CANONICAL FORMULAS FOR wK4

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Abstract. We generalize the theory of canonical formulas for **K4** (the logic of transitive frames) to **wK4** (the logic of weakly transitive frames). Our main result establishes that each logic over **wK4** is axiomatizable by canonical formulas, thus generalizing Zakharyaschev's theorem for logics over **K4**. The key new ingredients include the concepts of transitive and strongly cofinal subframes of weakly transitive spaces. This yields, along with the standard notions of subframe and cofinal subframe logics, the new notions of transitive subframe and strongly cofinal subframe logics over **wK4**. We obtain axiomatizations of all four kinds of subframe logics over **wK4**. We conclude by giving a number of examples of different kinds of subframe logics over **wK4**.

1. Introduction

Axiomatizability, the finite model property (FMP), and decidability are some of the most frequently studied properties of modal logics. There are a number of general results stating that large families of modal logics are finitely axiomatizable, have the FMP, and hence are decidable. Probably the first such general result was obtained by Scroggs [24] who proved that all logics over **S5** are finitely axiomatizable, have the FMP, and hence are decidable. Scroggs' proof was algebraic. Using frame-theoretic methods, Fine [15] proved that each logic over **S4.3** is finitely axiomatizable, has the FMP, and hence is decidable (that each such logic has the FMP was proved earlier by Bull [11] using algebraic methods). As **S5** is an extension of **S4.3**, Scroggs' result follows. In [16] Fine developed the technique of frame formulas for logics over **S4**, which allowed him to axiomatize large classes of logics over **S4**. This technique generalizes to logics over **K4**. A similar axiomatization of large classes of superintuitionistic logics was obtained earlier by Jankov [18] using algebraic methods. These classes of logics are known as splitting and join-splitting logics. Jankov's method was generalized to n-transitive modal logics by Rautenberg [23]. A number of deep results about the structure of the lattice of normal modal logics was obtained by Blok using the technique of splittings (see, e.g., [9, 10]). Further interesting results on splittings were obtained by Kracht (see, e.g., [19, 20]) and Wolter (see, e.g., [25, 26]).

In [17] Fine introduced the concept of a subframe logic over **K4**, axiomatized all subframe logics over **K4** by means of subframe formulas, and proved that each subframe logic over **K4** has the FMP. Fine's line of research was generalized by Zakharyaschev. In [29] Zakharyaschev introduced the concept of a cofinal subframe logic over **K4** (which generalizes the concept of a subframe logic over **K4**), axiomatized all cofinal subframe logics over **K4** by means of cofinal subframe formulas (which generalize subframe formulas), and proved that each cofinal subframe logic over **K4** has the FMP. This, in particular, implies the Bull-Fine theorem that all logics over **S4.3** have the FMP because each logic over **S4.3** is a cofinal subframe logic. In [28] Zakharyaschev developed the technique of canonical formulas for **K4** and proved that all logics over **K4** are axiomatizable by canonical formulas. The technique of canonical formulas for superintuitionistic logics was developed earlier by Zakharyaschev in [27], where it was shown that all superintuitionistic logics are axiomatizable by canonical formulas.

The algebraic technique of Jankov, Rautenberg, and Blok is closely related to the frame-theoretic technique of Fine and Zakharyaschev via the duality between modal algebras and modal spaces

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(descriptive Kripke frames). In particular, the splitting formulas developed by Jankov for superintuitionistic logics and generalized by Rautenberg to modal logics provide an algebraic version of Fine's frame formulas. Zakharyaschev's canonical formulas generalize Fine's frame formulas, as well as subframe and cofinal subframe formulas. An algebraic version of Zakharyaschev's canonical formulas for superintuitionistic logics was developed in [1], where Jankov's formulas were generalized and it was shown that the resulting formulas are equivalent to Zakharyaschev's canonical formulas for superintuitionistic logics via the generalized Esakia duality for Heyting algebras. The key ingredient of this generalized duality is the dual characterization of partial p-morphisms by means of $(\wedge, \to)$-preserving homomorphisms. An algebraic version of Zakharyaschev's canonical formulas for **K4** was developed in [2], where the Jankov-Rautenberg formulas for **K4** were generalized and it was shown that the resulting formulas are equivalent to Zakharyaschev's canonical formulas for **K4** via the generalized duality between modal algebras and modal spaces. The key ingredient of this generalized duality is the characterization of partial p-morphisms by means of relativized modal algebra homomorphisms.

The aim of this paper is to generalize the theory of canonical formulas for **K4** to **wK4**. It is well known that $\mathbf{K4} = \mathbf{K} + (\Diamond\Diamond p \to \Diamond p)$ is the logic of all transitive frames. On the other hand, $\mathbf{wK4} = \mathbf{K} + (\Diamond\Diamond p \to p \vee \Diamond p)$ is the logic of all weakly transitive frames, where a frame $\mathfrak{F} = (W, R)$ is weakly transitive if

$$(\forall w, v, u \in W)(wRv \wedge vRu \Rightarrow w = u \vee wRu).$$

In Rautenberg's terminology [23], **wK4** is the least 1-transitive modal logic.[1] The logic **wK4** plays an important role in the topological semantics of modal logic. As was shown by Esakia [14], if we interpret $\Diamond$ as topological derivative, then **wK4** is the logic of all topological spaces, while **K4** is the logic of all T_d-spaces (the spaces in which each point is locally closed). There are continuum many logics in the interval $[\mathbf{wK4}, \mathbf{K4}]$. In particular, the logic $\mathbf{wK4T_0}$ of all T_0-spaces, which was axiomatized in [3], belongs to this interval.

We view this paper as part of the program that develops an algebraic approach to canonical formulas. The key ingredient of the program is to find appropriate (generalized) dualities and put algebraic and frame-theoretic approaches in the context of these dualities, making them different sides of the same coin. As we already pointed out, an algebraic approach to canonical formulas for superintuitionistic logics was developed in [1]. It utilized the algebraic proof of the FMP for all (cofinal) subframe superintuitionistic logics given in [4]. An algebraic approach to canonical formulas for logics over **K4** was developed in [2]. It was based on the algebraic proof of the FMP for all (cofinal) subframe logics over **K4** developed in [5]. But [5] actually proved more, that all (cofinal) subframe logics over **wK4** have the FMP. This result will play a substantial role in our considerations.

Although our arguments mostly follow the same pattern as in [2], the generalization of the method developed for **K4** to **wK4** is not straightforward. Zakharyaschev's notion of a cofinal subframe that works for the transitive case is not sufficiently strong for the weakly transitive case. This is because each subframe of a transitive space (transitive descriptive Kripke frame) is automatically transitive, while there are subframes of weakly transitive spaces (weakly transitive descriptive Kripke frames) that are not transitive. Thus, we introduce a new notion of a transitive subframe of a weakly transitive space. We call a subframe of a weakly transitive space strongly cofinal if it is both transitive and cofinal. We give an algebraic characterization of strongly cofinal subframes, which allows us to introduce canonical formulas for finite subdirectly irreducible **wK4**-algebras. Using the results of [2] and [5], we prove that every logic over **wK4** is axiomatizable by canonical formulas, thus generalizing Zakharyaschev's theorem. We also give an algebraic characterization of transitive subframes, and show that negation-free canonical formulas for **wK4** are closely linked to

[1] Kracht [21, Sec. 2.5] calls an n-transitive modal logic weakly transitive, so his terminology is different from ours.

transitive subframes of weakly transitive spaces. As a result, we prove that every logic over **wK4** axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas.

Our considerations yield four different notions of subframes of weakly transitive spaces: subframes, transitive subframes, cofinal subframes, and strongly cofinal subframes. These four notions give rise to four different classes of subframe logics over **wK4**. We give algebraic characterizations of these four notions of subframes, as well as axiomatize the corresponding four classes of subframe logics. This provides a generalization of [2], where subframe and cofinal subframe logics over **K4** were axiomatized using algebraic methods. The key ingredient of the proof is the FMP for (cofinal) subframe logics over **wK4** [5] and the technique of frame-based formulas of [6, 7]. We conclude the paper by giving a number of examples of subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe logics over **wK4**. These examples underline similarities and differences between different kinds of subframe logics over **wK4** and **K4**.

The paper is organized as follows. In Section 2 we recall the generalized duality for modal algebras developed in [2]. In Section 3 we restrict our attention to **wK4**-algebras and weakly transitive spaces, and develop Zakharyaschev's closed domain condition (CDC) for **wK4**. In Section 4 we introduce the notions of transitive and strongly cofinal subframes of weakly transitive spaces, and establish their main properties. In Section 5 we define canonical formulas for finite subdirectly irreducible **wK4**-algebras, and show that every logic over **wK4** is axiomatizable by canonical formulas. In Section 6 we define negation-free canonical formulas, and show that every logic over **wK4** axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas. Section 7 consists of three subsections. In the first subsection we generalize the technique of frame-based formulas of [6, 7] to **wK4**; in the second subsection we discuss how to arrive at Rautenberg's axiomatization of splitting and join-splitting logics over **wK4** from our results; and in the third subsection we axiomatize subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe logics over **wK4**. In Section 8 we give several examples of subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe logics over **wK4**. Finally, in Section 9 we summarize our results and discuss possible venues for further research.

