietoris topology, which is tailored to our specific needs.

Proposition 4.3. *Let $\mathbb{X}$ be a Stone space. The topology on $\mathsf{VV}\mathbb{X}$ is generated by the clopen subbase*

$$\boxbar(a_1, \ldots, a_n; b) = \{W \in \mathsf{VV}\mathbb{X} \mid \exists w \in W \text{ s.t. } w \text{ witnesses } (a_1, \ldots, a_n; b)\},$$
$$\Diamondbar(a_1, \ldots, a_n; b) = \{W \in \mathsf{VV}\mathbb{X} \mid \forall w \in W \; w \text{ co-witnesses } (a_1, \ldots, a_n; b)\},$$

where the a_i, b range over the clopen subsets of $\mathbb{X}$.

Proof. The given sets are clopen in $\mathsf{VV}\mathbb{X}$, because

$$\boxbar(a_1, \ldots, a_n; b) = \Diamonddot(\Diamonddot a_1 \cap \cdots \cap \Diamonddot a_n \cap \boxdot b)$$

and

$$\Diamondbar(a_1, \ldots, a_n; b) = \boxdot(\boxdot a_1 \cup \cdots \cup \boxdot a_n \cup \Diamonddot b).$$

In order to show that they are a subbase for the topology on $\mathsf{VV}\mathbb{X}$, we must show that $\boxdot A$ and $\Diamonddot A$ are boolean combinations of clopens of the form $\boxbar(\boldsymbol{a}; b)$ and $\Diamondbar(\boldsymbol{a}; b)$, where A is clopen in $\mathsf{V}\mathbb{X}$. Note that A can be written as the finite intersection of finite unions of clopens in $\mathsf{V}\mathbb{X}$ of the form $\boxdot a$, $\Diamonddot a$. Moreover, we may assume that there is a single diamond in each finite union because diamonds distribute over unions. So we may write $A = \bigcap_{i=1}^{n} \big(\boxdot\, a_1 \cup \cdots \cup \boxdot a_{m_i} \cup \Diamonddot b_i \big)$. This implies

$$\boxdot A = \bigcap_{i=1}^{n} \boxdot \Big(\boxdot\, a_1 \cup \cdots \cup \boxdot a_{m_i} \cup \Diamonddot b_i \Big) = \bigcap_{i=1}^{n} \Diamondbar(a_1, \ldots, a_{m_i}; b_i).$$

Similarly, writing A as a finite union of finite intersections, $\Diamonddot A$ can be expressed as a finite union of clopens of the form $\boxbar(a_1, \ldots, a_n; b)$. □

We note that the statement of Proposition 4.3 holds for any topological space $\mathbb{X}$ if we require the a_i and b to be range over the *open* subsets of $\mathbb{X}$. The proof of this is slightly more involved because it crucially uses compactness of the elements of $\mathsf{V}\mathbb{X}$.

Remark 4.4. Recall that ultrafilters correspond bijectively to homomorphisms into 2, the two-element boolean algebra. For an ultrafilter u in B let $p : B \to 2$ be the corresponding homomorphism. Then $p(b) = \top$ iff $b \in u$. We will use these two perspectives interchangeably.

We now define a map $\xi : \mathsf{uf} \circ \mathsf{l} \circ \mathsf{clp}\mathbb{X} \to \mathsf{VV}\mathbb{X}$, where $\mathbb{X}$ is a Stone space. These will form the components of the intended natural transformation from

(2). Although such a map can be defined for any Stone space $\mathbb{X}$, and therefore the definition actually yields a **Stone**-indexed collection of maps $(\xi_{\mathbb{X}})_{\mathbb{X}\in\mathbf{Stone}}$, we shall refrain from writing this subscript until Theorem 4.11, where we prove that this collection is a natural transformation.

Intuitively, to an ultrafilter p we want to attach a closed subset W_p of $\mathsf{V}\mathbb{X}$ which satisfies $W_p \in \boxplus(a_1,\dots,a_n;b)$ if and only if $p(\Box(a_1,\dots,a_n;b)) = \top$. In our definition, we guarantee the implication from left to right by "killing the witnesses": If $p(\Box(a_1,\dots,a_n;b)) = \bot$, then we make sure that none of the witnesses of $(a_1,\dots,a_n;b)$ is in W_p. In other words, we stipulate that W_p be disjoint from $\Diamond a_1 \cap \dots \cap \Diamond a_n \cap \boxdot b$.

Definition 4.5. For a Stone space $\mathbb{X}$, define $\xi : \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X} \to \mathsf{V}\mathsf{V}\mathbb{X}$ by sending an ultrafilter p to

$$W_p = \mathsf{V}\mathbb{X} \setminus \bigcup\{\Diamond a_1 \cap \dots \cap \Diamond a_n \cap \boxdot b \mid p(\Box(a_1,\dots,a_n;b)) = \bot\}.$$

Since W_p is the complement of a union of clopen subsets of $\mathsf{V}\mathbb{X}$, it is closed in $\mathsf{V}\mathbb{X}$, hence an element of $\mathsf{V}\mathsf{V}\mathbb{X}$. Therefore ξ is well defined. For the converse direction we need the following definition.

Definition 4.6. For a Stone space $\mathbb{X}$, define

$$\theta : \mathsf{V}\mathsf{V}\mathbb{X} \to \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X} : W \mapsto p_W,$$

where $p_W : \mathsf{I}(\mathsf{clp}\mathbb{X}) \to 2$ is given on generators by

$$p_W : \mathsf{I} \circ \mathsf{clp}\mathbb{X} \to 2 : \Box(\boldsymbol{a};b) \mapsto \begin{cases} \top & \text{if } W \in \boxplus(\boldsymbol{a};b) \\ \bot & \text{otherwise} \end{cases}$$

Lemma 4.7. *The assignment θ is well defined.*

Proof. In order to show that θ is well defined, we need to show that p_W is an ultrafilter, that is, a boolean algebra homomorphism $\mathsf{I} \circ \mathsf{clp}\mathbb{X} \to 2$. Since I is defined by generators and relations it suffices to show that the images of the generators under p_W satisfy the relations I_1 through I_7. We leave this straightforward verification to the reader. $\Box$

The following lemma provides the key ingredient for proving that ξ and θ are continuous and inverses of each other.

