VARIETIES OF TWO-DIMENSIONAL CYLINDRIC ALGEBRAS

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1. Introduction

In this chapter we survey recent developments in the theory of two-dimensional cylindric algebras. In particular, we will deal with varieties of two-dimensional diagonal-free cylindric algebras and with varieties of two-dimensional cylindric algebras with the diagonal. It is well known that two-dimensional diagonal-free cylindric algebras correspond to the two variable equality-free fragment of classical first-order logic FOL, whereas two-dimensional cylindric algebras with the diagonal correspond to the two variable fragment of FOL with equality. It is also well known that one-dimensional cylindric algebras, also called Halmos monadic algebras, provide algebraic completeness for the one variable fragment of FOL. For a systematic discussion on the connection between various fragments of FOL and classes of (cylindric) algebras that correspond to these fragments we refer to [2].

The variety Df_1 of one-dimensional cylindric algebras has a lot of 'nice' properties: Df_1 is finitely axiomatizable, it is generated by its finite algebras, and has a decidable equational theory. Moreover, the lattice of subvarieties of Df_1 is rather simple: it is an $(\omega+1)$-chain. Every proper subvariety of Df_1 is finitely generated, finitely axiomatizable and has a decidable equational theory (see [22], [23] and [44]). In contrast to this, the three variable fragment of FOL corresponding to three-dimensional cylindric algebras is much more complicated and no longer has 'nice' properties. It has been shown by Maddux [28] that the equational theory of three-dimensional cylindric algebras is undecidable. Moreover, every subvariety in between the variety of all representable three-dimensional cylindric algebras and the variety of all three-dimensional cylindric algebras is undecidable. Kurucz [27] strengthened this by showing that none of these varieties are generated by their finite algebras. It follows from Monk [36] and Johnson [24] that varieties of all representable three-dimensional cylindric algebras with and without diagonals are not finitely axiomatizable.

Our aim is to show that the two-dimensional case is not as complicated as the three-dimensional one, but is not as simple as the one-dimensional case. It has been known for a long time that the variety Df_2 of two-dimensional diagonal-free cylindric algebras is generated by its finite algebras and has a decidable equational theory. Moreover, every Df_2-algebra is representable. The variety CA_2 of two-dimensional cylindric algebras with the diagonal is also generated by its finite algebras and has a decidable equational theory. However, not every CA_2-algebra is representable. Representable CA_2-algebras form a subvariety of CA_2 denoted by RCA_2. Unlike the three-dimensional case, RCA_2 can be axiomatized by adding only one axiom to the axiomatization of CA_2. In this chapter we will mainly concentrate on the lattices of subvarieties of Df_2, CA_2 and RCA_2, respectively. As we will see below, the lattice of subvarieties of Df_2, although more complicated than the one-dimensional case, is still countable. Moreover, every subvariety of Df_2 is finitely axiomatizable and has a decidable equational theory. The lattices of subvarieties of CA_2 and RCA_2, respectively, have a more complex structure. In particular, it is known that unlike Df_2, CA_2 and RCA_2 have continuum many subvarieties. As a result, there is a continuum of non-finitely axiomatizable and undecidable subvarieties of CA_2 and RCA_2. We will show that every proper subvariety of Df_2 is locally finite and every subvariety of Df_2 is generated by its finite algebras. On the other hand, we will prove that there are continuum many non-locally finite subvarieties of CA_2 and RCA_2. It is still an open problem whether every subvariety of CA_2 and RCA_2 is generated by its finite algebras.

Our main technique in studying varieties of two-dimensional cylindric algebras will be the duality between two-dimensional cylindric algebras and Stone spaces enriched with two commuting equivalence relations. The topology-free analogue of this duality is thoroughly discussed in [23, Section 2.7]. Stone-like topological dualities are extensively used to investigate modal logics (see, e.g, [10, 12]). In fact, many of the results discussed in this chapter are proved using techniques developed in modal logic. These techniques apply to Df_2 and CA_2-algebras since they are algebraic models of cylindric modal logic (see [Venema's chapter]).

The chapter is organized as follows: in Section 2 we recall the definition of Df_2 and CA_2-algebras and their topological representation. We also recall the functor from CA_2 to Df_2 that forgets the diagonal and discuss some of its basic properties. In Section 3 we recall representable cylindric algebras and a topological characterization of representable CA_2-algebras. We also construct rather simple non-representable CA_2-algebras. In Section 4 we discuss the cardinality of the lattices of subvarieties of CA_2 and RCA_2. Section 5 reviews a criterion of local finiteness for subvarieties of Df_2 and Section 6 discusses a classification of subvarieties of Df_2. In Section 7 we review local finiteness of subvarieties of CA_2 and RCA_2. Section 8 is devoted to finitely generated varieties of Df_2 and CA_2-algebras, and finally, we close the chapter by discussing some open problems.

Acknowledgment The author is grateful to Ian Hodkinson for his comments on the earlier version of this chapter. Special thanks go to the referee for valuable suggestions and pointers to the literature that substantially improved the presentation of this chapter.

2. Cylindric algebras and cylindric spaces

In this section we recall the basic duality for two-dimensional cylindric algebras that will be used throughout this chapter.

Definition 2.1. [23, Definition 1.1.2] *A triple $\mathcal{B} = \langle B, \mathsf{c}_1, \mathsf{c}_2\rangle$ is said to be a* two-dimensional diagonal-free cylindric algebra*, or a* Df_2-algebra *for short, if B is a Boolean algebra and $\mathsf{c}_i : B \to B$, $i = 1, 2$, are unary operations satisfying the following axioms for all $a, b \in B$:*

(C$_1$) $\mathsf{c}_i 0 = 0$,
(C$_2$) $a \leq \mathsf{c}_i a$,
(C$_3$) $\mathsf{c}_i(\mathsf{c}_i a \cdot b) = \mathsf{c}_i a \cdot \mathsf{c}_i b$,
(C$_4$) $\mathsf{c}_1\mathsf{c}_2 a = \mathsf{c}_2\mathsf{c}_1 a$.

Let Df_2 denote the variety of all two-dimensional diagonal-free cylindric algebras.

Definition 2.2. [23, Definition 1.1.1] *A quadruple $\mathfrak{B} = \langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$ is said to be a* two-dimensional cylindric algebra*, or a* CA_2-algebra *for short, if $\langle B, \mathsf{c}_1, \mathsf{c}_2\rangle$ is a Df_2-algebra and $\mathsf{d} \in B$ is a constant satisfying the following axioms for all $a \in B$ and $i = 1, 2$.*

(C$_5$) $\mathsf{c}_i(\mathsf{d}) = 1$,
(C$_6$) $\mathsf{c}_i(\mathsf{d} \cdot a) \leq -\mathsf{c}_i(\mathsf{d} \cdot -a)$.

Let CA_2 *denote the variety of all two-dimensional cylindric algebras.*

Cylindric algebras were first introduced in [13] without explicitely mentioning the term 'cylindric algebra'. The full definition later appeared in [25], [47] and [46].

We recall that a Df_1-algebra or a *Halmos* monadic algebra is a pair $\langle B, \mathsf{c}\rangle$ such that B is a Boolean algebra and c is a unary operator on B satisfying conditions C_1-C_3 of Definition 2.1; see, e.g., [22, p.40]. The unary operator c is called a *monadic operator*, and Df_1-algebras are widely known as *monadic algebras*. A systematic investigation of monadic algebras has been carried out by Halmos [22], Bass [3], Monk [37], and Kagan and Quackenbush [26].

Now we turn to a topological representation of two-dimensional cylindric algebras. This duality (see [5, Section 2.2] and [6, Section 2]) is based on a standard Jónsson-Tarski duality for modal algebras [25] (see also [10, Section 5.5] and [12, Section 8.2]). A topological duality for Df_1-algebras was developed by Halmos [22]. We recall that a *Stone space* is a compact, Hausdorff space with a basis of *clopen* (simultaneously closed and open) sets. For a Stone space X, we denote by $\mathcal{CP}(X)$ the set of all clopen subsets of X. For an arbitrary binary relation R on X, $x \in X$ and $A \subseteq X$, we let $R(x) = \{y \in X : xRy\}$, $R^{-1}(x) = \{y \in X : yRx\}$, $R(A) = \bigcup_{x\in A} R(x)$ and $R^{-1}(A) = \bigcup_{x\in A} R^{-1}(x)$. A relation R on a Stone space X is said to be *point-closed* if for each $x \in X$ the set $R(x)$ is closed, and R is called a *clopen* relation, if $A \in \mathcal{CP}(X)$ implies $R^{-1}(A) \in \mathcal{CP}(X)$. Note that if R is an equivalence relation, then $R(x) = R^{-1}(x)$ and $R(A) = R^{-1}(A)$. In such a case we call $R(A)$ the *R-saturation* of A.