2. Preliminaries

We recall (see, e.g., [8, 12, 21]) that a *modal algebra* is a pair $(A, \Diamond)$, where A is a Boolean algebra and $\Diamond$ is a unary function on A satisfying $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for all $a, b \in A$. As usual, we define $\Box : A \to A$ by $\Box a = \neg\Diamond\neg a$. Then it is easy to see that $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, and $\Diamond a = \neg\Box\neg a$. When no confusion arises we denote a modal algebra $(A, \Diamond)$ simply by A. Let A and B be modal algebras. We recall that a map $\eta : A \to B$ is a *modal algebra homomorphism* if η is a Boolean algebra homomorphism and $\eta(\Diamond a) = \Diamond\eta(a)$ for all $a \in A$. Let MA denote the category of modal algebras and modal algebra homomorphisms.

Let A be a modal algebra and $s \in A$. We recall (see, e.g., [25, 5, 2]) that the *relativization* of A to s is the modal algebra $A_s = \{x \in A : x \le s\}$, where $0_s = 0$, $1_s = s$, $x \wedge_s y = x \wedge y$, $x \vee_s y = x \vee y$, $\neg_s x = s \wedge \neg x$, and $\Diamond_s x = s \wedge \Diamond x$ for all $x, y \in A_s$. For modal algebras A and B, a map $\eta : A \to B$ is a *relativized modal algebra homomorphism* if η is a modal algebra homomorphism from A to $B_{\eta(1)}$. Clearly η is a modal algebra homomorphism iff $\eta(1) = 1$. As follows from [2, Sec. 3], $\eta : A \to B$ is a relativized modal algebra homomorphism iff $\eta(0) = 0$, $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$, $\eta(a \vee b) = \eta(a) \vee \eta(b)$, and $\eta(\Diamond a) = \Diamond_{\eta(1)}\eta(a)$. We let MA^{R} denote the category of modal algebras and relativized modal algebra homomorphisms. Then MA is a (non-full) subcategory of MA^{R}.

We recall (see, e.g., [5]) that a *modal space* (descriptive Kripke frame) is a pair (X, R), where X is a Stone space (zero-dimensional compact Hausdorff space) and R is a binary relation on X such that the R-image $R[x]$ of each $x \in X$ is closed and the R-inverse image $R^{-1}[U]$ of each clopen $U \subseteq X$ is clopen. When no confusion arises we denote a modal space (X, R) simply by X. Let X and Y be modal spaces. We recall that a map $f : X \to Y$ is a *p-morphism* if $f(R[x]) = R[f(x)]$

for all $x \in X$. We let MS denote the category of modal spaces and continuous p-morphisms. It is a classic result in modal logic that MA is dually equivalent to MS.

Let X and Y be modal spaces and $f : X \to Y$ be a partial map. We recall [2] that f is a *partial continuous map* if $\mathrm{dom}(f)$ is a clopen subset of X and f is a continuous map from $\mathrm{dom}(f)$ to Y. We also recall that f is a *partial continuous p-morphism* if f is a partial continuous map and $f(R[x]) = R[f(x)]$ for all $x \in \mathrm{dom}(f)$. We let MS^P denote the category of modal spaces and partial continuous p-morphisms. Then MS is a (non-full) subcategory of MA^P. Moreover, as follows from [2, Thm. 3.4], MA^R is dually equivalent to MS^P.

We briefly recall the functors $(-)_* : \mathsf{MA}^\mathsf{R} \to \mathsf{MS}^\mathsf{P}$ and $(-)^* : \mathsf{MS}^\mathsf{P} \to \mathsf{MA}^\mathsf{R}$ that establish a dual equivalence of MA^R and MS^P. For a modal algebra A, let X_A be the set of ultrafilters of A. For $a \in A$, let $\varphi(a) = \{x \in X_A : a \in x\}$. Then $\{\varphi(a) : a \in A\}$ is a basis for the topology τ_A on X_A. Define R_A on X_A by xR_Ay iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$. Then $A_* = (X_A, R_A)$ is a modal space. For a relativized modal algebra homomorphism $\eta : A \to B$, let $\eta_* : B_* \to A_*$ be the partial map such that $\mathrm{dom}(\eta_*) = \varphi(\eta(1))$ and for $x \in \mathrm{dom}(\eta_*)$ we have $\eta_*(x) = \eta^{-1}(x)$. Then $\eta_* : B_* \to A_*$ is a partial continuous p-morphism. This defines the functor $(-)_* : \mathsf{MA}^\mathsf{R} \to \mathsf{MS}^\mathsf{P}$.

For a modal space X, let $\mathsf{Cp}(X)$ be the Boolean algebra of clopen subsets of X, and let $X^* = (\mathsf{Cp}(X), \Diamond_R)$, where $\Diamond_R(U) = R^{-1}[U]$. Then X^* is a modal algebra. For a partial continuous p-morphism $f : X \to Y$, let $f^* : Y^* \to X^*$ be given by $f^*(U) = f^{-1}(U)$. Then $f^* : Y^* \to X^*$ is a relativized modal algebra homomorphism. This defines the functor $(-)^* : \mathsf{MS}^\mathsf{P} \to \mathsf{MA}^\mathsf{R}$. Moreover, for each $A \in \mathsf{MA}^\mathsf{R}$, we have $\varphi : A \to {A_*}^*$ is a natural isomorphism in MA^R. Also, for each $X \in \mathsf{MS}^\mathsf{P}$, let $\varepsilon : X \to {X^*}_*$ be given by $\varepsilon(x) = \{U \in X^* : x \in U\}$. Then ε is a natural isomorphism in MS^P, and so the functors $(-)_*$ and $(-)^*$ establish a dual equivalence between MA^R and MS^P.

3. wK4-algebras and the closed domain condition

Let A be a modal algebra. We recall (see, e.g., [5]) that A is a **wK4**-*algebra* if $\Diamond\Diamond a \le a \vee \Diamond a$, that A is a **K4**-*algebra* if $\Diamond\Diamond a \le \Diamond a$, and that A is an **S4**-*algebra* if A is a **K4**-algebra and $a \le \Diamond a$. Let $\mathsf{wK4}$ denote the category of **wK4**-algebras and modal algebra homomorphisms, $\mathsf{K4}$ denote the category of **K4**-algebras and modal algebra homomorphisms, and $\mathsf{S4}$ denote the category of **S4**-algebras and modal algebra homomorphisms. Clearly $\mathsf{S4}$ is a full subcategory of $\mathsf{K4}$, $\mathsf{K4}$ is a full subcategory of $\mathsf{wK4}$, and $\mathsf{wK4}$ is a full subcategory of MA.

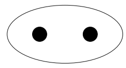
Let X be a modal space. We recall (see, e.g., [5]) that X is *weakly transitive* if R is weakly transitive, that X is *transitive* if R is transitive, and that X is *reflexive and transitive* if R is reflexive and transitive. Let wTS denote the category of weakly transitive spaces and continuous p-morphisms, TS denote the category of transitive spaces and continuous p-morphisms, and RTS denote the category of reflexive and transitive spaces and continuous p-morphisms. Clearly RTS is a full subcategory of TS, TS is a full subcategory of wTS, and wTS is a full subcategory of MS. The next theorem is well known (see, e.g., [5, Thm. 3.4]).

Theorem 3.1. $\mathsf{wK4}$ *is dually equivalent to* wTS*,* $\mathsf{K4}$ *is dually equivalent to* TS*, and* $\mathsf{S4}$ *is dually equivalent to* RTS*.*

We also have that a relativization of a **wK4**-algebra is a **wK4**-algebra, a relativization of a **K4**-algebra is a **K4**-algebra, and a relativization of an **S4**-algebra is an **S4**-algebra [5, Lem. 4.8].

Next we recall the closed domain condition for transitive spaces. Let X be a transitive space and let $R^+ = R \cup \{(x, x) : x \in X\}$ be the reflexive closure of R. We recall [2, p. 104] that $\mathfrak{d}$ is a *quasi-antichain* in X if xRy implies yRx for each $x, y \in \mathfrak{d}$. Let $\mathfrak{D}$ be a (possibly empty) set of quasi-antichains in X. Suppose that Y is a transitive space and $f : X \to Y$ is a partial continuous p-morphism. We say that f satisfies the *closed domain condition (CDC)* for $\mathfrak{D}$ if $x \notin \mathrm{dom}(f)$ implies $f(R[x]) \ne R^+[\mathfrak{d}]$ for each $\mathfrak{d} \in \mathfrak{D}$.