Lemma 4.8. *Let $\mathbb{X}$ be a Stone space. We have*

$$W_p \in \boxplus(\boldsymbol{a};b) \quad \textit{if and only if} \quad p(\Box(\boldsymbol{a};b)) = \top.$$

Proof. If $p(\Box(\boldsymbol{a};b)) = \bot$, then by construction $W_p \notin \boxplus(\boldsymbol{a};b)$. So suppose $W_p \notin \boxplus(\boldsymbol{a},b)$. Then for every witness w of $(\boldsymbol{a},b)$ there exists $(\boldsymbol{c}_w,d_w)$ which is witnessed by w and is such that $p(\Box(\boldsymbol{c}_w,d_w)) = \bot$.

The collection of witnesses is the set $A = \Diamond a_1 \cap \dots \cap \Diamond a_n \cap \boxdot b$. This is a closed set of $\mathsf{V}\mathbb{X}$, and it is covered by the collection

$$\{\Diamond c_{w,1} \cap \dots \cap \Diamond c_{w,m_w} \cap \boxdot d_w \mid w \in A\}.$$

Clearly this set is an open covering of A, so by compactness of $\mathsf{V}\mathbb{X}$ there must be a finite subcover of A. That is

$$A \subseteq \bigcup_{w \in A'} (\diamondsuit c_{w,1} \cap \dots \cap \diamondsuit c_{w,m_w} \cap \boxdot d_w),$$

where A' is some finite subset of A serving as an index. Now it follows from Lemma 4.9 below that $p(\Box(\boldsymbol{a}; b)) = \bot$. □

The following technical result is motivated by the proof of Lemma 4.8.

Lemma 4.9. *Let $\mathbb{X}$ be a Stone space, $a_i, b, c_j, d \in \mathsf{clp}\mathbb{X}$. Suppose $A = \diamondsuit a_1 \cap \dots \cap \diamondsuit a_n \cap \boxdot b$ is covered by the finite set $\{C_i = \diamondsuit c_{i,1} \cap \dots \cap \diamondsuit c_{i,n_i} \cap \boxdot d_i \mid 1 \leq i \leq m\}$. Suppose $p : \mathsf{I} \circ \mathsf{clp}\mathbb{X} \to 2$ is a point and $p(\Box(c_{i,1}, \dots, c_{i,n_i}; d_i)) = \bot$ for all C_i in the given cover. Then $p(\Box(a_1, \dots, a_n; b)) = \bot$.*

Proof. If $A = \emptyset$ the lemma is trivial, so henceforth we shall assume $A \neq \emptyset$.

Part 1. Since (clearly) $b \in A$ there must be a C_i containing b. Call this $C = \diamondsuit c_1 \cap \dots \cap \diamondsuit c_k \cap \boxdot d$. Now consider $b_j := b \setminus c_j$. This is not in C. If it is not in A, then we must have $c_j \supseteq a_i$ for some i, because clearly $b_j \subseteq b$. If it *is* in A, then it must be in another element of the cover, say, $C_j = \diamondsuit c_{j1} \cap \dots \cap \diamondsuit c_{jn_j} \cap \boxdot d_j$. Observe that $b_j = b \setminus c_j \subseteq d_j$.

Next, consider $b_{j,k} := b \setminus (c_j \cup c_{jk})$, where $1 \leq k \leq n_j$. If this is not in A, then we must have $a_i \subseteq c_j \cup c_{jk}$ for some i. If it is in A, then it must be in one of the elements of the cover, say, $C_{jk} = \diamondsuit c_{jk,1} \cap \dots \cap \diamondsuit c_{jk,n_{jk}} \cap \boxdot d_{jk}$. Note that $b_{j,k}$ is not in C and not in C_j by construction. Again, observe that $b_{j,k} \subseteq d_{jk}$. Continuing this way gives a tree, see the diagram below for intuition.

Each $b_{j_1,j_2,\dots,j_k} = b \setminus (c_{j_1} \cup c_{j_1 j_2} \cup \dots \cup c_{j_1 j_2 \cdots j_k})$ is in none of the preceding cover elements. Since we started with a finite cover, this process must terminate, i.e. the branches of our tree must be finite. That is, at some point $b_{j_1,j_2,\dots,j_k}$ is not in A, and since clearly $b_{j_1,j_2,\dots,j_k} \subseteq b$, it must be the case that $b_{j_1,j_2,\dots,j_k} \notin \diamondsuit a_i$ for some a_i, i.e. we must have $a_i \subseteq (c_{j_1} \cup c_{j_1 j_2} \cup \dots \cup c_{j_1 j_2 \cdots j_k})$.