A Df_2-*space* is a triple $\mathcal{X} = \langle X, E_1, E_2\rangle$, where X is a Stone space, and E_1 and E_2 are point-closed and clopen equivalence relations on X such that

$$(\forall x, y, z \in X)((xE_1y \wedge yE_2z) \to (\exists u \in X)(xE_2u \wedge uE_1z)).$$

Given two Df_2-spaces $\mathcal{X} = \langle X, E_1, E_2\rangle$ and $\mathcal{X}' = \langle X', E_1', E_2'\rangle$, a map $f : X \to X'$ is said to be a Df_2-*morphism* if f is continuous, and for each $x \in X$ and $i = 1, 2$ we have $fE_i(x) = E_i'f(x)$. We denote by $\mathbf{DS}$ the category of Df_2-spaces and Df_2-morphisms. Then Df_2 is dual (dually equivalent) to $\mathbf{DS}$. In particular, every Df_2-algebra $\langle B, \mathsf{c}_1, \mathsf{c}_2\rangle$ can be represented as $\langle \mathcal{CP}(X), E_1, E_2\rangle$, for the corresponding Df_2-space $\langle X, E_1, E_2\rangle$. We recall that $\langle X, E_1, E_2\rangle$ is constructed as follows: X is the set of all ultrafilters of B, $\varphi(a) = \{x \in X : a \in x\}$, $\{\varphi(a)\}_{a\in B}$ is a basis for the topology on X, and xE_iy if $(\mathsf{c}_ia \in x \Leftrightarrow \mathsf{c}_ia \in y)$, for each $a \in B$ and $i = 1, 2$. We call $\langle X, E_1, E_2\rangle$ the *dual* of $\langle B, \mathsf{c}_1, \mathsf{c}_2\rangle$. As an easy corollary of this duality, we obtain that the category $\mathrm{Fin}\mathsf{Df}_2$ of finite Df_2-algebras is dual to the category $\mathrm{Fin}\mathbf{DS}$ of finite Df_2-spaces with the discrete topology. Hence, every finite Df_2-algebra is represented as the algebra $\langle \mathcal{P}(X), E_1, E_2\rangle$ for the corresponding finite Df_2-space $\langle X, E_1, E_2\rangle$, where $\mathcal{P}(X)$ denotes the powerset of X (see [23, Theorem 2.7.34]). For $i = 1, 2$ we call the E_i-equivalence classes of X the E_i-*clusters*.

The dual spaces of CA_2-algebras are obtained as easy extensions of Df_2-spaces. A quadruple $\langle X, E_1, E_2, D\rangle$ is said to be a CA_2-*space* if $\langle X, E_1, E_2\rangle$ is a Df_2-space and D is a clopen subset of X such that for each $i = 1, 2$, each E_i-cluster of X contains a unique point from D. This implies that in each CA_2-space $\langle X, E_1, E_2, D\rangle$, unlike Df_2-spaces, the cardinality of the set of all E_1-clusters of X is always equal to the cardinality of the set of all E_2-clusters of X. Given two CA_2-spaces $\langle X, E_1, E_2, D\rangle$ and $\langle X', E_1', E_2', D'\rangle$, a map $f : X \to X'$ is said to be a CA_2-*morphism* if f is a Df_2-morphism and in addition $f^{-1}(D') = D$. We denote the category of CA_2-spaces and CA_2-morphisms by $\mathbf{CS}$. Then CA_2 is dual to $\mathbf{CS}$. In particular, every CA_2-algebra $\mathfrak{B} = \langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$ can be represented as $\langle \mathcal{CP}(X), E_1, E_2, D\rangle$ for the corresponding

CA_2-space $\langle X, E_1, E_2, D\rangle$. The construction of $\langle X, E_1, E_2, D\rangle$ is the same as in the Df_2-case with the addition that we let $D = \varphi(\mathsf{d}) = \{x \in X : \mathsf{d} \in x\}$. We call $\langle X, E_1, E_2, D\rangle$ the *dual* of $\langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$. As an easy corollary of this duality we obtain that the category FinCA_2 of finite CA_2-algebras is dual to the category Fin**CS** of finite CA_2-spaces with the discrete topology. In particular, every finite CA_2-algebra is represented as the algebra $\langle \mathcal{P}(X), E_1, E_2, D\rangle$ for the corresponding finite CA_2-space $\langle X, E_1, E_2, D\rangle$ (see [23, Theorem 2.7.34]).

Remark 2.3. We point out a close connection between Df_2 and CA_2-spaces and cylindric atom structures defined in [23]. Recall from [23, Definition 2.7.32] that if $\mathfrak{B} = \langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$ is a CA_2-algebra (the case of Df_2-algebras is similar) such that B is an atomic Boolean algebra, then the cylindric atom structure of $\mathfrak{B}$ is defined as the quadruple $\mathfrak{AtB} = \langle AtB, E_1, E_2, D\rangle$, where AtB is the set of all atoms of B; E_i is defined by setting: xE_iy if $\mathsf{c}_i x = \mathsf{c}_i y$, for $x, y \in AtB$, $i = 1, 2$; and $D = \{x \in AtB : x \leq \mathsf{d}\}$.

Now suppose $\mathfrak{B} = \langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$ is an arbitrary CA_2-algebra. Let $\mathfrak{B}^\sigma = \langle B^\sigma, \mathsf{c}_1^\sigma, \mathsf{c}_2^\sigma, \mathsf{d}^\sigma\rangle$ be the canonical extension of $\mathfrak{B}$, and let $i : B \to B^\sigma$ be the canonical embedding [23, Definition 2.7.4]. Then it is well known that B^σ is complete and atomic. Let $\mathfrak{AtB}^\sigma$ be the cylindric atom structure of $\mathfrak{B}^\sigma$. For $a \in B$ let $O_a = \{x \in AtB^\sigma : x \leq i(a)\}$. We make $\mathfrak{AtB}^\sigma$ into a topological space by letting $\{O_a\}_{a\in B}$ to be a basis for the topology τ. Then it can be shown that $\langle \mathfrak{AtB}^\sigma, \tau\rangle$ is a CA_2-space, and that $\langle \mathfrak{AtB}^\sigma, \tau\rangle$ is isomorphic to the dual of $\mathfrak{B}$. We note that this connection applies not only to cylindric algebras, but, in general, to any Boolean algebra with operators; see, e.g, [49, Section 5].

Having this duality at hand, we can obtain dual descriptions of important algebraic concepts of Df_2 and CA_2-algebras. A subset U of a Df_2-space (resp. of a CA_2-space) is said to be *saturated* if $E_1(U) = E_2(U) = U$. For each Df_2-algebra $\mathcal{B}$ (resp. CA_2-algebra $\mathfrak{B}$), the lattice of all congruences of $\mathcal{B}$ (resp. $\mathfrak{B}$) is dually isomorphic to the lattice of all closed saturated subsets of its dual space. A Df_2-space $\langle X, E_1, E_2\rangle$ (resp. a CA_2-space $\langle X, E_1, E_2, D\rangle$) is called *rooted* if

$$(\forall x, y \in X)(\exists z \in X)(xE_1z \wedge zE_2y).$$

(By the commutativity of E_1 and E_2, if such a z exists, then there exists $u \in X$ such that xE_2u and uE_1y). Rooted spaces correspond to subdirectly irreducible and simple cylindric algebras. In particular, a Df_2-algebra (resp. a CA_2-algebra) is subdirectly irreducible iff it is simple iff its dual space is rooted [23, Theorems 2.4.43 and 2.4.14]. As a result of the above, we obtain that both Df_2 and CA_2 are semi-simple and congruence distributive varieties with the congruence extension property. As shown in [41], these results also follow from the fact that Df_2 and CA_2 are discriminator varieties. The aforementioned properties are also discussed in [49, Section 4] in a wider context of Boolean algebras with operators.