Let $S \subseteq Y$. We recall (see, e.g., [2, Sec. 4]) that $x \in S$ is *minimal* if yRx implies xRy for each $y \in S$. Let $\min(S)$ denote the set of minimal points of S. For U a clopen subset of Y,



X_2^{ir}

FIGURE 1. The weakly transitive space X_2^{ir}

let $\mathfrak{D}_U = \{\min f(R[x]) : f(R[x]) \cap U \neq \emptyset\}$. By [2, Lem. 4.3], the following four conditions are equivalent: (i) f satisfies (CDC) for $\mathfrak{D}_U$, (ii) $x \notin \mathrm{dom}(f)$ implies $\min f(R[x]) \notin \mathfrak{D}_U$, (iii) $x \notin \mathrm{dom}(f)$ implies $f(R[x]) \cap U = \emptyset$, and (iv) $\Diamond_R(f^{-1}(U)) \subseteq f^{-1}(\Diamond_R(U))$. The key ingredient of the proof is that if X and Y are transitive spaces and $f : X \to Y$ is a partial continuous p-morphism, then $f(R[x]) = R^+[\min f(R[x])]$ for each $x \in X$. As the next example shows, this property does not necessarily hold in weakly transitive spaces.

Example 3.2. Let X_2^{ir} consist of two irreflexive points x and y such that xRy and yRx; see Figure 1. Clearly X_2^{ir} is a non-transitive weakly transitive space. Let $f : X_2^{\mathrm{ir}} \to X_2^{\mathrm{ir}}$ be the identity map. Then f is a partial (continuous) p-morphism, which is total. Moreover, $R[x] = \{y\}$, $\min(R[x]) = \{y\}$, and $R^+[\min(R[x])] = X_2^{\mathrm{ir}}$. Thus, $f(R[x]) = \{y\}$ and $R^+[\min f(R[x])] = X_2^{\mathrm{ir}}$, implying that $f(R[x]) \neq R^+[\min f(R[x])]$.

Nevertheless, the following weaker statement holds for weakly transitive spaces.

Lemma 3.3. *Let X and Y be weakly transitive spaces and let $f : X \to Y$ be a partial continuous p-morphism. If $x \notin \mathrm{dom}(f)$, then $f(R[x]) = R^+[\min f(R[x])]$.*

Proof. As $R[x]$ is closed and $\mathrm{dom}(f)$ is clopen, $R[x] \cap \mathrm{dom}(f)$ is closed in $\mathrm{dom}(f)$. Therefore, $f(R[x])$ is closed in Y. As (Y, R^+) is a reflexive and transitive space [5, Lem. 3.6] and $f(R[x])$ is closed in Y, for each $u \in f(R[x])$ there exists $v \in \min f(R[x])$ such that vR^+u [13, Sec. III.2]. Thus, $f(R[x]) \subseteq R^+[\min f(R[x])]$. Conversely, let $u \in R^+[\min f(R[x])]$. Then there exists $v \in \min f(R[x])$ such that vR^+u. Therefore, $v = u$ or vRu. If $v = u$, then $u \in f(R[x])$. Suppose that vRu. As $v \in \min f(R[x]) \subseteq f(R[x])$, there exists $y \in \mathrm{dom}(f)$ such that xRy and $f(y) = v$. Since f is a partial p-morphism, there exists $z \in \mathrm{dom}(f)$ such that yRz and $f(z) = u$. Because R is weakly transitive, xRy and yRz imply $x = z$ or xRz. As $x \notin \mathrm{dom}(f)$ and $z \in \mathrm{dom}(f)$, we have $x \neq z$. Therefore, xRz. Thus, $u \in f(R[x])$, and so $R^+[\min f(R[x])] \subseteq f(R[x])$. Consequently, $f(R[x]) = R^+[\min f(R[x])]$. □

Lemma 3.4. *Let X and Y be weakly transitive spaces, $f : X \to Y$ be a partial continuous p-morphism, U be a clopen subset of Y, and $\mathfrak{D}_U = \{\min f(R[x]) : f(R[x]) \cap U \neq \emptyset\}$. Then the following four conditions are equivalent.*

(1) $x \notin \mathrm{dom}(f)$ *implies* $f(R[x]) \neq R^+[\mathfrak{d}]$ *for each* $\mathfrak{d} \in \mathfrak{D}_U$.
(2) $x \notin \mathrm{dom}(f)$ *implies* $\min f(R[x]) \notin \mathfrak{D}_U$.
(3) $x \notin \mathrm{dom}(f)$ *implies* $f(R[x]) \cap U = \emptyset$.
(4) $\Diamond_R(f^{-1}(U)) \subseteq f^{-1}(\Diamond_R(U))$.

Proof. The proof is the same as the proof of [2, Lem. 4.3], but rests on Lemma 3.3. □

Note that condition (1) of Lemma 3.4 is Zakharyaschev's (CDC) for $\mathfrak{D}_U$. Thus, each of the four conditions of Lemma 3.4 can be taken as the definition of (CDC) for weakly transitive spaces. We choose condition (3) of Lemma 3.4 as our definition of (CDC) for weakly transitive spaces.

Definition 3.5. Let X and Y be weakly transitive spaces, $f : X \to Y$ be a partial continuous p-morphism, and U be a clopen subset of Y. We say that f satisfies *the closed domain condition* (*CDC*) for U if $x \notin \mathrm{dom}(f)$ implies $f(R[x]) \cap U = \emptyset$.

Theorem 3.6. *Let A and B be* **wK4***-algebras, $\eta : A \to B$ be a relativized modal algebra homomorphism, and $a \in A$. Then the following two conditions are equivalent:*

(1) $\eta(\Diamond a) = \Diamond\eta(a)$.
(2) $\eta_* : B_* \to A_*$ *satisfies* (CDC) *for* $\varphi(a)$.

Proof. Since $\eta(\Diamond a) \le \Diamond\eta(a)$ always holds [2, Lem. 3.3], the theorem follows from Lemma 3.4 and the generalized duality for modal algebras. □

Corollary 3.7. *Let A and B be* **wK4***-algebras, $\eta : A \to B$ be a relativized modal algebra homomorphism, and $D \subseteq A$. Then the following two conditions are equivalent:*

(1) $\eta(\Diamond a) = \Diamond\eta(a)$ *for each* $a \in D$.
(2) $\eta_* : B_* \to A_*$ *satisfies* (CDC) *for* $\mathfrak{D} = \{\varphi(a) : a \in D\}$.

It follows that (CDC) amounts to preserving $\Diamond$ for some selected set of elements of A.

4. Transitive and strongly cofinal subframes of weakly transitive spaces

We recall (see, e.g., [5, Def. 4.1]) that a *subframe* of a modal space X is a clopen subset of X. Let X be a transitive space. By [2, Def. 4.7], a subframe S of X is *cofinal* if $X = (R^+)^{-1}[S]$. Zakharyaschev's definition of a cofinal subframe is slightly weaker, namely that $R[S] \subseteq (R^+)^{-1}[S]$ [12, p. 295]. Clearly if $X = (R^+)^{-1}[S]$, then $R[S] \subseteq (R^+)^{-1}[S]$. Although the converse is not true in general, we do have that if $R[S] \subseteq (R^+)^{-1}[S]$, then S is a cofinal subframe of the closed upset $R^+[S]$ of X.[2] Since for the theory of canonical formulas (see Section 5) this difference between two notions of cofinality is negligible, throughout the paper we will always assume that a subframe is cofinal whenever $X = (R^+)^{-1}[S]$.

Note that if $f : X \to Y$ is a partial continuous p-morphism, then $\mathrm{dom}(f)$ is a subframe of X. We call a partial continuous p-morphism f between weakly transitive spaces X and Y *cofinal* if $\mathrm{dom}(f)$ is a cofinal subframe of X. Cofinal partial continuous p-morphisms between transitive spaces play a crucial role in Zakharyaschev's development of the theory of canonical formulas for **K4**. For weakly transitive spaces this notion turns out to be too weak. This is because if X is a transitive space, then each subframe of X is automatically transitive (see Definition 4.1 below); however, if X is weakly transitive, then there exist subframes of X that are not transitive (see Theorem 4.2 below). As we will see in Section 5, it is the notions of transitive cofinal subframes and transitive cofinal partial continuous p-morphisms that play the same role for weakly transitive spaces as the notions of cofinal subframes and cofinal partial continuous p-morphisms for transitive spaces.

Definition 4.1. Let X be a weakly transitive space and let S be a subframe of X. We call S *transitive* if $R^{-1}[R^{-1}[S]] \subseteq R^{-1}[S]$, and we call S *strongly cofinal* if S is transitive and cofinal.