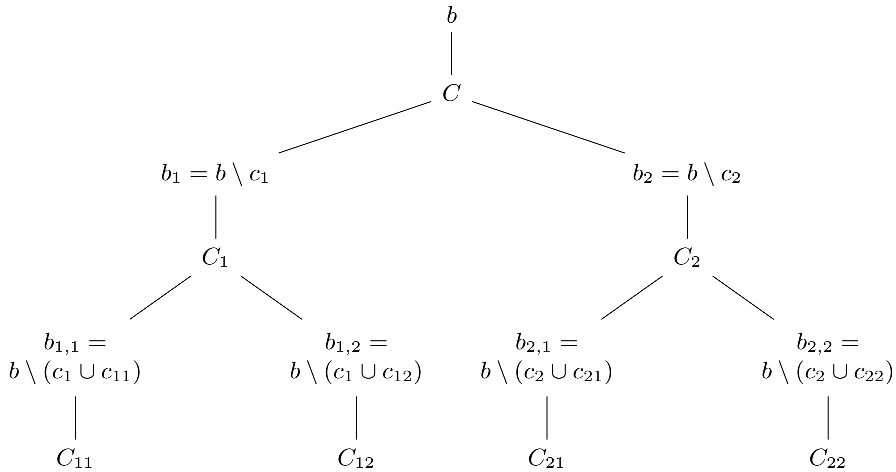
Part 2. Now we have set ourselves up for the proof of the proposition. We will use rule I_5 finitely many times. The first step is:

$$\Box(\boldsymbol{a}; b) \leq \Box(\boldsymbol{a}, c_1; b) \vee \Box(\boldsymbol{a}; b \setminus c_1). \tag{3}$$

We continue using I_5 as follows: given an element of the form $\Box(\boldsymbol{a}, \boldsymbol{c}; b_{j_1,j_2,\dots,j_k})$ we verify what is the lowest entry ℓ such that $c_{j_1 j_2 \cdots j_k \ell}$ is not in $\boldsymbol{c}$, and apply I_5 using this. It can be seen that (3) above is also obtained in this way. Thus the first two iterations are:

$$\begin{aligned} \Box(\boldsymbol{a}; b) &\leq \Box(\boldsymbol{a}, c_1; b) && \vee \Box(\boldsymbol{a}; b \setminus c_1) \\ &\leq \Box(\boldsymbol{a}, c_1, c_2; b) \vee \Box(\boldsymbol{a}, c_1; b_2) && \vee \Box(\boldsymbol{a}, c_{11}; b_1) \vee \Box(\boldsymbol{a}; b_{1,11}) \end{aligned}$$

For an entry, we cannot proceed if either all $c_{j_1 \cdots j_k, \ell}$ from a $C_{j_1 \cdots j_k}$ in the tree already occur in $\boldsymbol{c}$, or if the thing we subtract from b, i.e. $c_{j_1} \cup c_{j_1 j_2} \cup \dots \cup c_{j_1 j_2 \cdots j_k}$



contains one of the a_i (for then $c_{j_1 j_2 \cdots j_k, \ell}$ is not defined). In the first case, we have

$$\begin{aligned}\Box(\boldsymbol{a}, \boldsymbol{c}; b \setminus (c_{j_1} \cup c_{j_1 j_2} \cup \cdots \cup c_{j_1 j_2 \cdots j_k})) \\ \leq \Box(c_{j_1 \cdots j_k, 1}, \ldots, c_{j_1 \cdots j_k, n_{j_1 \cdots j_k}}; d_{j_1 \cdots j_k}).\end{aligned} \tag{4}$$

The inequality follows from using I_6 a lot, and applying I_3 to the fact that $b_{j_1, j_2, \ldots, j_k} \subseteq d_{j_1 \cdots j_k}$. As the right hand side of (4) is one of the elements in the cover, we get

$$\begin{aligned}p(\Box(\boldsymbol{a}, \boldsymbol{c}; b \setminus (c_{j_1} \cup c_{j_1 j_2} \cup \cdots \cup c_{j_1 j_2 \cdots j_k}))) \\ \leq p(\Box(c_{j_1 \cdots j_k, 1}, \ldots, c_{j_1 \cdots j_k, n_{j_1 \cdots j_k}}; d_{j_1 \cdots j_k})) = \bot.\end{aligned}$$

In the second case, we get $\bot$ because the intersection of one of the a_i and $b_{j_1 j_2 \cdots j_k}$ is empty, and we use I_4 and I_1.

Since this procedure is finite, this yields $\Box(\boldsymbol{a}; b) \leq \bot$, as desired. □

As a corollary of Lemma 4.8 we obtain the following lemma.

Lemma 4.10. *The maps θ and ξ are continuous and each others inverses. Hence ξ is a homeomorphism.*

Proof. We first prove continuity. The open subsets of $\mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X}$ are generated by $(\!| \Box(\boldsymbol{a}; b) |\!) = \{p \mid p(\Box(\boldsymbol{a}; b)) = \top\}$, where $(\boldsymbol{a}; b) = (a_1, \ldots, a_n; b)$ is an $(n+1)$-tuple of clopen subsets of $\mathbb{X}$. We have

$$\begin{aligned}\theta^{-1}(\!| \Box(\boldsymbol{a}; b) |\!) &= \theta^{-1}(\{p \mid p(\Box(\boldsymbol{a}; b)) = \top\}) \\ &= \{W \in \mathsf{VV}\mathbb{X} \mid W \in \boxminus(\boldsymbol{a}; b)\} = \boxminus(\boldsymbol{a}; b),\end{aligned}$$

which is clopen in $\mathsf{VV}\mathbb{X}$. Similarly $\xi^{-1}(\boxminus(\boldsymbol{a}; b)) = \{p \mid W_p \in \boxminus(\boldsymbol{a}; b)\} = (\!| \Box(\boldsymbol{a}; b) |\!)$.

We now show prove $\xi \circ \theta$ and $\theta \circ \xi$ are identities. For the former, observe that Lemma 4.8 implies $p(\Box(\boldsymbol{a}; b)) = \top$ iff $W_p \in \boxbar(\boldsymbol{a}; b)$ iff $p_{W_p}(\Box(\boldsymbol{a}; b)) = \top$. So p and p_W coincide on the generators of $\mathsf{I}(\mathsf{clp}\mathbb{X})$, therefore $p = p_{W_p}$ and hence $\theta \circ \xi = \mathrm{id}_{\mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X}}$.