Now we are ready to recall the definition of the reduct functor $\mathfrak{Df} : \mathsf{CA}_2 \to \mathsf{Df}_2$ [23, Definition 1.1.2] (see also [6, Section 2.2]). For each CA_2-algebra $\langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle$ we set

$$\mathfrak{Df}\langle B, \mathsf{c}_1, \mathsf{c}_2, \mathsf{d}\rangle = \langle B, \mathsf{c}_1, \mathsf{c}_2\rangle.$$

Thus, $\mathfrak{Df}$ forgets the diagonal element d from the signature of CA_2-algebras.

Next we will give a simple argument showing that $\mathfrak{Df}$ is not onto (see [23, Corollary 5.1.4(ii)]). In fact, the set of isomorphism types of $\mathsf{Df}_2 - \mathfrak{Df}(\mathsf{CA}_2)$ is infinite. For this, we define the reduct functor $\mathfrak{Rd} : \mathbf{CS} \to \mathbf{DS}$. For each CA_2-space $\langle X, E_1, E_2, D\rangle$ we set

$$\mathfrak{Rd}\langle X, E_1, E_2, D\rangle = \langle X, E_1, E_2\rangle.$$

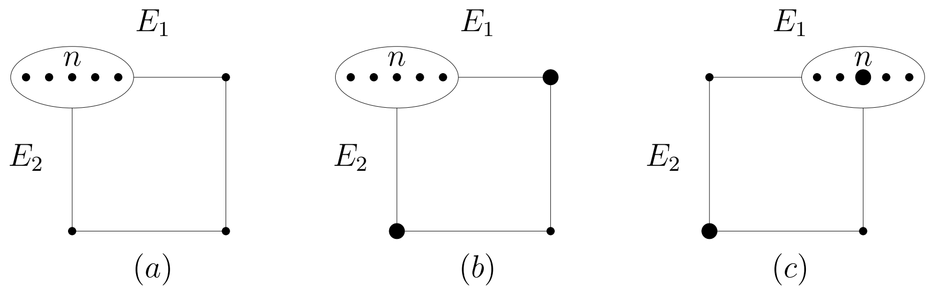

FIGURE 1. Some CA_2-spaces and their reducts

Suppose a Df_2-space $\langle Y, E_1, E_2 \rangle$ is rooted. We call $\langle Y, E_1, E_2 \rangle$ a *quasi-square* if the cardinalities of the sets of all E_1 and E_2-clusters of Y coincide. It is not hard to see that a rooted $\langle Y, E_1, E_2 \rangle$ is a reduct of some CA_2-space iff it is a quasi-square. Note that not every rooted space from $\mathbf{DS}$ is a quasi-square. The simplest examples of rooted Df_2-spaces that are not quasi-squares are rectangle Df_2-spaces (for the definition of a rectangle consult the next section). Since there are infinitely many rectangle Df_2-spaces (as we will see below), the set $\mathbf{DS} - \mathfrak{Rd}(\mathbf{CS})$ is infinite.

We call a Df_2-algebra $\mathcal{B}$ a *quasi-square algebra* if its dual space is a quasi-square. As follows from the above, for each simple CA_2-algebra $\mathfrak{B}$, its Df_2-reduct is a quasi-square algebra. Therefore, the set $\mathsf{Df}_2 - \mathfrak{Df}(\mathsf{CA}_2)$ is infinite. Moreover, one Df_2-algebra can be the reduct of many non-isomorphic CA_2-algebras. For instance, a Df_2-algebra whose dual space is shown in Figure 1(a) is the reduct of the CA_2-algebras whose dual CA_2-spaces are shown in Figures 1(b) and 1(c), where dots represent points of the spaces, while big dots represent the points belonging to the (diagonal) set D.

More algebraic properties of Df_2 are discussed in [23, Section 5.1] and [5]. In particular, a characterization of finitely approximable Df_2-algebras, projective and injective Df_2-algebras, and absolute retracts in Df_2 is given in [5, Sections 3.1 and 3.2].

3. Representable cylindric algebras

In this section we recall some basic facts about representable two-dimensional cylindric algebras.

Let W and W' be sets. We define on the Cartesian product $W \times W'$ two equivalence relations E_1 and E_2 by setting

$$(w, w')E_1(v, v') \text{ if } w' = v',$$
$$(w, w')E_2(v, v') \text{ if } w = v,$$

for $w, v \in W$ and $w', v' \in W'$. We call $\langle W \times W', E_1, E_2 \rangle$ a *rectangle*. If $W = W'$ then we call $\langle W \times W, E_1, E_2 \rangle$ a *square*. For a square $\langle W \times W, E_1, E_2 \rangle$ we set $D = \{(w, w) : w \in W\}$. We call $\langle W \times W, E_1, E_2, D \rangle$ a *cylindric square*. A *generalized rectangle* is a disjoint union of rectangles, a *generalized square* is a disjoint union of squares, and a *generalized cylindric square* is a disjoint union of cylindric squares. It is easy to see that for each (generalized) rectangle $\langle U, E_1, E_2 \rangle$, the algebra $\langle \mathcal{P}(U), E_1, E_2 \rangle$ is a Df_2-algebra and that for each (generalized) cylindric square $\langle U, E_1, E_2, D \rangle$ the algebra $\langle \mathcal{P}(U), E_1, E_2, D \rangle$ is a CA_2-algebra.

We call a Df_2-algebra $\langle \mathcal{P}(U), E_1, E_2\rangle$ a (*generalized*) *rectangular* Df_2-*algebra*,[1] if $\langle U, E_1, E_2\rangle$ is a (generalized) rectangle and we call $\langle \mathcal{P}(U), E_1, E_2\rangle$ a (generalized) *square* Df_2-*algebra* if $\langle U, E_1, E_2\rangle$ is a (generalized) square. We call the CA_2-algebra $\langle \mathcal{P}(U), E_1, E_2, D\rangle$ a (*generalized*) *square* CA_2-*algebra*, if $\langle U, E_1, E_2, D\rangle$ is a (generalized) square. Let ($\mathbb{GRECT}$) $\mathbb{RECT}$ denote the class of all (generalized) rectangular Df_2-algebras, let ($\mathbb{GSQ}$) $\mathbb{SQ}$ denote the class of all (generalized) square algebras, and let ($\mathbb{GCSQ}$) $\mathbb{CSQ}$ denote the class of all (generalized) square CA_2-algebras. Finally, we let Fin$\mathbb{RECT}$, Fin$\mathbb{SQ}$ and Fin$\mathbb{CSQ}$ denote the classes of all finite rectangular, finite square and finite cylindric square algebras, respectively.

For a class K of algebras, we denote by $\mathbf{H}(\mathsf{K})$, $\mathbf{S}(\mathsf{K})$ and $\mathbf{P}(\mathsf{K})$ the closure of K under homomorphic images, subalgebras and products, respectively. We say that a variety V is generated by a class K of algebras if $\mathsf{V} = \mathbf{HSP}(\mathsf{K})$. The classes $\mathbf{S}(\mathbb{RECT})$, $\mathbf{S}(\mathbb{SQ})$, $\mathbf{S}(\mathbb{CSQ})$, $\mathbf{S}(\mathbb{GRECT})$, and $\mathbf{S}(\mathbb{GCSQ})$ in [23, Definitions 3.1.1 and 5.1.33] are denoted by Csdf_2, Csudf_2, Cs_2, Gsdf_2, and Gs_2, respectively. The algebras in these classes are called *two-dimensional*: *diagonal-free cylindric set algebras*, *diagonal-free uniform cylindric set algebras*, *cylindric set algebras*, *diagonal-free generalized cylindric set algebras*, and *generalized cylindric set algebras*, respectively. Since we only work with two-dimensional cylindric algebras, we find our terminology and notations more suggestive.

Definition 3.1. [23, Definitions 5.1.33(v), 3.1.1(vii) and Remark 1.1.13]

(i) *A* Df_2-*algebra* $\mathcal{B}$ *is said to be* representable *if* $\mathcal{B} \in \mathbf{S}(\mathbb{GRECT})$.