Our immediate goal is to characterize transitive subframes of weakly transitive spaces. Let X be a weakly transitive space and let $S \subseteq X$. We recall (see, e.g., [5, Sec. 3]) that $x \in S$ is *maximal* if xRy implies yRx for each $y \in S$. Let $\max(S)$ denote the set of maximal points of S. Moreover, let $\mu(S) = \{x \in S : R[x] \cap S = \emptyset\}$. Clearly $\mu(S) \subseteq \max(S)$, but not vice versa. Furthermore, we recall (see, e.g., [12, Sec. 3.2]) that $C \subseteq X$ is a *cluster* if for each $x, y \in C$ we have $x \neq y$ implies xRy; a cluster C of X is *proper* if it consists of more than one point, C is *simple* if it consists of a single reflexive point, and C is *degenerate* if it consists of a single irreflexive point. For $x \in X$, let $C(x) = \{x\} \cup \{y \in X : xRy \text{ and } yRx\}$ be the *cluster generated by* x.

Theorem 4.2. *Let X be a weakly transitive space and let S be a subframe of X. Then S is transitive iff for each proper cluster C of X, if $C \cap \max(S) = \{x\}$, then x is reflexive.*

Proof. First suppose that S is transitive. Let C be a proper cluster of X with $C \cap \max(S) = \{x\}$. As C is proper, $x \in R^{-1}[R^{-1}[S]]$. Since S is transitive, $x \in R^{-1}[S]$. Therefore, there is $y \in S$ such that xRy. But as $C \cap \max(S) = \{x\}$, we have $y = x$, and so x is reflexive.

[2] We recall that U is an *upset* of X if $x \in U$ and xRy imply $y \in U$, and that by the standard duality between MA and MS, homomorphic images of a modal algebra A correspond to closed upsets of the dual space A_* of A.

Next suppose that for each proper cluster C of X, if $C \cap \max(S) = \{s\}$, then s is reflexive. Let $x \in R^{-1}[R^{-1}[S]]$. Then there exist $y \in X$ and $z \in S$ such that $xRyRz$. If $x = y$, then xRz, and so $x \in R^{-1}[S]$. Thus, without loss of generality we may assume that $x \neq y$. As (X, R^+) is a reflexive and transitive space, S is closed in X, and $z \in S$, by [13, Sec. III.2], there exists $u \in \max(S)$ such that zR^+u. Therefore, xR^+u, and so xRu or $x = u$. If xRu, then $x \in R^{-1}[S]$. Suppose that $x = u$. Then $x, y \in C(x)$, implying that $C(x)$ is proper. If $C(x)$ contains a point from $S - \{x\}$, then $x \in R^{-1}[S]$. On the other hand, if $C(x) \cap S = \{x\}$, then x is reflexive, and again $x \in R^{-1}[S]$. Thus, $x \in R^{-1}[R^{-1}[S]]$ implies $x \in R^{-1}[S]$, and so S is transitive. □

As an immediate consequence of Theorem 4.2, we obtain:

Corollary 4.3. *Let X be a weakly transitive space.*

1. *X is a transitive subframe of X.*
2. *If each point of a subframe S of X is reflexive, then S is transitive.*
3. *If a subframe S of X is a cluster, then S is transitive.*
4. *If X is transitive, then each subframe of X is transitive.*
5. *If X is transitive, then a subframe S of X is cofinal iff S is strongly cofinal.*

As follows from Definition 4.1, each strongly cofinal subframe of a weakly transitive space is both transitive and cofinal. On the other hand, the two notions of transitive and cofinal subframes are independent. Since there exist subframes of transitive spaces that are not cofinal, it follows from Corollary 4.3.4 that not every transitive subframe is cofinal. Consequently, not every transitive subframe is strongly cofinal. That not every cofinal subframe is transitive follows from the next example, which also shows that there exist cofinal subframes that are not strongly cofinal.

Example 4.4. Let X_2^{ir} be the weakly transitive space of Example 3.2 and let $S = \{y\}$. It is obvious that S is a cofinal subframe of X_2^{ir}. On the other hand, as follows from Theorem 4.2, S is not transitive, hence S is not strongly cofinal.

Theorem 4.5. *Let X be a weakly transitive space and let S be a subframe of X. Then S is strongly cofinal iff the following two conditions are satisfied:*

1. *$\mu(X) \subseteq S$.*
2. *$x \notin \mu(X)$ implies $x \in R^{-1}[S]$.*

Proof. First suppose that S is strongly cofinal. Let $x \in \mu(X)$. As S is cofinal, $X = (R^+)^{-1}[S]$, so $x \in (R^+)^{-1}[S]$. But since $x \in \mu(X)$, we have $R[x] = \emptyset$, hence $x \in S$, and so $\mu(X) \subseteq S$. Thus, condition (1) is satisfied. Now let $x \notin \mu(X)$. Then there exists $y \in X$ such that xRy. As S is cofinal, $y \in (R^+)^{-1}[S]$. Therefore, there exists $z \in S$ such that yR^+z. If $y = z$, then $x \in R^{-1}[S]$. On the other hand, if yRz, then $x \in R^{-1}[R^{-1}[S]]$. As S is transitive, $R^{-1}[R^{-1}[S]] \subseteq R^{-1}[S]$, implying that $x \in R^{-1}[S]$. Thus, condition (2) is also satisfied.

Next suppose that $\mu(X) \subseteq S$ and $x \notin \mu(X)$ implies $x \in R^{-1}[S]$. To see that S is transitive, let $x \in R^{-1}[R^{-1}[S]]$. As x has an R-successor, $x \notin \mu(X)$. Therefore, $x \in R^{-1}[S]$, and so S is transitive. To see that S is cofinal, let $x \in X$. Then $x \in \mu(X)$ or $x \in R^{-1}[S]$. As $\mu(X) \subseteq S$, in either case we see that $x \in (R^+)^{-1}[S]$, and so S is cofinal. □

Proposition 4.6. *Let A be a* **wK4***-algebra, $s \in A$, and X be the dual weakly transitive space of A.*

1. *The subframe $\varphi(s)$ of X is transitive iff $\Diamond\Diamond s \leq \Diamond s$.*
2. *The subframe $\varphi(s)$ of X is cofinal iff $\Diamond^+ s = 1$.*
3. *The following conditions are equivalent:*
   (a) *The subframe $\varphi(s)$ of X is strongly cofinal.*
   (b) *$\Diamond\Diamond s \leq \Diamond s$ and $\Diamond^+ s = 1$.*
   (c) *$\Box 0 \leq s$ and $\Box 0 \vee \Diamond s = 1$.*

Proof. (1) We have $\varphi(s)$ is transitive iff $R^{-1}[R^{-1}[\varphi(s)]] \subseteq R^{-1}[\varphi(s)]$ iff $\varphi(\Diamond\Diamond s) \subseteq \varphi(\Diamond s)$ iff $\Diamond\Diamond s \leq \Diamond s$.

(2) We have $\varphi(s)$ is cofinal iff $(R^+)^{-1}[\varphi(s)] = X$ iff $\varphi(\Diamond^+ s) = \varphi(1)$ iff $\Diamond^+ s = 1$.

(3) That $\varphi(s)$ is strongly cofinal iff $\Diamond\Diamond s \leq \Diamond s$ and $\Diamond^+ s = 1$ follows from (1) and (2). By Theorem 4.5, $\varphi(s)$ is strongly cofinal iff $\mu(X) \subseteq \varphi(s)$ and $x \notin \mu(X)$ implies $x \in R^{-1}[\varphi(s)]$. As $\mu(X) = \varphi(\Box 0)$, we have $\mu(X) \subseteq \varphi(s)$ iff $\varphi(\Box 0) \subseteq \varphi(s)$, which happens iff $\Box 0 \leq s$. Also, $x \notin \mu(X)$ implies $x \in R^{-1}[\varphi(s)]$ is equivalent to $\mu(X) \cup R^{-1}[\varphi(s)] = X$, which is equivalent to $\varphi(\Box 0 \vee \Diamond s) = \varphi(1)$, which happens iff $\Box 0 \vee \Diamond s = 1$. Thus, $\varphi(s)$ is strongly cofinal iff $\Box 0 \leq s$ and $\Box 0 \vee \Diamond s = 1$. □

Definition 4.7. Let A be a **wK4**-algebra and $s \in A$.

(1) We call s *transitive* if $\Diamond\Diamond s \leq \Diamond s$, *cofinal* if $\Diamond^+ s = 1$, and *strongly cofinal* if $\Diamond\Diamond s \leq \Diamond s$ and $\Diamond^+ s = 1$.
(2) We call the relativization A_s *transitive* if s is transitive, *cofinal* if s is cofinal, and *strongly cofinal* if s is strongly cofinal.

As follows from Proposition 4.6.3, $s \in A$ is strongly cofinal iff $\Box 0 \leq s$ and $\Box 0 \vee \Diamond s = 1$.

Definition 4.8. Let X and Y be weakly transitive spaces and let $f : X \to Y$ be a partial continuous p-morphism.