To see that $\xi \circ \theta = \mathrm{id}_{\mathsf{VV}\mathbb{X}}$, note that it suffices to show that W and W_{p_W} are in the same generating opens of the topology. Since the diamond is dual to the box it suffices to show that $W \notin \boxbar(\boldsymbol{a}; b)$ iff $W_{p_W} \notin \boxbar(\boldsymbol{a}; b)$. This is easy: $W \notin \boxbar(\boldsymbol{a}; b)$ iff $p_W(\Box(\boldsymbol{a}, b)) = \bot$ iff $W_{p_W} \notin \boxbar(\boldsymbol{a}; b)$. The first "iff" holds by definition of θ, the second by Lemma 4.8. □

Each Stone space $\mathbb{X}$ gives rise to a homeomorphism ξ, so we get a transformation $(\xi_{\mathbb{X}})_{\mathbb{X} \in \mathbf{Stone}} : \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp} \to \mathsf{VV}$. This transformation is in fact natural:

Lemma 4.11. *The collection* $\xi = (\xi_{\mathbb{X}})_{\mathbb{X} \in \mathbf{Stone}} : \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp} \to \mathsf{VV}$ *is a natural isomorphism.*

Proof. We have already seen that $\xi_{\mathbb{X}}$ is a homeomorphism for every Stone space $\mathbb{X}$ (i.e. an isomorphism in **Stone**), so it is left to show naturality. That is, where $f : \mathbb{X} \to \mathbb{X}'$ is a continuous function we need to show that

$$\begin{array}{ccc} \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X} & \xrightarrow{\ \xi_{\mathbb{X}}\ } & \mathsf{VV}\mathbb{X} \\ {\scriptstyle \mathsf{uf}\circ\mathsf{I}\circ\mathsf{clp}f}\downarrow & & \downarrow{\scriptstyle \mathsf{VV}f} \\ \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X}' & \xrightarrow{\ \xi_{\mathbb{X}}\ } & \mathsf{VV}\mathbb{X}' \end{array}$$

commutes. Since elements of a Stone space are uniquely determined by the clopen sets in which they are contained, it suffices to show that for all $p \in \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}\mathbb{X}$ and $a_i, b \in \mathsf{clp}\mathbb{X}$ we have

$$\mathsf{VV}f(\xi_{\mathbb{X}}(p)) \in \boxbar(\boldsymbol{a}; b) \quad \text{iff} \quad \xi_{\mathbb{X}'}(\mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}f(p)) \in \boxbar(\boldsymbol{a}; b).$$

This follows from a straightforward computation:

$$\begin{array}{rll} \mathsf{VV}f(\xi_{\mathbb{X}}(p)) \in \boxbar(\boldsymbol{a}; b) & \text{iff} & \xi_{\mathbb{X}}(p) \in (\mathsf{VV}f)^{-1}(\boxbar(\boldsymbol{a}; b)) \\ & \text{iff} & \xi_{\mathbb{X}}(p) \in \boxbar(f^{-1}\boldsymbol{a}; f^{-1}b) \\ & \text{iff} & p(\Box(f^{-1}\boldsymbol{a}; f^{-1}b)) = \top \\ & \text{iff} & p(\mathsf{I} \circ \mathsf{clp}f(\Box(\boldsymbol{a}; b))) = \top \\ & \text{iff} & \mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}f(p)(\Box(\boldsymbol{a}; b)) = \top \\ & \text{iff} & \xi_{\mathbb{X}'}(\mathsf{uf} \circ \mathsf{I} \circ \mathsf{clp}f(p)) \in \boxbar(\boldsymbol{a}; b) \end{array}$$

We conclude that $(\xi_{\mathbb{X}})_{\mathbb{X} \in \mathbf{Stone}}$ is indeed a natural isomorphism. □

As an immediate corollary we obtain the main theorem of this section.

Theorem 4.12. *We have a dual equivalence*

$$\mathbf{Alg}(\mathsf{I}) \equiv^{\mathrm{op}} \mathbf{Coalg}(\mathsf{VV}).$$

As V is dual to the functor M on boolean algebras, we obtain the following corollary which confirms the intuition that INL is "modal logic taken twice".

Corollary 4.13. *The functor* I *is naturally isomorphic to the composition* $\mathsf{M} \circ \mathsf{M}$.

5 Descriptive frames as coalgebras

We show that the descriptive frames from Definition 3.7 are precisely coalgebras for the double Vietoris functor VV.

Theorem 5.1. *We have*

$$\mathbf{D\text{-}INL} \cong \mathbf{Coalg}(\mathsf{VV}).$$

A helpful tool in the proof of Theorem 5.1 is the notion of the *largest representative* of a set $d \subseteq X$ in a general INL-frame (X, N, A).

Definition 5.2. Let (X, N, A) be a general INL-frame and $d \subseteq X$. Then we define the *largest representative* of d to be

$$\overline{d} = \bigcap \{a \in A \mid d \subseteq a\}.$$

This is of course the topological closure of d in the topology on X generated by the clopen base A. It enjoys the following useful properties.

Lemma 5.3. *Let (X, N, A) be a general INL-frame, $d \subseteq X$ and $a_1, \ldots, a_n, b \in A$. Then d witnesses $(a_1, \ldots, a_n; b)$ if and only if $\overline{d}$ does.*

Proof. We show that $d \cap a_i \neq \emptyset$ iff $\overline{d} \cap a_i \neq \emptyset$ and $d \subseteq b$ iff $\overline{d} \subseteq b$. It is easy to see that this proves the lemma. Suppose $d \cap a_i \neq \emptyset$. Since $d \subseteq \overline{d}$ we also have $\overline{d} \cap a_i \neq \emptyset$. Conversely, if $d \cap a_i = \emptyset$ then $d \subseteq X \setminus a_i$ and since the latter is in A we have $\overline{d} \subseteq X \setminus a_i$. This implies $\overline{d} \cap a_i = \emptyset$. Next suppose $d \subseteq b$, then by definition $\overline{d} \subseteq b$, because $b \in A$. Conversely, if $\overline{d} \subseteq b$ we have $d \subseteq \overline{d} \subseteq b$. □

Lemma 5.4. *Let (X, N, A) be a descriptive INL-frame, $d \subseteq X$ and $x \in X$. Then $d \in N(x)$ if and only if $\overline{d} \in N(x)$.*

Proof. This follows directly from the proof of Lemma 5.3. □

The following two lemmas describe the object part of the isomorphism from Theorem 5.1.