(ii) *A* CA_2-*algebra* $\mathfrak{B}$ *is said to be* representable *if* $\mathfrak{B} \in \mathbf{S}(\mathbb{GCSQ})$.

The classes of representable Df_2 and CA_2-algebras are usually denoted by RDf_2 and RCA_2, respectively. For the proof of the next theorem we refer to [23, Corollary 5.1.35, Theorems 5.1.43 and 5.1.47] for (1) (see also [17, Corollary 5.10]), to [23, Corollary 3.1.108] for (2), and to [23, Lemmas 2.6.41 and 2.6.42] for (3).

Theorem 3.2.

(i) $\mathsf{RDf}_2 = \mathsf{Df}_2 = \mathbf{HSP}(\mathbb{RECT}) = \mathbf{HSP}(\mathbb{SQ}) = \mathbf{SP}(\mathbb{RECT}) = \mathbf{SP}(\mathbb{SQ}) = \mathbf{S}(\mathbb{GSQ})$.

(ii) $\mathsf{RCA}_2 = \mathbf{HSP}(\mathbb{CSQ}) = \mathbf{SP}(\mathbb{CSQ})$.

(iii) $\mathsf{RCA}_2 \subsetneq \mathsf{CA}_2$.

Let

$$(\mathrm{H}) = \mathsf{c}_i(a \cdot -b \cdot \mathsf{c}_j(a \cdot b)) \le \mathsf{c}_j(-\mathsf{d} \cdot \mathsf{c}_i a), \quad i \neq j, \quad i, j = 1, 2.$$

and

$$(\mathrm{V}) = \mathsf{d} \cdot \mathsf{c}_i(-a \cdot \mathsf{c}_j a) \le \mathsf{c}_j(-\mathsf{d} \cdot \mathsf{c}_i a), \quad i \neq j, \quad i, j = 1, 2.$$

We call (H) and (V) the *Henkin* and *Venema axioms*, respectively. Then RCA_2 is axiomatized by adding either of these axioms to the axiomatization of CA_2; see, e.g., [23, Theorem 3.2.65(ii)] or [48, Proposition 3.5.8]). We denote by $\mathsf{V}+(\mathrm{Ax})$ the addition of the axiom (Ax) to the axiomatization of a variety V. Then

Theorem 3.3. $\mathsf{RCA}_2 = \mathsf{CA}_2+(\mathrm{H}) = \mathsf{CA}_2+(\mathrm{V})$.

Next we recall from [48] and [6] a dual characterization of representable CA_2-algebras, and construct rather simple finite non-representable CA_2-algebras. For this purpose we recall that

[1] Note that the concept of a 'rectangular algebra' is different from the one of a 'rectangular element' defined in [23, Definition 1.10.6].

on each Df_2-space $\langle X, E_1, E_2\rangle$ and on each CA_2-space $\langle X, E_1, E_2, D\rangle$ we can define yet another equivalence relation that naturally arises from E_1 and E_2. We define E_0 by: xE_0y if xE_1y and xE_2y, for each $x, y \in X$. In other words, $E_0 = E_1 \cap E_2$. We call E_0-equivalence classes E_0-*clusters*. Suppose $\langle X, E_1, E_2, D\rangle$ is a CA_2-space. We call $x \in D$ a *diagonal point*, and we call $x \in X - D$ a *non-diagonal point*. We also call an E_0-cluster C a *diagonal E_0-cluster* if it contains a diagonal point. Otherwise we call C a *non-diagonal E_0-cluster*.

Definition 3.4. ([6, Definition 3.3]) *A CA_2-space $\langle X, E_1, E_2, D\rangle$ is said to satisfy $(*)$ if there exists a diagonal point $x_0 \in D$ such that $E_0(x_0) = \{x_0\}$ and there exists a non-singleton E_0-cluster C whose elements are either E_1 or E_2-related to x_0.*

The $(*)$ condition is equivalent to Venema's condition NH7 of [48, Definition 3.2.5] (see also [Definition 1.3, Venema's chapter]). In the terminology of [23] a CA_2-space satisfies the condition $(*)$ of Definition 3.4 iff the corresponding CA_2-algebra has at least one so-called defective atom (for details see [23, Lemma 3.2.59]). Next we recall a dual characterization of representable CA_2-algebras. There are at least three different proofs of Theorem 3.5. The one given in [23, Lemma 3.2.59] uses Henkin's axioms, the proof of [48, Theorem 3.2.6] (see also [Proposition 1.5, Venema's chapter]) is based on a powerful technique of modal logic called Sahlqvist correspondence and [6, Theorem 3.4] applies Venema's axioms and order-topological methods.

Theorem 3.5. *A CA_2-algebra $\mathfrak{B}$ is representable iff its dual CA_2-space $\mathcal{X}$ does not satisfy $(*)$.*

Using this criterion it is easy to see that the CA_2-algebras corresponding to the CA_2-spaces shown in Figure 1(c) (big spots denote the diagonal points) are representable, while the CA_2-algebras corresponding to the CA_2-spaces shown in Figure 1(b) are not. Moreover, the smallest non-representable CA_2-algebra is the algebra corresponding to the CA_2-space shown in Figure 1(b), where the non-singleton E_0-cluster contains only two points.

4. The finite model property and cardinality of lattices of varieties

There is a wide variety of proofs available for the decidability of classical first-order logic with two variables. Equivalent results were stated and proved using quite different methods in first-order, modal and algebraic logic. We present a short historic overview.

Decidability of the validity of equality-free first-order sentences in two variables was proved by Scott [43]. The proof uses a reduction to the set of prenex formulas of the form $\exists^2\forall^n\varphi$, whose validity is decidable by Gödel [18]. The result was stated with equality in the language, because at that time it was still believed that the validity problem for $\exists^2\forall^n$ formulas containing equality is decidable. This belief was however refuted in Goldfarb [19]. Scott's result was extended by Mortimer [39], who included equality in the language and showed that such sentences cannot enforce infinite models, obtaining decidability as a corollary. A simpler proof was provided in Grädel et al. [20]. They showed that any satisfiable formula can actually be satisfied in a model whose size is single exponential in the length of the formula. Adding two unary function symbols to the language with only one variable leads to undecidability, as shown in Gurevich [21]. Segerberg [45] proved the finite model property and decidability for so-called 'two-dimensional modal logic', which is essentially cylindric modal logic (see [Venema's chapter]) enriched with the operation of involution. For an algebraic proof we refer to [23, Lemma 5.1.24 and Theorem 5.1.64]. A mosaic type proof can be found in Marx and Mikulás [33]. A proof using quasimodels is provided in [17, Theorem 5.22], and [35, Proposition 7.4.3] and [7, Theorem 6.1.1] give simple proofs via the filtration method.

The fact that CA_2 is generated by its finite algebras and has a decidable equational theory was first proved by Henkin [23, Lemma 2.5.4 and Theorem 4.2.7] (see also [7, Theorem 7.1.1] for a proof using the filtration method). That RCA_2 is generated by its finite algebras and has a decidable equational theory follows from Mortimer [39] (see also [23, Theorem 4.2.9], [34, Theorem 2.3.5] and [31]). Summing all this up we arrive at the following result.

Theorem 4.1. Df_2, CA_2 *and* RCA_2 *are generated by their finite algebras and have decidable equational theories.*

However, Df_2 is not only generated by its finite algebras, but it is also generated by its finite rectangular and finite square algebras. This result follows from Segerberg [45]. A short algebraic proof can be found in Andréka and Németi [1]. For a frame-theoretic proof using quasimodels see [17, Theorem 5.25]. Another frame-theoretic proof is given in [7, Theorem 6.11]. All these proofs can be adjusted to show that RCA_2 is generated by finite cylindric square algebras. This result also follows from Mortimer [39]. Thus, we arrive at the following theorem.

Theorem 4.2.
(i) $\mathsf{Df}_2 = \mathbf{HSP}(\mathrm{Fin}\mathbb{RECT}) = \mathbf{HSP}(\mathrm{Fin}\mathbb{SQ})$.
(ii) $\mathsf{RCA}_2 = \mathbf{HSP}(\mathrm{Fin}\mathbb{CSQ})$.