(1) We call f *transitive* if $\mathrm{dom}(f)$ is a transitive subframe of X.
(2) We call f *strongly cofinal* if $\mathrm{dom}(f)$ is a strongly cofinal subframe of X.

Definition 4.9. Let A and B be **wK4**-algebras and let $\eta : A \to B$ be a relativized modal algebra homomorphism.

(1) We call η *transitive* if $\Diamond\Diamond\eta(1) \leq \Diamond\eta(1)$.
(2) We call η *cofinal* if $\Diamond^+\eta(1) = 1$.
(3) We call η *strongly cofinal* if $\Diamond\Diamond\eta(1) \leq \Diamond\eta(1)$ and $\Diamond^+\eta(1) = 1$.

Proposition 4.10. *Let A and B be* **wK4**-*algebras and let $\eta : A \to B$ be a relativized modal algebra homomorphism.*

(1) *η is transitive iff $\eta_* : B_* \to A_*$ is transitive.*
(2) *η is cofinal iff $\eta_* : B_* \to A_*$ is cofinal.*
(3) *The following conditions are equivalent:*
   (a) *η is strongly cofinal.*
   (b) *$\Box 0 \leq \eta(1)$ and $\Box 0 \vee \Diamond\eta(1) = 1$.*
   (c) *$\eta_* : B_* \to A_*$ is strongly cofinal.*

Proof. Since $\mathrm{dom}(\eta_*) = \varphi(\eta(1))$ (see [2, Claim 3.5]), the result is immediate from Proposition 4.6 and Definitions 4.8 and 4.9. □

5. Canonical formulas for **wK4**

Let A be a **wK4**-algebra. For $a \in A$, let $\Diamond^+ a = a \vee \Diamond a$. Then $\Box^+ a = a \wedge \Box a$. It is well known (see, e.g., [5, Lem. 3.6]) that $(A, \Diamond^+)$ is an **S4**-algebra (and if (X, R) is the dual space of A, then (X, R^+) is the dual space of $(A, \Diamond^+)$). It follows that $H := \Box^+(A) = \{\Box^+ a : a \in A\}$ is a Heyting algebra, where for $h, g \in H$, we have $h \underset{H}{\to} g = \Box^+(h \to g)$.

We recall that a filter F of a modal algebra A is a $\Box$-*filter* if $a \in F$ implies $\Box a \in F$, and that congruences of A correspond to $\Box$-filters of A. Therefore, A is subdirectly irreducible iff there exists a least $\Box$-filter of A properly containing the $\Box$-filter $\{1\}$. Since the $\Box$-filters of a **wK4**-algebra A are in 1-1 correspondence with the filters of H, we obtain that A is subdirectly irreducible iff H is subdirectly irreducible, which is equivalent to H having the second largest element.

We assume that modal formulas are built from propositional variables and the constant $\top$ by means of the connectives $\neg, \vee$ and the modal operator $\Diamond$. The constant $\bot$, the connectives $\wedge, \to$, $\leftrightarrow$, and the modal operator $\Box$ are the standard abbreviations: $\bot := \neg\top$, $p \wedge q := \neg(\neg p \vee \neg q)$, $p \to q := \neg p \vee q$, $p \leftrightarrow q := (p \to q) \wedge (q \to p)$, and $\Box p := \neg\Diamond\neg p$. For modal formulas α and β, we use the following abbreviation: $\Diamond_\alpha \beta := \alpha \wedge \Diamond\beta$.

Let A be a finite subdirectly irreducible **wK4**-algebra. Then $H = \Box^+(A)$ is a subdirectly irreducible Heyting algebra, hence H has the second largest element which we denote by t. Let D be a subset of A. For each $a \in A$ we introduce a new variable p_a and define the *canonical formula* $\alpha(A, D)$ associated with A and D as follows:

$$\begin{aligned}\alpha(A, D) =\Box^+ \Big[& (\Diamond\Diamond p_1 \to \Diamond p_1) \wedge (\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in D\} \Big] \to (p_1 \to p_t).\end{aligned}$$

Remark 5.1. If A happens to be a **K4**-algebra, then $\alpha_{\mathbf{K4}}(A, D)$ is obtained from $\alpha(A, D)$ by deleting the conjunct $\Diamond\Diamond p_1 \to \Diamond p_1$ (see [2, Sec. 5.1]), which is redundant in the transitive case.

Our goal is to show that each logic over **wK4** (that is, each normal extension of **wK4**) is axiomatizable by canonical formulas, thus generalizing Zakharyaschev's theorem. Our strategy is the same as in [2], where we gave an algebraic proof of Zakharyaschev's theorem for **K4**. In fact, the theorems of this section and their proofs are direct generalizations of the corresponding theorems and proofs for the **K4**-case developed in [2, Sec. 5]. In each of the proofs given below, we describe in detail exactly where the corresponding proof from [2, Sec. 5] requires a generalization, and supply the details about how the required generalization works.

We start by the following generalization of [2, Lem. 4.1].

Lemma 5.2. *Let A be a* **wK4***-algebra, $a, b \in A$, and $\Box^+ a \not\leq b$. Then there exists a subdirectly irreducible* **wK4***-algebra B and an onto modal algebra homomorphism $\eta : A \to B$ such that $\eta(\Box^+ a) = 1$ and $\eta(b) \neq 1$.*

Proof. The only place in the proof of [2, Lem. 4.1] where it is used that A is a **K4**-algebra is in showing that the filter generated by $\Box^+ a$ is a $\Box$-filter. But the filter generated by $\Box^+ a$ is a $\Box$-filter already in a **wK4**-algebra. To see this, let F be the filter generated by $\Box^+ a$ and let $x \in F$. Then $\Box^+ a \leq x$. Therefore, $\Box\Box^+ a \leq \Box x$. As A is a **wK4**-algebra, $\Box^+ a = a \wedge \Box a \leq \Box\Box a$. Also, $\Box^+ a = a \wedge \Box a \leq \Box a$. Thus, $\Box^+ a \leq \Box a \wedge \Box\Box a = \Box(a \wedge \Box a) = \Box\Box^+ a$. Consequently, $\Box^+ a \leq \Box x$, so $\Box x \in F$, and so F is a $\Box$-filter. The rest of the proof is the same as the proof of [2, Lem. 4.1]. $\Box$

Next we generalize [2, Thm. 5.2].

Theorem 5.3. *Let A be a finite subdirectly irreducible* **wK4***-algebra, $D \subseteq A$, and B be a* **wK4***-algebra. Then $B \not\models \alpha(A, D)$ iff there exist a homomorphic image C of B and a 1-1 modal algebra homomorphism η from A into a strongly cofinal relativization C_s of C such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$.*

Proof. The proof follows the same path as the proof of [2, Thm. 5.2] with some slight modifications. First suppose that there exist a homomorphic image C of B and a 1-1 modal algebra homomorphism η from A into a strongly cofinal relativization C_s of C such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$. Define a valuation ν on A by $\nu(p_a) = a$. As $\Diamond 1_A \leq 1_A$, we have $\Diamond\Diamond 1_A \leq \Diamond 1_A$. Therefore, $\nu(\Diamond\Diamond p_1 \to \Diamond p_1) = \Diamond\Diamond\nu(p_1) \to \Diamond\nu(p_1) = \Diamond\Diamond 1_A \to \Diamond 1_A = 1_A$. Moreover, $\nu(\top \leftrightarrow \Diamond^+ p_1) = 1_A \leftrightarrow \Diamond^+ 1_A = 1_A$.

Thus, $\nu(\alpha(A,D)) = \Box^+ 1_A \to (1_A \to t) = t$, and so $A \not\models \alpha(A,D)$. Next define a valuation μ on C by $\mu(p_a) = \eta \circ \nu(p_a) = \eta(a)$ for each $a \in A$. Since s is strongly cofinal, s is transitive and cofinal. As s is cofinal, it follows from the proof of [2, Thm. 5.2] that $\mu(\top \leftrightarrow \Diamond^+ p_1) = 1_C$. As s is transitive, $\mu(\Diamond\Diamond p_1 \to \Diamond p_1) = \Diamond\Diamond\mu(p_1) \to \Diamond\mu(p_1) = \Diamond\Diamond s \to \Diamond s = 1_C$. Now the same argument as in the proof of [2, Thm. 5.2] gives $\mu(\alpha(A,D)) = \eta(1_A) \to \eta(t) \neq 1_C$. Consequently, $\alpha(A,D)$ is refuted on C. As C is a homomorphic image of B, we also have that $\alpha(A,D)$ is refuted on B.