Lemma 5.5. *Let $(\mathbb{X}, \gamma)$ be a VV-coalgebra. Write X for the space underlying $\mathbb{X}$ and let $N_\gamma(x) = \{d \subseteq X \mid \overline{d} \in \gamma(x)\}$. Then $(X, N_\gamma, \mathsf{clp}\mathbb{X})$ is a descriptive INL frame.*

Proof. We know that $\mathsf{clp}X$ is closed under boolean operations and it follows from continuity of γ that $\mathsf{clp}X$ is closed under $m_\Box$. Furthermore, $(X, N_\gamma, \mathsf{clp}\mathbb{X})$ is differentiated because $\mathbb{X}$ is Hausdorff and compact because $\mathbb{X}$ is compact.

Lastly, we show that it is crowded. Suppose $c \notin N_\gamma(x)$. Without loss of generality we may assume c to be closed, hence an element of $\mathsf{V}\mathbb{X}$, because we know from Lemma 5.3 that c and $\overline{c}$ witness precisely the same tuples. It follows from the definition of N_γ that $c \notin \gamma(x)$. Since $\gamma(x)$ is a closed subset of $\mathsf{V}\mathbb{X}$, there must be a basic clopen $\diamondsuit a_1 \cap \cdots \cap \diamondsuit a_n \cap \boxdot b$ containing c and disjoint from $\gamma(x)$. Therefore c witnesses $(a_1, \ldots, a_n; b)$ while none of the elements in $\gamma(x)$ witness $(a_1, \ldots, a_n; b)$. It then follows from the definition of N_γ and Lemma 5.3 that none of the $d \in N_\gamma(x)$ witness $(a_1, \ldots, a_n; b)$. Therefore $(X, N_\gamma, \mathsf{clp}\mathbb{X})$ is crowded. □

Lemma 5.6. *Let (X, N, A) be a descriptive INL-frame, write $\mathbb{X}$ for the set X topologized by the clopen subbase A and let $\gamma_N : \mathbb{X} \to \mathsf{VV}\mathbb{X} : x \mapsto \{c \in \mathsf{V}\mathbb{X} \mid c \in N(x)\}$. Then $(\mathbb{X}, \gamma_N)$ is a* VV*-coalgebra.*

Proof. The topological space $\mathbb{X}$ is zero-dimensional because A is closed under complementation (hence is a *clopen* base). Moreover, $\mathbb{X}$ is compact Hausdorff because (X, N, A) is compact and differentiated, so $\mathbb{X}$ is a Stone space.

In order to show that γ_N is well defined, we need to show that $\gamma_N(x)$ is a closed subset of $\mathsf{V}\mathbb{X}$ for every $x \in \mathbb{X}$. Suppose $c \in \mathsf{V}\mathbb{X}$ and $c \notin \gamma_N(x)$. Then $c \notin N(x)$, and because (X, N, A) is crowded we can find $(a_1, \dots, a_n; b)$ which is witnessed by c but by none of the elements in $N(x)$. This implies $c \in \Diamond a_1 \cap \dots \cap \Diamond a_n \cap \boxdot b$ and $\Diamond a_1 \cap \dots \cap \Diamond a_n \cap \boxdot b$ is disjoint from $\gamma_N(x)$. Thus we have found an open neighbourhood of c disjoint from $\gamma_N(x)$ so $\gamma_N(x)$ is closed in $\mathsf{V}\mathbb{X}$.

For continuity of γ_N, it suffices to show that $\gamma_N^{-1}(\boxplus(a_1, \dots, a_n; b))$ is clopen in $\mathbb{X}$ for all $a_1, \dots, a_n, b \in A$. This is a consequence of the fact that A is closed under $m_\Box$, because

$$\gamma_N^{-1}(\boxplus(a_1, \dots, a_n; b)) = m_\Box(a_1, \dots, a_n; b).$$

We conclude that $(\mathbb{X}, \gamma_N)$ is a VV-coalgebra. □

We proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1. First we verify that the assignments from Lemmas 5.5 and 5.6 define a bijection between descriptive INL-frames and VV-coalgebras. Let $(\mathbb{X}, \gamma)$ be a VV-coalgebra. Lemma 5.5 assigns to this the descriptive INL-frame $(X, N_\gamma, \mathsf{clp}\mathbb{X})$. We know that the topology on X generated by $\mathsf{clp}\mathbb{X}$ yields the topological space $\mathbb{X}$, so applying Lemma 5.6 to $(X, N_\gamma, \mathsf{clp}\mathbb{X})$ yields the VV-coalgebra $(\mathbb{X}, \gamma_{N_\gamma})$. Furthermore, for a closed set $c \in \mathsf{V}\mathbb{X}$ we have $c \in \gamma_{N_\gamma}(x)$ iff $c \in N_\gamma(x)$ iff $c \in \gamma(x)$, hence $\gamma = \gamma_{N_\gamma}$ and $(\mathbb{X}, \gamma) = (\mathbb{X}, \gamma_{N_\gamma})$.