Now we turn to lattices of subvarieties of two-dimensional cylindric algebras. Let $\Lambda(\mathsf{Df}_2)$ denote the lattice of subvarieties of Df_2, $\Lambda(\mathsf{CA}_2)$ denote the lattice of subvarieties of CA_2 and $\Lambda(\mathsf{RCA}_2)$ denote the lattice of subvarieties of RCA_2. We also let $\Lambda(\mathsf{Df}_1)$ denote the lattice of subvarieties of Df_1-algebras. This lattice is easy to describe. The lattice of all subvarieties of Df_1 is an $(\omega+1)$-chain that converges to Df_1 (see [44, Theorem 4] and [23, Theorem 4.1.22]). As we will see below, the lattice $\Lambda(\mathsf{Df}_2)$ is also countable, although much more complex than $\Lambda(\mathsf{Df}_1)$. The lattices $\Lambda(\mathsf{CA}_2)$ and $\Lambda(\mathsf{RCA}_2)$, however, are not countable [23, Theorem 4.1.27 and Remark 4.1.28]. We will sketch the proof of this fact by using the technique of Jankov-Fine formulas, which is a standard tool in modal logic. For an overview on Jankov-Fine formulas we refer to [12, Section 9.4], [10, Section 3.4] or [7, Section 3.4]. This proof will also underline the difference between the finite square Df_2 and CA_2-algebras.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be simple CA_2-algebras. We write

$$\mathfrak{B} \leq \mathfrak{B}' \quad \text{iff} \quad \mathfrak{B} \text{ is a subalgebra of } \mathfrak{B}'.$$

Next we construct simple $\leq$-antichains of finite simple RCA_2-algebras. It follows from [8] that there do not exist any infinite $\leq$-antichains of finite simple Df_2-algebras. For the proof of the next lemma we refer to [23, Theorem 4.1.27] (see also [6, Lemma 4.1] or [7, Lemma 7.1.13]).

Lemma 4.3. *Every two non-isomorphic finite cylindric square algebras are $\leq$-incomparable.*

Applying the standard technique of Jankov-Fine formulas to two-dimensional cylindric algebras [7, Section 7.1.2], as an immediate consequence of Lemma 4.3, we obtain the following result ([23, Theorem 4.1.27], (see also [6, Theorem 4.2] or [7, Theorem 7.1.14]).

Theorem 4.4. *The cardinality of* $\Lambda(\mathsf{RCA}_2)$ *is that of the continuum.*

Moreover, by replacing a singleton non-diagonal E_0-cluster with a two-element E_0-cluster in each finite cylindric square, we obtain an infinite $\leq$-antichain of finite rooted CA_2-spaces satisfying $(*)$. Therefore, the corresponding algebras do not belong to RCA_2. Applying Jankov-Fine formulas again we obtain the following result [23, Remark 4.1.28], (see also [6, Theorem 4.4, Corollary 4.5] or [7, Theorem 7.16, Corollary 7.1.17]).

Theorem 4.5.
(i) *The cardinality of* $\Lambda(\mathsf{CA}_2) - \Lambda(\mathsf{RCA}_2)$ *is that of the continuum.*
(ii) *There exists a continuum of varieties in between* RCA_2 *and* CA_2.

Note that there are only countably many finitely axiomatizable varieties and there are only countably many varieties with a decidable equational theory. Therefore, Theorem 4.4 also implies that there exists a continuum of non-finitely axiomatizable subvarieties of CA_2 (resp. of RCA_2), and there exists a continuum of subvarieties of CA_2 (resp. of RCA_2) with an undecidable equational theory. We also remark that for an uncountable α, CA_α has 2^α many subvarieties [40, p. 246]. It is still an open problem whether the same result holds for RCA_α [23, Problem 4.2] and [38]. We will see in the next section that $\Lambda(\mathsf{Df}_2)$ is countable.

5. Locally finite subvarieties of Df_2

In this section we investigate locally finite subvarieties of Df_2. We recall that a variety V is called *locally finite* if every finitely generated V-algebra is finite. It is well known (see e.g., Halmos [22]) that Df_1 is locally finite. It was Tarski who first noticed that Df_2 is not locally finite. Detailed proofs of this fact can be found in [23, Theorem 2.1.11], [22, p.92], [15], [7, Example 6.2.1].

Let $\mathcal{B}$ be a simple Df_2-algebra and $\mathcal{X}$ its dual rooted Df_2-space. Let also $i = 1, 2$ and $n > 0$. We say that $\mathcal{X}$ is of *E_i-depth n* if the number of E_i-clusters of $\mathcal{X}$ is exactly n. The E_i-depth of $\mathcal{X}$ is said to be *infinite* if $\mathcal{X}$ has infinitely many E_i-clusters. $\mathcal{B}$ is said to be of *E_i-depth $n < \omega$* if the E_i-depth of $\mathcal{X}$ is n. The E_i-depth of $\mathcal{B}$ is said to be *infinite* if $\mathcal{X}$ is of infinite E_i-depth. $\mathsf{V} \subseteq \mathsf{Df}_2$ is said to be of *E_i-depth $n < \omega$* if n is the maximal E_i-depth of the simple members of V, and V is of *E_i-depth ω* if there is no bound on the E_i-depth of simple members of V. For a simple Df_2-algebra $\mathcal{B}$ and its dual $\mathcal{X}$, let $d_i(\mathcal{B})$ and $d_i(\mathcal{X})$ denote the E_i-depth of $\mathcal{B}$ and $\mathcal{X}$, respectively. Similarly, let $d_i(\mathsf{V})$ denote the E_i-depth of a variety $\mathsf{V} \subseteq \mathsf{Df}_2$. We note that there exists a formula measuring the depth of a subvariety of Df_2 (see [5, Theorem 4.2] or [7, Theorem 6.2.4]).

Definition 5.1. *For a variety* V, *let* $\mathrm{SI}(\mathsf{V})$ *and* $\mathrm{S}(\mathsf{V})$ *denote the classes of all subdirectly irreducible and simple* V*-algebras, respectively. Let also* $\mathrm{FinSI}(\mathsf{V})$ *and* $\mathrm{FinS}(\mathsf{V})$ *denote the class of all finite subdirectly irreducible and simple* V*-algebras, respectively.*

We recall from [4] a criterion of local finiteness.

Theorem 5.2. *A variety* V *of a finite signature is locally finite iff the class* $\mathrm{SI}(\mathsf{V})$ *is uniformly locally finite; that is, for each natural number n there is a natural number $M(n)$ such that for each n-generated* $\mathcal{A} \in \mathrm{SI}(\mathsf{V})$ *we have* $|\mathcal{A}| \leq M(n)$.

The next theorem is an important tool in characterizing locally finite subvarieties of Df_2. Its proof, which can be found in [5, Lemma 4.4] or [7, Lemma 6.2.7], relies on the fact that the variety of Df_1-algebras is locally finite.

Theorem 5.3. *Every subvariety* $\mathsf{V} \subseteq \mathsf{Df}_2$ *such that* $d_1(\mathsf{V}) < \omega$ *or* $d_2(\mathsf{V}) < \omega$ *is locally finite.*

Now we are in a position to prove that every proper subvariety of Df_2 is locally finite.

Theorem 5.4. *If a variety* $\mathsf{V} \subseteq \mathsf{Df}_2$ *is not locally finite, then* $\mathsf{V} = \mathsf{Df}_2$.

Proof. We sketch the main idea of the proof. Suppose V is not locally finite. Then there exists a finitely generated infinite V-algebra $\mathcal{B}$. Let $\mathcal{X}$ be the dual of $\mathcal{B}$. Then either there exists an infinite rooted saturated subset of $\mathcal{X}$, or $\mathcal{X}$ consists of infinitely many finite rooted saturated subsets.