Next suppose that $B \not\models \alpha(A,D)$. Then there exists a valuation μ on B such that $\mu(\alpha(A,D)) \neq 1_B$. Letting Γ denote the subformula of $\alpha(A,D)$ in the scope of $\Box^+$, we obtain $\mu(\alpha(A,D)) = \Box^+\mu(\Gamma) \to (\mu(p_1) \to \mu(p_t)) \neq 1_B$. Therefore, $\Box^+\mu(\Gamma) \not\leq \mu(p_1) \to \mu(p_t)$. By Lemma 5.2, there exist a subdirectly irreducible **wK4**-algebra C and an onto homomorphism $\theta : B \to C$ such that $\theta(\Box^+\mu(\Gamma)) = 1_C$ and $\theta(\mu(p_1) \to \mu(p_t)) \neq 1_C$. Clearly $\nu = \theta \circ \mu$ is a valuation on C such that $\Box^+\nu(\Gamma) = 1_C$ and $\nu(p_1) \to \nu(p_t) \neq 1_C$. It follows that $\nu(\Gamma) = 1_C$. Next define $\eta : A \to C$ by $\eta(a) = \nu(p_a)$ for each $a \in A$. Let $s = \eta(1_A)$. Then the same argument as in the proof of [2, Thm. 5.2] gives that s is cofinal. As $\nu(\Diamond\Diamond p_1 \to \Diamond p_1) = 1_C$, we have $\Diamond\Diamond\eta(1_A) \to \Diamond\eta(1_A) = 1_C$, implying that s is transitive. Thus, s is strongly cofinal. Moreover, the same argument as in the proof of [2, Thm. 5.2] gives that η is a 1-1 modal algebra homomorphism from A into C_s such that $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$, thus completing the proof. □

The following corollary generalizes [2, Cor. 5.3].

Corollary 5.4. *Let A be a finite subdirectly irreducible* **wK4***-algebra, $D \subseteq A$, and $\mathfrak{D} = \{\varphi(a) : a \in D\}$. For each weakly transitive space X, we have $X \not\models \alpha(A,D)$ iff there exist a closed upset Y of X and an onto strongly cofinal partial continuous p-morphism $f : Y \to A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

Proof. Since homomorphic images of modal algebras correspond to closed upsets of their dual spaces, the corollary is a consequence of Theorem 5.3, Proposition 4.10, Corollary 3.7, and Theorem 3.1. □

We next generalize [2, Lem. 4.14].

Lemma 5.5. *Let A and B be* **wK4***-algebras, $s \in A$, and $\eta : A_s \to B$ be an onto modal algebra homomorphism. Then there exists a* **wK4***-algebra C and an onto modal algebra homomorphism $\theta : A \to C$ such that B is isomorphic to the relativization of C to $\theta(s)$. Moreover, if s is cofinal in A, then $\theta(s)$ is cofinal in C; if s is transitive in A, then $\theta(s)$ is transitive in C; and if s is strongly cofinal in A, then $\theta(s)$ is strongly cofinal in C.*

Proof. The proof of the first half of the lemma is the same as that of [2, Lem. 4.14]. It also follows from [2, Lem. 4.14] that if s is cofinal in A, then $\theta(s)$ is cofinal in C. It is straightforward to see that if s is transitive in A, then $\theta(s)$ is transitive in C. Consequently, if s is strongly cofinal in A, then $\theta(s)$ is strongly cofinal in C. □

The following key theorem generalizes [2, Thm. 5.5].

Theorem 5.6. *If* **wK4** $\not\vdash \alpha(p_1,\dots,p_n)$, *then there exist $(A_1,D_1),\dots,(A_m,D_m)$ such that each A_i is a finite subdirectly irreducible* **wK4***-algebra, $D_i \subseteq A_i$, and for each* **wK4***-algebra B we have $B \not\models \alpha(p_1,\dots,p_n)$ iff there exist $i \leq m$, a homomorphic image C of B, and a modal algebra homomorphism η_i from A_i into a strongly cofinal relativization C_u of C such that $\eta_i(\Diamond_i a) = \Diamond\eta_i(a)$ for each $a \in D_i$.*

Proof. Our proof follows the same path as the proof of [2, Thm. 5.5], but some modifications are needed. Let F_n be the free n-generated **wK4**-algebra and let $g_1,\dots,g_n$ be the generators of F_n. Since **wK4** $\not\vdash \alpha(p_1,\dots,p_n)$, we have $F_n \not\models \alpha(p_1,\dots,p_n)$. Therefore, $\alpha(g_1,\dots,g_n) \neq 1_{F_n}$. By [5, Main Lemma], there exist $s \in F_n$ and a finite modal subalgebra B_s of $(F_n)_s$ such that $B_s \not\models \alpha(p_1,\dots,p_n)$. We briefly recall the construction of s. Let B_α be the Boolean subalgebra of

F_n generated by the subpolynomials of $\alpha(g_1,\dots,g_n)$. Then B_α is finite. Let A_α denote the set of atoms of B_α. Let also $H_n = \Box^+(F_n)$. Then H_n is a Heyting algebra, where $\xrightarrow[H_n]{}$ denotes the Heyting implication in H_n. Let H_α be the $(\wedge, \xrightarrow[H_n]{})$-subalgebra of H_n generated by $\Box^+(B_\alpha)\cup\{\Box^+\neg(a\wedge\Diamond a) : a \in A_\alpha\}$. By Diego's Theorem, H_α is finite. Set

$$s = \bigvee_{a\in A_\alpha}\bigwedge_{h\in H_\alpha}\left(h_a \vee \Box_a^+\neg_a h_a\right).$$

Let B be the Boolean subalgebra of F_n generated by $B_\alpha \cup H_\alpha$, and let $B_s = \{b_s : b \in B\}$, where $b_s = s \wedge b$. Clearly B_s is finite. By [5, Main Lemma], B_s is a modal subalgebra of $(F_n)_s$ and $B_s \not\models \alpha(p_1,\dots,p_n)$. Moreover, by [5, Lem. 5.3 and 5.4], $\max((F_n)_*) \subseteq \varphi(s)$, and so $\varphi(s)$ is a cofinal subframe of $(F_n)_*$. By Theorem 4.2, $\varphi(s)$ is also a transitive subframe of $(F_n)_*$. Thus, $\varphi(s)$ is a strongly cofinal subframe of $(F_n)_*$, and so s is strongly cofinal in F_n.

Let $A_1,\dots,A_m$ be the subdirectly irreducible homomorphic images of B_s refuting $\alpha(p_1,\dots,p_n)$, and let $\theta_i : B_s \to A_i$ be the corresponding onto homomorphisms. Since each A_i refutes $\alpha(p_1,\dots,p_n)$, there exist $a_1,\dots,a_n \in A_i$ such that $\alpha(a_1,\dots,a_n) \neq 1_{A_i}$. Let A_i^α be the Boolean subalgebra of A_i generated by the subpolynomials of $\alpha(a_1,\dots,a_n)$. We set $D_i = \{\neg a \in A_i^\alpha : \Diamond_i a \in A_i^\alpha\}$.[3]

Let B be an arbitrary **wK4**-algebra. We show that $B \not\models \alpha(p_1,\dots,p_n)$ *iff there is* $i \leq m$*, a homomorphic image* C *of* B*, and a modal algebra homomorphism* η_i *from* A_i *into a strongly cofinal relativization* C_u *of* C *such that* $\eta_i(\Diamond_i d) = \Diamond\eta_i(d)$ *for each* $d \in D_i$.

($\Leftarrow$): First suppose there exist $i \leq m$, a homomorphic image C of B, and a modal algebra homomorphism η_i from A_i into a strongly cofinal relativization C_u of C such that $\eta_i(\Diamond_i d) = \Diamond\eta_i(d)$ for each $d \in D_i$. Since $\eta_i : A_i \to C_u$ is a 1-1 modal algebra homomorphism, the formula $\alpha(p_1,\dots,p_n)$ is refuted on C_u. We show that $\alpha(p_1,\dots,p_n)$ is also refuted on C. For this we need the following generalization of [2, Lem. 5.6].

Lemma 5.7. *Suppose that* B *is a* **wK4***-algebra and* $u \in B$ *is strongly cofinal. Let* B_u *be the relativization of* B *to* u*. Let also* A *be a* **wK4***-algebra such that* $\alpha(a_1,\dots,a_n) \neq 1_A$ *for some* $a_1,\dots,a_n \in A$*. We let* A_α *be the Boolean subalgebra of* A *generated by the subpolynomials of* $\alpha(a_1,\dots,a_n)$*, and* $D = \{\neg a \in A_\alpha : \Diamond a \in A_\alpha\}$*. If there is a 1-1 modal algebra homomorphism* η *from* A *into* B_u *satisfying* $\Diamond\eta(d) = \eta(\Diamond d)$ *for each* $d \in D$*, then* $\alpha(\eta(a_1),\dots,\eta(a_n)) \neq 1_B$.

The proof of the lemma is the same as the proof of [2, Lem. 5.6], but we need the following generalization of [2, Claim 5.7].