Conversely, suppose given a descriptive INL-frame (X, N, A). Write τ_A for the topology on X generated by the (clopen) base A and let $\mathbb{X} = (X, \tau_A)$. Then Lemma 5.6 sends (X, N, A) to $(\mathbb{X}, \gamma_N)$, which is in turn send to $(X, N_{\gamma_N}, \mathsf{clp}\mathbb{X})$ by Lemma 5.5. We know that the clopen sets of τ_A are precisely the sets in A, so $\mathsf{clp}\mathbb{X} = A$. Comparing the neighbourhood functions gives

$$\begin{aligned}
d \in N(x) \quad &\text{iff} \quad \overline{d} \in N(x) && \text{(Lemma 5.4)} \\
&\text{iff} \quad \overline{d} \in \gamma_N(x) && \text{(Lemma 5.6)} \\
&\text{iff} \quad \overline{d} \in N_{\gamma_N}(x) && \text{(Lemma 5.5)} \\
&\text{iff} \quad d \in N_{\gamma_N}(x) && \text{(Lemma 5.4)}
\end{aligned}$$

and therefore $(X, N, A) = (X, N_{\gamma_N}, \mathsf{clp}\mathbb{X})$. This proves the isomorphism on objects.

Let $(\mathbb{X}, \gamma)$ and $(\mathbb{X}', \gamma')$ be two VV-coalgebras and $f : X \to X'$ a function. We claim that f is a VV-coalgebra morphism if and only if it is a general INL-morphism. If f is a general INL-morphism then clearly it is continuous. Since it is an INL-morphism moreover the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle N_\gamma} & & \downarrow{\scriptstyle N_{\gamma'}} \\ \mathsf{PP}X & \xrightarrow{\mathsf{PP}f} & \mathsf{PP}X' \end{array} \tag{5}$$

commutes. It follows immediately that

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma'} \\ \mathsf{VV}X & \xrightarrow{\mathsf{VV}f} & \mathsf{VV}X' \end{array} \tag{6}$$

commutes, so f is an VV-coalgebra morphism.

Conversely, if f is continuous and (6) commutes, then (5) commutes because $d \in N_{\gamma'}(f(x))$ iff $\overline{d} \in N_{\gamma'}(f(x))$ iff $\overline{d} \in \gamma'(f(x))$ iff $\overline{d} \in \mathsf{VV}f(\gamma(x))$ iff $\overline{d} \in \mathsf{PP}f(N_\gamma(x))$ iff $d \in \mathsf{PP}f(N_\gamma(x))$. The last "iff" follows from Lemma 5.4. It is a general INL-morphism because continuity implies that f^{-1} sends admissible subsets to admissible subsets. □

We have now completed the strategy outlined in diagram (1). As a corollary we obtain Theorem 3.9, whose formulation we copy here for the reader's convenience.

Theorem 3.9. *We have a dual equivalence*

$$\mathbf{D\text{-}INL} \equiv^{\mathrm{op}} \mathbf{BAIO}.$$

Remark 5.7. Careful inspection of the definitions shows that the duality in Theorem 3.9 is given on objects by the construction in Example 3.6 and the subsequent paragraph.

6 Applications

We will give two applications of our results for completeness of INL-based logic and theory of INL-bisimulations.

6.1 Completeness

An *extension* of INL is any set of INL-formulas which contains INL (that is, all the INL-formulas valid on all INL-frames) and is closed under the rules of Modus Ponens and (RE). The latter states that for formulas α, β and φ, if $\alpha \leftrightarrow \beta$, then $\varphi[\alpha/\beta]$ holds, where $\varphi[\alpha/\beta]$ is the result of possibly replacing some occurrences of α in φ by β (see [3]).

It is well known that every modal logic is sound and complete with respect to its algebraic semantics [10, 7, 22]. From the main completeness result of [3] it follows that BAIOs provide an algebraic semantics for INL. Then the standard argument yields that every extension L of INL is sound and complete with respect to the class of BAIOs validating L. Moreover, as a direct corollary of Theorem 3.9, we obtain:

Theorem 6.1. *Every extension of INL is sound and complete with respect to descriptive INL-frames.*

6.2 Bisimulations

We briefly discuss bisimulations for INL and derive a Hennessy-Milner property for descriptive INL-frames. We work in a setting without proposition letters, but all results carry over to the setting *with* proposition letters.

Recall the definition of a bisimulation, in its coalgebraic form:

Definition 6.2. Let (X, N) and (X', N') be neighbourhood frames. A relation $B \subseteq X \times X'$ is an *INL-bisimulation* if there exists a neighbourhood function $M : B \to \mathsf{PP}B$ such that

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & B & \xrightarrow{\pi'} & X' \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle N'} \\ \mathsf{PP}X & \xleftarrow[\mathsf{PP}\pi]{} & \mathsf{PP}B & \xrightarrow[\mathsf{PP}\pi']{} & \mathsf{PP}X' \end{array}$$

commutes. Two states are called bisimilar if they are linked by a bisimulation.

Since PP weakly preserves pullbacks, bisimilarity and behavioural equivalence coincide. In particular, if $(X, N) \xrightarrow{f} (Z, M) \xleftarrow{f'} (X', N')$ is a cospan in **INL** (witnessing behavioural equivalence of some states), then the pullback of f and f' in **Set** is a bisimulation.

Define bisimulations between descriptive frames as follows:

Definition 6.3. A *descriptive INL-bisimulation* between $(\mathbb{X}, \gamma)$ and $(\mathbb{X}', \gamma')$ is a subspace $B \subseteq \mathbb{X} \times \mathbb{X}'$ such that B is a bisimulation between the underlying neighbourhood frames.

In [11] the notion of Λ-bisimulation is introduced, where Λ is a so-called (characteristic) modal signature for an endofunctor on **Stone**. It is straightforward to see that the interpretation of INL in descriptive INL-frames can be translated to the setting used in *op. cit.* Moreover, an easy computation shows that every descriptive INL-bisimulation in the sense of Definition 6.3 is a Λ-bisimulation. We expect that the converse holds as well but at present do not have a proof of this. We leave it as an interesting open question.

We now prove a Hennessey-Milner property for descriptive INL-frames. This is the INL analogue of [5, Corollary 3.9].