First suppose that $\mathcal{X}$ contains an infinite rooted saturated subset $\mathcal{X}_0$. If either the E_1 or E_2-depth of $\mathcal{X}_0$ is finite, then the Df_2-algebra, call it $\mathcal{B}_0$, corresponding to $\mathcal{X}_0$ belongs to some variety $\mathsf{V}' \subseteq \mathsf{Df}_2$ of a finite E_1 or E_2-depth. Then $\mathcal{B}_0$ is a homomorphic image of $\mathcal{B}$ and is finitely generated. Moreover, by our assumption, $\mathcal{X}_0$ and hence $\mathcal{B}_0$ is infinite. This is a contradiction, since by Theorem 5.3, V' is locally finite. Thus, both the E_1 and E_2-depths of $\mathcal{X}_0$ are infinite. Next we can show (see [5, Claim 4.7] or [7, Claim 6.2.10]) that for each $n \in \omega$, the square $\langle W \times W, E_1, E_2\rangle$ with $|W| = n$ is a Df_2-morphic image of $\mathcal{X}_0$. By duality, this means that the algebra $\langle \mathcal{P}(W \times W), E_1, E_2\rangle$ is a subalgebra of $\langle \mathcal{CP}(\mathcal{X}_0), E_1, E_2\rangle$ for each $n < \omega$. Since Df_2 is generated by finite square algebras (see Theorem 3.2) this implies that $\mathsf{V} = \mathsf{Df}_2$.

Now suppose that $\mathcal{X}$ consists of infinitely many finite rooted spaces which we denote by $\{\mathcal{X}_j\}_{j\in J}$. If either the E_1 or E_2-depth of the members of $\{\mathcal{X}_j\}_{j\in J}$ is bounded by some integer n, then their corresponding algebras belong to some variety $\mathsf{V}' \subseteq \mathsf{Df}_2$ with $d_1(\mathsf{V}') < n$ or $d_2(\mathsf{V}') < n$. This means that there is an infinite finitely generated algebra in V'. By Theorem 5.3, this is a contradiction. Therefore, we can assume that neither the E_1 nor E_2-depth of $\{\mathcal{X}_j\}_{j\in J}$ is bounded by any integer. Then we can again show that every finite square algebra is a subalgebra of $\langle \mathcal{CP}(\mathcal{X}_0), E_1, E_2\rangle$. This, by Theorem 3.2, means that $\mathsf{V} = \mathsf{Df}_2$.

Thus, if V is not locally finite, then $\mathsf{V} = \mathsf{Df}_2$, which completes the proof of the theorem. □

Recall that a variety V is called *pre-locally finite* if V is not locally finite but every proper subvariety of V is locally finite. We also recall that every locally finite variety is generated by its finite algebras. Therefore, we arrive at the following theorem.

Corollary 5.5.

(i) $\mathsf{V} \in \Lambda(\mathsf{Df}_2)$ *is locally finite iff* V *is a proper subvariety of* Df_2.

(ii) Df_2 *is the only pre-locally finite subvariety of* Df_2.

(iii) *Every variety* $\mathsf{V} \subseteq \mathsf{Df}_2$ *is generated by its finite algebras.*

In fact, Corollary 5.5(iii) can be significantly strengthened. It is proved in [9] (see also [7, Section 8.2]) that the (bi-)modal logic corresponding to every proper subvariety of Df_2 has the poly-size model property. Moreover, using combinatorial set theory, namely, the theory of better-quasi-orderings, [8] (see also [7, Section 8.1]) proves that every subvariety of Df_2 is finitely axiomatizable. Combining this with Corollary 5.5(iii) gives us that the equational theory of every subvariety of Df_2 is decidable. However, even more is true. It is proved in [8] (see also [7, Section 8.4]) that the equational theory of every subvariety of Df_2 is NP-complete.

We finish this section by mentioning the analogy of these results with those obtained by Monk [37] for two-dimensional polyadic algebras. Two-dimensional polyadic algebras are obtained by adding four extra unary operations to the signature of Df_2-algebras (see [23, Definition 5.4.1] or [37]). Using the methods very similar to ours Monk [37] proves that the variety PA_2 of two-dimensional polyadic algebras has only countably many subvarieties, each subvariety is finitely axiomatizable, is determined by its finite members and has a decidable equational theory. Moreover, similarly to Df_2, each proper subvariety of PA_2 is locally finite. As was noted in [23, Theorem 5.4.5 and Remark 5.4.6], the results on subvarieties of PA_2 do not immediately transfer to subvarieties of Df_2. For example, the subvarieties of Df_2 axiomatized by the equations $(\mathsf{c}_1 x = x)$ and $(\mathsf{c}_2 x = x)$, respectively, are distinct, while the subvarieties of PA_2 axiomatized by these equations coincide. This also means that PA_2 is not a conservative extension of Df_2.

6. Classification of subvarieties of Df_2

In this section we will see that Corollary 5.5 enables us to give a classification of subvarieties of Df_2 in terms of E_1 and E_2-depths (see [5, Section 4] or [7, Section 6.3]). It follows from Corollary 5.5(iii) that every subvariety V of Df_2 is generated by $\mathrm{FinS}(\mathsf{V})$.

Theorem 6.1. *For every proper subvariety* V *of* Df_2 *there exists a natural number* n *such that* $\mathrm{FinS}(\mathsf{V})$ *can be divided into three disjoint sets* $\mathrm{FinS}(\mathsf{V}) = \mathsf{F}_1 \uplus \mathsf{F}_2 \uplus \mathsf{F}_3$*, where* $d_2(\mathsf{F}_1),\ d_1(\mathsf{F}_2) \le n$ *and* $d_1(\mathsf{F}_3),\ d_2(\mathsf{F}_3) \le n$*. (Note that any two of the sets* $\mathsf{F}_1, \mathsf{F}_2$ *and* F_3 *may be empty.)*

Proof. We sketch the proof. Suppose V is a proper subvariety of Df_2. By Theorem 3.2, Df_2 is generated by finite square algebra. Therefore, there exists $n \in \omega$ and a square $\langle W \times W, E_1, E_2 \rangle$ such that $|W| = n$ and $\langle \mathcal{P}(W \times W), E_1, E_2 \rangle \notin \mathrm{FinS}(\mathsf{V})$. Let n be the minimal such number. We consider three subclasses of $\mathrm{FinS}(\mathsf{V})$: $\mathsf{F}_1 = \{\mathcal{B} \in \mathrm{FinS}(\mathsf{V}) : d_1(\mathcal{B}) > n\}$, $\mathsf{F}_2 = \{\mathcal{B} \in \mathrm{FinS}(\mathsf{V}) : d_2(\mathcal{B}) > n\}$ and $\mathsf{F}_3 = \{\mathcal{B} \in \mathrm{FinS}(\mathsf{V}) : d_1(\mathcal{B}), d_2(\mathcal{B}) \le n\}$. It is obvious that $\mathrm{FinS}(\mathsf{V}) = \mathsf{F}_1 \cup \mathsf{F}_2 \cup \mathsf{F}_3$. We prove that F_1, F_2 and F_3 are disjoint.

Let us show that if $\mathcal{B} \in \mathsf{F}_1$, then $d_2(\mathcal{B}) \le n$ and if $\mathcal{B} \in \mathsf{F}_2$, then $d_1(\mathcal{B}) \le n$. Suppose $\mathcal{B} \in \mathsf{F}_1 \cup \mathsf{F}_2$, $d_1(\mathcal{B}) = k$, $d_2(\mathcal{B}) = m$ and both $k, m > n$. Let $\mathcal{X}$ be the dual of $\mathcal{B}$. Then we can show that a finite square $\langle W \times W, E_1, E_2 \rangle$ such that $|W| = n$ is a Df_2-morphic image of $\mathcal{X}$. By duality, this means that the square algebra $\langle \mathcal{P}(W \times W), E_1, E_2 \rangle$ is a subalgebra of $\mathcal{B}$ and therefore belongs to $\mathrm{FinS}(\mathsf{V})$, which is a contradiction. Thus, $\mathcal{B} \in \mathsf{F}_1$ implies $d_1(\mathcal{B}) > n$ and $d_2(\mathcal{B}) \le n$, and $\mathcal{B} \in \mathsf{F}_2$ implies $d_1(\mathcal{B}) \le n$ and $d_2(\mathcal{B}) > n$. Also, if $\mathcal{B} \in \mathsf{F}_3$, then $d_1(\mathcal{B}), d_2(\mathcal{B}) \le n$. This shows that all the three sets are disjoint. □

From this theorem we obtain the following classification of subvarieties of Df_2 (see [5, Theorem 4.10] or [7, Theorem 6.3.4]).