Claim 5.8. *Let* B_u^α *be the Boolean subalgebra of* B_u *generated by the subpolynomials of* $\alpha_{B_u}(\eta(a_1),\dots,\eta(a_n))$*. If* $R[x]\cap\varphi(u) \subseteq \varphi(\neg d)$ *for each* $d \in D$ *and* $x \notin \varphi(u)$*, then*

$$u \wedge \alpha(b_1,\dots,b_n) = \alpha_{B_u}(b_1,\dots,b_n)$$

for each $b_1,\dots,b_n \in B_u^\alpha$*. Consequently, if there exist* $b_1,\dots,b_n \in B_u^\alpha$ *such that* $\alpha_{B_u}(b_1,\dots,b_n) \neq u$*, then* $\alpha(b_1,\dots,b_n) \neq 1_B$.

Proof. The proof is by induction on the complexity of $\alpha(b_1,\dots,b_n)$. The cases when $\alpha(b_1,\dots,b_n) = b_i$, $\alpha(b_1,\dots,b_n) = \beta\vee\gamma$, and $\alpha(b_1,\dots,b_n) = \neg\beta$ are proved as in [2, Claim 5.7]. Let $\alpha(b_1,\dots,b_n) = \Diamond\beta$. Then

$$u \wedge \alpha(b_1,\dots,b_n) = u \wedge \Diamond\beta$$

and

$$\alpha_{B_u}(b_1,\dots,b_n) = \Diamond_u\beta_u = u \wedge \Diamond(u\wedge\beta).$$

We show that $u\wedge\Diamond\beta = u\wedge\Diamond(u\wedge\beta)$. It is obvious that $u\wedge\Diamond(u\wedge\beta) \leq u\wedge\Diamond\beta$. Conversely, let $x \in \varphi(u\wedge\Diamond\beta)$. Then $x \in \varphi(u)$ and $R[x]\cap\varphi(\beta) \neq \emptyset$. So there exists $y \in B_*$ such that xRy and $y \in \varphi(\beta)$. If $y \in \varphi(u)$, then $x \in \varphi(u\wedge\Diamond(u\wedge\beta))$. Suppose $y \notin \varphi(u)$. As u is cofinal,

[3] D_i could alternatively be defined as $\{a \in A_i^\alpha : \Box_i a \in A_i^\alpha\}$.

$(R^+)^{-1}\varphi(u) = B_*$. Therefore, there exists $z \in \varphi(u)$ such that yR^+z. But $y \neq z$ as $z \in \varphi(u)$ and $y \notin \varphi(u)$. Thus, yRz. Since $\Diamond\beta \in B_u^\alpha$, we have $\neg\beta \in D$. As $y \notin \varphi(u)$, by the assumption of the claim, $R[y] \cap \varphi(u) \subseteq \varphi(\beta)$. Consequently, $z \in \varphi(\beta)$. If $x \neq z$, then as R is weakly transitive, xRz, and so $x \in \varphi(u \wedge \Diamond(u \wedge \beta))$. If $x = z$, then xRy and yRx. So x belongs to a proper cluster C. We claim that $R[x] \cap \varphi(u) \neq \emptyset$. If $R[x] \cap \varphi(u) = \emptyset$, then $C \cap \varphi(u) = \{x\}$. Therefore, $C \cap \max(\varphi(u)) = \{x\}$. This, by Theorem 4.2, means that x is reflexive. Thus, $x \in R[x] \cap \varphi(u)$, a contradiction. Consequently, there exists $w \in \varphi(u)$ such that xRw. As R is weakly transitive, $yRxRw$, $w \in \varphi(u)$, and $y \notin \varphi(u)$, we have yRw. Since $R[y]\cap\varphi(u) \subseteq \varphi(\beta)$, we have $w \in \varphi(\beta)$. Thus, $x \in \varphi(u \wedge \Diamond(u \wedge \beta))$. This implies that $u \wedge \Diamond\beta \leq u \wedge \Diamond(u \wedge \beta)$. Consequently, $u \wedge \Diamond\beta = u \wedge \Diamond(u \wedge \beta)$, and hence by induction we can conclude that $u \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_u}(b_1, \dots, b_n)$.

Finally, if $\alpha_{B_u}(b_1, \dots, b_n) \neq u$, then as $u \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_u}(b_1, \dots, b_n) \neq u$, we obtain that $\alpha(b_1, \dots, b_n) \neq 1_B$. □

Consequently, $\alpha(p_1, \dots, p_n)$ is refuted on C. Since C is a homomorphic image of B, it follows that $\alpha(p_1, \dots, p_n)$ is also refuted on B.

($\Rightarrow$): Next suppose that $B \not\models \alpha(p_1, \dots, p_n)$. Then there exist $a_1, \dots, a_n \in B$ such that $\alpha(a_1, \dots, a_n) \neq 1_B$. Let S_n be the subalgebra of B generated by $a_1, \dots, a_n$. Then S_n is an n-generated **wK4**-algebra, and so S_n is a homomorphic image of F_n. Let $\theta : F_n \to S_n$ be the onto homomorphism and let S_α be the Boolean subalgebra of S_n generated by the subpolynomials of $\alpha(a_1, \dots, a_n)$. We construct a strongly cofinal u and B_u in S_n exactly the same way we constructed s and B_s in F_n. We also let $D = \{\neg a \in S_\alpha : \Diamond a \in S_\alpha\}$. Clearly $\theta(s) = u$. Also, by [5, Lem. 5.7], $\Diamond_u b_u = u \wedge \Diamond b$ for each $b \in S_\alpha$. Let $k : B_u \to (S_n)_u$, $l : (S_n)_u \to S_n$, and $m : S_n \to B$ be the corresponding embeddings. Then k and m are modal algebra homomorphisms, while l is a relativized modal algebra homomorphism. Moreover, the embedding $m \circ l \circ k : B_u \to B$ satisfies $\Diamond mlk(a) = mlk(\Diamond_u a)$ for each $a \in S_\alpha$.

Since $\theta : F_n \to S_n$ is an onto homomorphism and $\theta(s) = u$, the restriction of θ to B_s is a homomorphism from B_s onto B_u. As $B_u \not\models \alpha(p_1, \dots, p_n)$, there is a subdirectly irreducible homomorphic image of B_u refuting $\alpha(p_1, \dots, p_n)$. Since each homomorphic image of B_u is also a homomorphic image of B_s, we obtain that the subdirectly irreducible homomorphic image of B_u refuting $\alpha(p_1, \dots, p_n)$ is A_i for some $i \leq m$. Let $\theta_i : B_u \to A_i$ be the onto homomorphism. Then, by [2, Lem. 2.1], there exists a **wK4**-algebra T, an onto homomorphism $\zeta : (S_n)_u \to T$, and a 1-1 homomorphism $n : A_i \to T$ such that $\zeta \circ k = n \circ \theta_i$. By Lemma 5.5, there exists a **wK4**-algebra E and an onto homomorphism $\xi : S_n \to E$ such that T is isomorphic to the relativization of E to $\xi(u)$. Moreover, as u is strongly cofinal in S_n, we also have that $\xi(u)$ is strongly cofinal in E. Let $p : T \to E$ be the corresponding relativized modal algebra homomorphism from T into E. Then $\xi \circ l = p \circ \zeta$. Applying [2, Lem. 2.1] again, we obtain a **wK4**-algebra C, an onto homomorphism $\eta : A \to C$, and a 1-1 homomorphism $q : E \to C$ such that $\eta \circ m = q \circ \xi$. Therefore, we arrive at the following commutative diagram.

$$\begin{array}{ccccccc}
B_u & \overset{k}{\rightarrowtail} & (S_n)_u & \overset{l}{\rightarrowtail} & S_n & \overset{m}{\rightarrowtail} & B \\
\downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\
A_i & \overset{n}{\rightarrowtail} & T & \overset{p}{\rightarrowtail} & E & \overset{q}{\rightarrowtail} & C
\end{array}$$

Let $\eta_i = q \circ p \circ n$ and let $(A_i)_\alpha$ be the Boolean subalgebra of A_i generated by the subpolynomials of $\alpha(\theta_i(a_1), \dots, \theta_i(a_n))$. Then $(A_i)_\alpha = \theta_i[S_\alpha]$. Let $a \in (A_i)_\alpha$. Then there exists $b \in S_\alpha$ such that $a = \theta_i(b)$. As the diagram commutes and $\Diamond_B mlk(b) = mlk(\Diamond_u b)$ for each $b \in S_\alpha$, we have $\eta_i(\Diamond_i a) = \eta_i(\Diamond_i \theta_i(b)) = \eta_i\theta_i(\Diamond_u b) = \eta mlk(\Diamond_u b) = \eta\Diamond_B mlk(b) = \Diamond_C \eta mlk(b) = \Diamond_C \eta_i\theta_i(b) = \Diamond_C \eta_i(a)$. In particular, $\eta_i(\Diamond_i d) = \Diamond_C \eta_i(d)$ for each $d \in D$. Thus, we have found $i \leq m$, a homomorphic image C of B, and a relativized modal algebra homomorphism η_i from A_i into a strongly cofinal relativization $C_{\eta(u)}$ of C such that $\eta_i(\Diamond_i d) = \Diamond\eta_i(d)$ for each $d \in D_i$. □

The following corollary generalizes [2, Cor. 5.8].