Theorem 6.4. *Let* $(\mathbb{X}, \gamma)$ *and* $(\mathbb{X}', \gamma')$ *be descriptive frames. Then* $x \in \mathbb{X}$ *and* $x' \in \mathbb{X}'$ *are logically equivalent iff they are behaviourally equivalent iff they are linked by a descriptive INL-bisimulation.*

Proof. Logical equivalence implies behavioural equivalence because every two logically equivalent states are identified by the theory map to the canonical model (i.e. the final object in **D-INL**). If x and x' are behaviourally equivalent then there are morphisms in **D-INL** such that $f(x) = f'(x')$. The pullback of f and f' viewed as functions in **Set** is a bisimulation between the underlying neighbourhood frames (by the text following Definition 6.2). Moreover, this pullback is closed in $\mathbb{X} \times \mathbb{X}'$ because pullbacks in **Stone** are computed as in **Set**. Hence behavioural equivalence implies bisimilarity. Lastly, bisimilarity implies logical equivalence by design. □

We can apply this theorem to all INL-frames that carry a descriptive structure, that is, to $(X, N) \in$ **INL** such that there exists A making (X, N, A) a descriptive INL-frame.

Corollary 6.5. *Suppose the INL-frames* (X, N) *and* (X', N') *both carry a descriptive structure. Then between these frames, bisimilarity coincides with logical equivalence.*

Restricting this corollary entails [3, Theorem 3.1], the Hennessy-Milner property for finite frames, because all finite neighbourhood frames carry a descriptive frame structure (see Proposition 3.8).

Theorem 6.6. *Let* (X, N) *and* (X', N') *be finite neighbourhood frames. Then bisimilarity coincides with logical equivalence.*

7 Conclusion and future work

In this paper we introduced descriptive frames for the Instantial Neighbourhood Logic (INL) and showed that these frames are dual to BAIOs, the algebras for INL. Coalgebra provided a key for obtaining this duality. We first presented BAIOs as algebras for the functor I on the category of boolean algebras. We also represented descriptive INL-frames as coalgebras for the double Vietoris functor VV on the category of Stone spaces. Finally, we showed that the category of I-algebras is dual to the category of VV-coalgebras, leading to the desired duality result. As a corollary we obtained that every extension of INL is sound and complete with respect to descriptive INL-frames. One interesting question for future work is whether one can obtain an analogue of the celebrated Sahlqvist completeness and correspondence result for extensions of INL.

We recall that the Vietoris functor V is dual to the functor M on the category of boolean algebras (Section 2). Intuitively M freely adds one layer of normal modalities to a boolean algebra. We showed in this paper that BAIOs can be represented as algebras for $\mathsf{M} \circ \mathsf{M}$ (Section 4). Therefore, BAIOs can be seen as “modal algebras squared” and INL itself is, in a way, “the basic modal logic squared". This provokes the question of what "modal algebras cubed" looks like, i.e. what logic and algebras correspond to the functor $\mathsf{M} \circ \mathsf{M} \circ \mathsf{M}$, and similar questions for the n-fold composition of M.

Recall that monotone neighbourhood logic EM algebraically corresponds to a functor N on boolean algebras which intuitively adds one layer of monotone modalities to a boolean algebra [17, 18]. This generates a question whether NN-algebras and the corresponding logic admit an "INL-style axiomatization".

Another related formalism that would be interesting to investigate is that of *positive* INL. The algebras for this logic are *distributive lattices with instantial operators* (DLIOs):

Definition 7.1. A *distributive lattice with instantial operators* (DLIO) is a tuple $(D, (f_n)_{n\in\omega}, (g_n)_{n\in\omega})$ consisting of a distributive lattice D and two collections of ω-indexed maps $f_n, g_n : D^{n+1} \to D$ such that: (1) The f_n satisfy (B_1) to (B_7) from Definition 3.1, where, in absence of negation, we reformulate (B_5) as

$$f_n(a_1, \dots, a_n; b) \leq f_{n+1}(a_1, \dots, a_n, c; b) \vee f_n(a_1, \dots, a_n; b \wedge d),$$

whenever $c \vee d = \top$; (2) The g_n satisfy relations dual to the ones for f_n; and (3) The f_n and g_n satisfy the duality axioms

$$g_{n+1}(a_1, \dots, a_n, b'; b) \wedge \bigwedge_{i=1}^{m} g_n(a_1, \dots, a_n; a'_i \vee b) \leq g_n(a_1, \dots, a_n; b) \vee f_m(a'_1, \dots, a'_m; b') \tag{D_1}$$

and

$$f_n(a_1, \dots, a_n; b) \wedge g_m(a'_1, \dots, a'_m; b') \leq f_{n+1}(a_1, \dots, a_n, b'; b) \vee \bigvee_{i=1}^{m} f_n(a_1, \dots, a_n; a'_i \wedge b). \tag{D_2}$$

These are of course algebras for a endofunctor J on the category **DL** of distributive lattices and homomorphisms. In analogy with the results of this paper, one would expect that descriptive frames for positive INL are isomorphic to coalgebras for the double convex Vietoris functor $\mathsf{V_c}$ on the category of Priestley spaces, as this is the Priestley space analogue of the Vietoris functor [27, 6, 33]. We expect the following duality result:

Conjecture 7.2. *We have a dual equivalence*

$$\mathbf{Alg}(\mathsf{J}) \equiv^{\mathrm{op}} \mathbf{Coalg}(\mathsf{V_c}\mathsf{V_c}).$$

Finally, related to positive INL, an interesting question is to consider the geometric logic analogue of INL and to verify a slogan of [4], which in the case of INL will read as

$$\text{Geometric INL} = \text{Positive INL} + \text{Scott continuity}.$$

If correct, this may also provide a novel algebraic presentation of the double Vietoris powerlocale studied extensively by Vickers [35, 36, 34].