Theorem 6.2. *For each* $\mathsf{V} \in \Lambda(\mathsf{Df}_2)$*, either* $\mathsf{V} = \mathsf{Df}_2$*, or* $\mathsf{V} = \bigvee_{i \in S} \mathsf{V}_i$ *for some* $S \subseteq \{1, 2, 3\}$*, where* $d_1(\mathsf{V}_1), d_2(\mathsf{V}_2), d_1(\mathsf{V}_3), d_2(\mathsf{V}_3) < \omega$*.*

Proof. The proof follows Theorem 6.1, by setting $\mathsf{V}_i = \mathbf{HSP}(\mathsf{F}_i)$ for $i = 1, 2, 3$. □

We close this section by recalling from [5, Section 6] a characterization of rectangularly and square representable subvarieties of Df_2. First we give a general definition of representability for varieties of Df_2-algebras.

Definition 6.3. *A variety* $\mathsf{V} \subseteq \mathsf{Df}_2$ *is called* representable by (algebras from class) $\mathsf{K} \subseteq \mathsf{Df}_2$ *if* $\mathsf{V} = \mathbf{SP}(\mathsf{K} \cap \mathsf{V})$*.*

For a variety $\mathsf{V} \subseteq \mathsf{Df}_2$, we denote by $(\mathbb{GRECT}_{\mathsf{V}})$ $\mathbb{RECT}_{\mathsf{V}}$ and $(\mathbb{GSQ}_{\mathsf{V}})$ $\mathbb{SQ}_{\mathsf{V}}$ the classes of (generalized) rectangular and (generalized) square V-algebras, respectively. By Definition 6.3, $\mathsf{V} \subseteq \mathsf{Df}_2$ is *rectangularly representable* if $\mathsf{V} = \mathbf{SP}(\mathbb{RECT}_{\mathsf{V}})$ and V is *square representable* if $\mathsf{V} =$

$\mathbf{SP}(\mathbb{SQ}_{\mathsf{V}})$. Moreover, we have that $\mathsf{V} \subseteq \mathsf{Df}_2$ is *rectangularly representable* iff $\mathsf{V} = \mathbf{S}(\mathbb{GRECT}_{\mathsf{V}})$ and V is *square representable* iff $\mathsf{V} = \mathbf{S}(\mathbb{GSQ}_{\mathsf{V}})$. (This is a consequence of a general result of modal logic concerning a duality between products of complete and atomic modal algebras and disjoint unions of corresponding frames; see e.g., [49, Section 5.6].). Thus, square and rectangular representability in the subvarieties of Df_2 are restricted versions of the general definition of representability (Definition 3.1).

For positive integers m and n let $\mathcal{P}(m \times n)$ denote the rectangular algebra $\langle \mathcal{P}(W \times W'), E_1, E_2 \rangle$ such that $|W| = m$ and $|W'| = n$. Let n_1 and n_2 be positive integers. We let $\mathsf{V}_{(\omega, n_1)} = \mathbf{HSP}(\{\mathcal{P}(m \times n_1)\}_{m \in \omega})$ and $\mathsf{V}_{(n_2, \omega)} = \mathbf{HSP}(\{\mathcal{P}(n_2 \times m)\}_{m \in \omega})$. The next theorem provides a full characterization of rectangular and square representable subvarieties of Df_2.

Theorem 6.4. *Let* $\mathsf{V} \in \Lambda(\mathsf{Df}_2)$.

(i) V *is square representable iff* $\mathsf{V} = \mathsf{Df}_2$ *or* $\mathsf{V} = \mathbf{HSP}(\mathcal{P}(n \times n))$ *for some* $n \in \omega$.

(ii) V *is rectangularly representable iff* $\mathsf{V} = \mathsf{Df}_2$ *or* $\mathsf{V} = \mathsf{V}_{(\omega, n_1)} \vee \mathsf{V}_{(n_2, \omega)} \vee \mathsf{V}'$, *where* $\mathsf{V}' = \bigvee_{i=1}^{r} \mathbf{HSP}(\mathcal{P}(m_i \times k_i))$ *for some* $m_i, k_i, r \in \omega$.

7. Locally finite subvarieties of CA_2

In the previous section we proved that Df_2 is pre-locally finite. It is known (see, e.g., [23, Theorem 2.1.11]) that RCA_2, and hence every variety in the interval $[\mathsf{RCA}_2, \mathsf{CA}_2]$, is not locally finite. In this section, we present a criterion of local finiteness for varieties of CA_2-algebras (see [6, Section 5] or [7, Section 7.2]). We also show that there exists exactly one pre-locally finite subvariety of CA_2. The E_1 and E_2-depths of simple CA_2-algebras are defined as in the Df_2-case. Since the number of the E_1 and E_2-clusters in every CA_2-space is the same, for each simple CA_2-algebra $\mathfrak{B}$, we have $d_1(\mathfrak{B}) = d_2(\mathfrak{B})$. We denote it by $d(\mathfrak{B})$ and call it the *depth* of $\mathfrak{B}$. The depth of a variety of CA_2-algebras we denote by $d(\mathsf{V})$. Our goal is to show that a variety V of CA_2-algebras is locally finite iff $d(\mathsf{V}) < \omega$. For this we need the following definition.

Definition 7.1.

(i) *Call a rooted* CA_2*-space* $\mathcal{X}$ uniform *if every non-diagonal* E_0*-cluster of* $\mathcal{X}$ *is a singleton set, and every diagonal* E_0*-cluster of* $\mathcal{X}$ *contains only two points.*

(ii) *Call a simple* CA_2*-algebra* $\mathfrak{B}$ uniform *if its dual rooted* CA_2*-space* $\mathcal{X}$ *is uniform.*

Finite uniform rooted spaces are shown in Figure 2, where big dots denote the diagonal points. Let $\mathcal{X}_n$ denote the uniform rooted space of depth n. Also let $\mathfrak{B}_n$ denote the uniform CA_2-algebra of depth n. It is obvious that $\mathcal{X}_n$ is (isomorphic to) the dual CA_2-space of $\mathfrak{B}_n$. Let U denote the variety generated by all finite uniform CA_2-algebras; that is $\mathsf{U} = \mathbf{HSP}(\{\mathfrak{B}_n\}_{n \in \omega})$. Applying the criterion of Theorem 3.5, it is easy to check that $\mathsf{U} \subseteq \mathsf{RCA}_2$. For the proof of the next lemma we refer to [6, Lemma 5.2] or [7, Lemma 7.2.4].

Lemma 7.2.

(i) *If* $\mathfrak{B}$ *is a simple cylindric algebra of infinite depth, then each* $\mathfrak{B}_n$ *is a subalgebra of* $\mathfrak{B}$.

(ii) *If* $\mathfrak{B}$ *is a simple cylindric algebra of depth* $2n$, *then* $\mathfrak{B}_n$ *is a subalgebra of* $\mathfrak{B}$.

Now we characterize varieties of CA_2-algebras of infinite depth in terms of U.

Theorem 7.3. *For a variety* V *of* CA_2*-algebras,* $d(\mathsf{V}) = \omega$ *iff* $\mathsf{U} \subseteq \mathsf{V}$.

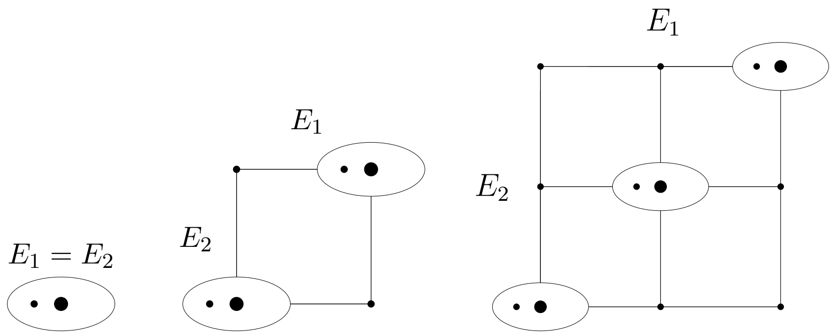

FIGURE 2. Uniform rooted CA_2-spaces

Proof. It is obvious that $d(\mathsf{U}) = \omega$. So, if $\mathsf{U} \subseteq \mathsf{V}$, then obviously $d(\mathsf{V}) = \omega$. Conversely, suppose $d(\mathsf{V}) = \omega$. We want to show that every finite uniform CA_2-algebra belongs to V. Since $d(\mathsf{V}) = \omega$, the depth of the simple members of V is not bounded by any integer. So, either there exists a family of simple V-algebras of increasing finite depth, or there exists a simple V-algebra of infinite depth. In either case, it follows from Lemma 7.2 that $\{\mathfrak{B}_n\}_{n\in\omega} \subseteq \mathsf{V}$. Therefore, $\mathsf{U} \subseteq \mathsf{V}$, since $\{\mathfrak{B}_n\}_{n\in\omega}$ generates U. □

Our next task is to show that U is not locally finite. For this we first need to observe that every finite uniform algebra is 1-generated (see [6, Lemma 5.6] or [7, Lemma 7.2.6]). Now Theorem 5.2 immediately implies that U is not locally finite. Next, using the fact that the variety of Boolean algebras is locally finite, we show that varieties of CA_2-algebras of finite depth are locally finite. For the proof of this result we refer to [6, Theorem 5.10] or [7, Theorem 7.2.9].

Theorem 7.4. *If $d(\mathsf{V}) < \omega$, then V is locally finite.*

We note that Theorem 7.4 is a CA_2-analogue of Theorem 5.3. Its proof, however, relies on local finiteness of Boolean algebras, whereas the proof of Theorem 5.3 uses the fact that Df_1 is locally finite. Finally, combining Theorems 7.3 and 7.4, we obtain the following characterization of locally finite varieties of CA_2-algebras.

Theorem 7.5.
(i) *For $\mathsf{V} \in \Lambda(\mathsf{CA}_2)$ the following conditions are equivalent:*
 (*a*) V *is locally finite,*
 (*b*) $d(\mathsf{V}) < \omega$,
 (*c*) $\mathsf{U} \not\subseteq \mathsf{V}$.
(ii) U *is the only pre-locally finite subvariety of* CA_2.

Therefore, in contrast to the diagonal-free case, there exist uncountably many subvarieties of CA_2 (RCA_2) which are not locally finite. Since every locally finite variety is generated by its finite algebras we obtain from Theorem 7.5 that every subvariety of CA_2 of finite depth is generated by its finite algebras. We leave it as an open problem whether each subvariety of CA_2 is generated by its finite algebras.

8. Finitely generated varieties of cylindric algebras

Recall that a variety is called *finitely generated* if it is generated by a single finite algebra, and that a variety is called *pre-finitely generated* if it is not finitely generated, but all its proper subvarieties are finitely generated. In studying a lattice of subvarieties of a given variety, finitely generated and pre-finitely generated varieties play an important role. Finitely generated varieties constitute the 'lower' part of this lattice, whereas pre-finitely generated ones are borderlines between the finitely generated and non-finitely generated ones. Pre-finitely generated varieties are minimal among non-finitely generated ones. Moreover, an explicit description of pre-finitely generated varieties provides a criterion for characterizing finitely generated varieties. For varieties of modal and Heyting algebras there are few well-known characterizations of pre-finitely generated varieties. Maksimova [29] showed that there are exactly three pre-finitely generated varieties of Heyting algebras. Maksimova [30] and Esakia and Meskhi [16] proved that there are exactly five pre-finitely generated varieties of **S4**-algebras. Blok [11] showed that there is a continuum of pre-finitely generated varieties of **K4**-algebras. On the other hand, Df_1 is the only pre-finitely generated variety in the lattice of subvarieties of Df_1. For Df_2 and CA_2 the picture is more complex than for Df_1. As follows from [5, Theorem 5.4] and [7, Corollary 6.4.7] there are exactly six pre-finitely generated varieties in $\Lambda(\mathsf{Df}_2)$, there are exactly fifteen pre-finitely generated varieties in $\Lambda(\mathsf{CA}_2)$, and six of them belong to $\Lambda(\mathsf{RCA}_2)$ (see [6, Corollary 6.6] and [7, Corollary 7.3.7]). These results yield a characterization of finitely generated varieties of Df_2 and CA_2-algebras. A variety V is finitely generated iff none of the pre-finitely generated varieties is a subvariety of V. Another characterization of finitely generated varieties of Df_2 and CA_2-algebras can be found in [5, Section 5] and [6, Section 7].

In Section 6 we gave a classification of subvarieties of Df_2. We close this section with a very rough description of $\Lambda(\mathsf{CA}_2)$. We need the following notation: Let FG denote the class of all finitely generated subvarieties of CA_2. Also let D_F denote the class of varieties of CA_2-algebras of finite depth which are not finitely generated varieties and let D_ω denote the class of varieties of CA_2-algebras of infinite depth.

It follows from the results discussed in Sections 7 and 8 that the variety V_{tr} generated by the trivial CA_2-algebra is the least element of FG, that FG does not have maximal elements, that D_F has precisely fifteen minimal elements, that D_F does not have maximal elements, and that U and CA_2 are the least and greatest elements of D_ω, respectively.

The detailed investigation of the 'lower' part of $\Lambda(\mathsf{CA}_2)$ can be found in [6, Section 7]. In particular, a complete characterization of the lattice structure of the extensions of CA_2 of depth one is given in [6, Section 7.1]. Using the reduct functor $\mathfrak{Df} : \mathsf{CA}_2 \to \mathsf{Df}_2$, we can define a reduct functor from the lattice $\Lambda(\mathsf{CA}_2)$ into the lattice $\Lambda(\mathsf{Df}_2)$. This reduct functor and the properties that are preserved and reflected by it are investigated in [6, Section 7].

9. Open problems

We close this chapter by listing some open problems.

(i) As we saw in Section 5, every subvariety of Df_2 is generated by its finite members. Moreover, corresponding logical systems have the poly-size model property and NP-complete satisfiability problem. The same question for CA_2-algebras remains open. Every subvariety of CA_2 of finite depth is locally finite and therefore is generated by its finite algebras. However, it is still an open (and rather complicated) question whether every subvariety of CA_2 and RCA_2 of infinite depth is generated by its finite algebras.

(ii) Subvarieties of CA_α (for both infinite and finite α) are investigated in [40, Section 1.1]. In particular, a characterization of subvarieties of CA_ω with decidable equational theories is given in [40]. An existence of such a characterization for finite α (especially in the case $\alpha = 2$) remains an open problem.

(iii) That Df_2 (resp. CA_2 and RCA_2) does not have the amalgamation property was first noticed by Comer [14] (see also Sain [42] and Marx [32]). In particular, it follows from the proof of this result that every subvariety V of Df_2 (resp. of CA_2 and RCA_2) such that $d_1(\mathsf{V}) > 2$ or $d_2(\mathsf{V}) > 2$ lacks the amalgamation property. We leave it as an open problem to give a full characterization of subvarieties of Df_2 (resp. CA_2 and RCA_2) with the amalgamation property.

(iv) In this chapter we considered two types of two-dimensional algebras: diagonal-free cylindric algebras and cylindric algebras with the diagonal. However, in order to get the full two-variable fragment of FOL (with substitution), we could have added to our signature four extra unary operations (analogous to substitutions of first-order variables) used in polyadic algebras. Then in addition to Df_2-algebras and CA_2-algebras we would have two more similarity types of two-dimensional cylindric-like algebras: *two-dimensional polyadic algebras* (PA_2-algebras) and *two-dimensional polyadic equality algebras* (PEA_2-algebras), see [23, Section 5.4]. A PA_2-like similarity type was also considered in [45].

As was pointed out earlier, it was shown by Monk [37] that subvarieties of the variety of PA_2-algebras have very similar (good) properties as subvarieties of Df_2-algebras. We leave it as an open problem to investigate the lattice of varieties of PEA_2-algebras and compare it with the lattices of varieties of PA_2-algebras, Df_2-algebras and CA_2-algebras. We also suggest studying the obvious reduct functors arising between these lattices. We conjecture that in the same way most of the properties of (varieties of) Df_2-algebras hold for (varieties of) PA_2-algebras, most of the properties of (varieties of) CA_2-algebras would hold for (varieties of) PEA_2-algebras.

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