References

1. J. van Benthem, N. Bezhanishvili, and S. Enqvist. A new game equivalence, its logic and algebra. *Journal of Philosophical Logic*, 48(4):649–684, 2019.
2. J. van Benthem, N. Bezhanishvili, and S. Enqvist. A propositional dynamic logic for instantial neighbourhood semantics. *Studia Logica*, 107(4):719–751, 2019.
3. J. van Benthem, N. Bezhanishvili, S. Enqvist, and J. Yu. Instantial neighbourhood logic. *Review of Symbolic Logic*, 10(1):116–144, 2017.
4. N. Bezhanishvili, J. de Groot, and Y. Venema. Coalgebraic Geometric Logic. In M. Roggenbach and A. Sokolova, editors, *8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019)*, volume 139 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 7:1–7:18, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
5. N. Bezhanishvili, G. Fontaine, and Y. Venema. Vietoris bisimulations. *Journal of Logic and Computation*, 20(5):1017–1040, 2010.
6. N. Bezhanishvili and A. Kurz. Free modal algebras: A coalgebraic perspective. In T. Mossakowski, U. Montanari, and M. Haveraaen, editors, *Algebra and Coalgebra in Computer Science*, pages 143–157, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.
7. P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.
8. S. Celani and R. Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of the IGPL*, 7:683–715, 12 1999.
9. S.A. Celani and R. Jansana. Easkia duality and its extensions. In G. Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, pages 63–98. Springer Netherlands, Dordrecht, 2014.
10. A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, Oxford, 1997.
11. S. Enqvist and S. Sourabh. Bisimulations for coalgebras on Stone spaces. *Journal of Logic and Computation*, 28(6):991–1010, 2018.
12. L.L. Esakia. Topological Kripke models. *Soviet Mathematics Doklady*, 15:147–151, 1974.
13. L.L. Esakia. The problem of dualism in the intuitionistic logic and Browerian lattices. In *V International Congress of Logic, Methodology and Philosophy of Science*, pages 7–8, Canada, 1975.
14. L.L. Esakia. *Heyting Algebras*. Trends in Logic. Springer International Publishing, Springer, 2019. Translated by A. Evseev.
15. R. Furber, D. Kozen, K.G. Larsen, R. Mardare, and P. Panangaden. Unrestricted stone duality for Markov processes. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '17, pages 1–9, Reykjavík, Iceland, 2017. IEEE Press.
16. R.I. Goldblatt. Metamathematics of modal logic I. *Reports on Mathematical Logic*, 6:41–78, 1976.
17. H.H. Hansen. Monotonic modal logics, 2003. Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam.
18. H.H. Hansen and C. Kupke. A coalgebraic perspective on monotone modal logic. *Electronic Notes in Theoretical Computer Science*, 106:121–143, 2004.
19. P.T. Johnstone. *Stone Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982.

20. B. Jónsson and A. Tarski. Boolean algebras with operators, Part I. *American Journal of Mathematics*, 73:891–939, 1951.
21. K. Kapulkin, A. Kurz, and J. Velebil. Expressiveness of positive coalgebraic logic. *Advances in Modal Logic*, 9, 2014.
22. C. Kupke, A. Kurz, and D. Pattinson. Algebraic semantics for coalgebraic logics. *Electronic Notes in Theoretical Computer Science*, 106:219–241, 2004. Proceedings of the Workshop on Coalgebraic Methods in Computer Science (CMCS).
23. C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. *Theoretical Computer Science*, 327(1):109–134, 2004. Selected Papers of CMCS '03.
24. C. Kupke and D. Pattinson. Coalgebraic semantics of modal logics: An overview. *Theoretical Computer Science*, 412(38):5070–5094, 2011. CMCS Tenth Anniversary Meeting.
25. M. Mio, R. Furber, and R. Mardare. Riesz modal logic for markov processes. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12, 06 2017.
26. L.S. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96(1):277–317, 1999.
27. A. Palmigiano. A coalgebraic view on positive modal logic. *Theoretical Computer Science*, 327:175–195, 10 2004.
28. H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society*, 2(2):186–190, 1970.
29. J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249((1)):3–80, 2000.
30. G. Sambin and V. Vaccaro. A new proof of Sahlqvist's theorem on modal definability and completeness. *The Journal of Symbolic Logic*, 54(3):992–999, 1989.
31. M.H. Stone. The theory of representations for boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.
32. O. Tuyt. Canonical rules on neighbourhood frames, 2016. Master's thesis, ILLC, University of Amsterdam.
33. Y. Venema and J. Vosmaer. Modal logic and the Vietoris functor. In G. Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, pages 119–153, Dordrecht, 2014. Springer Netherlands.
34. S. Vickers. Constructive points of powerlocales. *Mathematical Proceedings of the Cambridge Philosophical Society*, 122(2):207—-222, 1997.
35. S.J. Vickers. The double powerlocale and exponentiation: A case study in geometric logic. *Theory and Applications of Categories*, 12(13):272–422, 2004.
36. S.J. Vickers and C.F. Townsend. A universal characterization of the double powerlocale. *Theoretical Computer Science*, 316(1):297–321, 2004. Recent Developments in Domain Theory: A collection of papers in honour of Dana S. Scott.
37. L. Vietoris. Bereiche zweiter Ordnung. *Monatshefte für Mathematik und Physik*, 32(1):258–280, Dec 1922.
38. K. Yosida. On the representation of vector lattices. *Proceedings of the Imperial Academy*, 18(7):339–342, 1942.
39. J. Yu. A tableau system for instantial neighborhood logic. In *Logical Foundations of Computer Science - International Symposium, LFCS 2018, Deerfield Beach, FL, USA, January 8-11, 2018, Proceedings*, pages 337–353, 2018.
40. J. Yu. Lyndon interpolation theorem of instantial neighborhood logic - constructively via a sequent calculus. *Annals of Pure and Applied Logic*, 171(1), 2020.