Corollary 5.9. *If* $\mathbf{wK4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each weakly transitive space* X*, we have* $X \not\models \alpha(p_1, \dots, p_n)$ *iff there exist* $i \le m$*, a closed upset* Y *of* X*, and a strongly cofinal partial continuous p-morphism* f_i *from* Y *onto* $(A_i)_*$ *satisfying (CDC) for* $\mathfrak{D}_i = \{\varphi(a) : a \in D_i\}$*.*

Proof. Since homomorphic images of modal algebras correspond to closed upsets of their dual spaces, the corollary is a consequence of Theorem 5.6, Proposition 4.10, Corollary 3.7, and Theorem 3.1. □

Next we generalize [2, Cor. 5.9 and 5.10]. The proof is similar and we skip it.

Corollary 5.10.

(1) *If* $\mathbf{wK4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each* **wK4***-algebra* B*, we have:*

$$B \models \alpha(p_1, \dots, p_n) \text{ iff } B \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

(2) *If* $\mathbf{K4} \not\vdash \alpha(p_1, \dots, p_n)$*, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each weakly transitive space* X*, we have:*

$$X \models \alpha(p_1, \dots, p_n) \text{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

As a consequence of Corollary 5.10.1, we obtain that every logic over **wK4** is axiomatizable by canonical formulas, thus generalizing Zakharyaschev's theorem.

Theorem 5.11 (Main Theorem)**.** *Each logic* L *over* **wK4** *is axiomatizable by canonical formulas. Moreover, if* L *is finitely axiomatizable, then* L *is axiomatizable by finitely many canonical formulas.*

Proof. Let L be a logic over **wK4**. Then $L = \mathbf{wK4} + \{\alpha_k : k \in I\}$, for some index set I. Without loss of generality we may assume that $\mathbf{wK4} \not\vdash \alpha_k$ for each $k \in I$. Therefore, by Corollary 5.10.1, for each α_k there exist $(A_{k_1}, D_{k_1}), \dots, (A_{k_m}, D_{k_m})$ such that each A_{k_i} is a finite subdirectly irreducible **wK4**-algebra, $D_{k_i} \subseteq A_{k_i}$, and for each **wK4**-algebra B, we have:

$$B \models \alpha_k \text{ iff } B \models \bigwedge_{i=k_1}^{k_m} \alpha(A_i, D_i).$$

As every modal logic is complete with respect to algebraic semantics, we obtain that $L = \mathbf{wK4} + \{\bigwedge_{i=k_1}^{k_m} \alpha(A_i, D_i) : k \in I\}$. Thus, L is axiomatizable by canonical formulas. Clearly if the index set I is finite, then this axiomatization is finite as well. □

Remark 5.12. We could have written canonical formulas for **wK4** in a slightly different fashion. Namely, if A is a finite subdirectly irreducible **wK4**-algebra and $D \subseteq A$, then let

$$\begin{aligned} \alpha'(A, D) = \Box^+ \Big[& (\Box\bot \to p_1) \wedge (\Box\bot \vee \Diamond p_1 \leftrightarrow \top) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in D\} \Big] \to (p_1 \to p_t). \end{aligned}$$

Using Proposition 4.10 it is easy to see that for each **wK4**-algebra B, we have $B \models \alpha(A, D)$ iff $B \models \alpha'(A, D)$. Thus, one can alternatively axiomatize all logics over **wK4** by replacing $\alpha(A, D)$ with $\alpha'(A, D)$.

6. Negation-free canonical formulas for wK4

The results of this section generalize the corresponding results of [2, Sec. 6.1] about negation-free canonical formulas for **K4**.

Suppose that A is a finite subdirectly irreducible **wK4**-algebra, $H = \Box^+(A)$, t is the second largest element of H, and $D \subseteq A$. For each $a \in A$, we introduce a new variable p_a and define the *negation-free canonical formula* $\beta(A, D)$ associated with A and D as

$$\begin{aligned}\beta(A, D) =\Box^+ \Big[&(\Diamond\Diamond p_1 \to \Diamond p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ &\bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ &\bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ &\bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \wedge \\ &\bigwedge \{\Diamond p_a \leftrightarrow p_{\Diamond a} : a \in D\} \Big] \to (p_1 \to p_t).\end{aligned}$$

Thus, $\beta(A, D)$ is obtained from $\alpha(A, D)$ by deleting the conjunct $\top \leftrightarrow \Diamond^+ p_1$.

Theorem 6.1. *Let A be a finite subdirectly irreducible* **wK4***-algebra, $D \subseteq A$, and B be a* **wK4***-algebra. Then $B \not\models \beta(A, D)$ iff there exist a homomorphic image C of B and a relativized modal algebra homomorphism η from A into a transitive relativization C_s of C satisfying $\eta(\Diamond a) = \Diamond\eta(a)$ for each $a \in D$.*

Proof. The proof is a simplified version of the proof of Theorem 5.3. $\square$

Corollary 6.2. *Let A be a finite subdirectly irreducible* **wK4***-algebra, $D \subseteq A$, and $\mathfrak{D} = \{\varphi(a) : a \in D\}$. Then for each weakly transitive space X, we have $X \not\models \beta(A, D)$ iff there exist a closed upset Y of X and an onto transitive partial continuous p-morphism $f : Y \to A_*$ such that f satisfies (CDC) for $\mathfrak{D}$.*

We recall that a modal formula α is *negation-free* if α is built from propositional variables and the constants $\top, \bot$ by means of $\wedge$, $\vee$, and $\Diamond$. The next theorem is an analogue of Theorem 5.6 for negation-free canonical formulas.

Theorem 6.3. *If* **wK4** $\nvdash \alpha(p_1, \dots, p_n)$*, where $\alpha(p_1, \dots, p_n)$ is negation-free, then there exist $(A_1, D_1), \dots, (A_m, D_m)$ such that each A_i is a finite subdirectly irreducible* **wK4***-algebra, $D_i \subseteq A_i$, and for each* **wK4***-algebra B, we have $B \not\models \alpha(p_1, \dots, p_n)$ iff there exist $i \le m$, a homomorphic image C of B, and a modal algebra homomorphism η from A_i into a transitive relativization C_s of C.*

The proof of the theorem is largely the same as the proof of Theorem 5.6. Since the s constructed in the proof of Theorem 5.6 is strongly cofinal, it is transitive. The only real difference in the proof is that we need the following version of Claim 5.8 for negation-free canonical formulas.

Claim 6.4. *Let $s \in B$ be transitive and let B_s^α be the Boolean subalgebra of B_s generated by the subpolynomials of $\alpha_{B_s}(\eta(a_1), \dots, \eta(a_n))$. If $R[x] \cap \varphi(s) \subseteq \varphi(\neg d)$ for each $d \in D$ and $x \notin \varphi(s)$, then*

$$s \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_s}(b_1, \dots, b_n)$$

for each $b_1, \dots, b_n \in B_s^\alpha$. Consequently, if there exist $b_1, \dots, b_n \in B_s^\alpha$ such that $\alpha_{B_s}(b_1, \dots, b_n) \neq s$, then $\alpha(b_1, \dots, b_n) \neq 1_B$.

Proof. We prove the claim by induction on the complexity of $\alpha(b_1, \dots, b_n)$. The cases $\alpha(b_1, \dots, b_n) = b_i$ and $\alpha(b_1, \dots, b_n) = \beta \vee \gamma$ are proved as in Claim 5.8. The case $\alpha(b_1, \dots, b_n) = \beta \wedge \gamma$ is proved similarly.

Let $\alpha(b_1, \dots, b_n) = \Diamond\beta$. It is sufficient to prove that $s \wedge \Diamond\beta \leq s \wedge \Diamond(s \wedge \beta)$. Let $x \in \varphi(s \wedge \Diamond\beta)$. Then $x \in \varphi(s)$ and there exists $y \in B_*$ such that xRy and $y \in \varphi(\beta)$. If $y \in \varphi(s)$, then we are done. Suppose that $y \notin \varphi(s)$. If $R[y] \cap \varphi(s) \neq \emptyset$, we proceed as in the proof of Claim 5.8. Namely, consider $z \in R[y] \cap \varphi(s)$. If $x \neq z$, then xRz, and we follow the argument of Claim 5.8. If $x = z$, then xRy and yRx. So x belongs to a proper cluster C. Since s is transitive, as in Claim 5.8, using Theorem 4.2 we can show that there exists $w \in \varphi(s)$ such that xRw. As R is weakly transitive, this implies that yRw, and we can proceed as in the proof of Claim 5.8.

Suppose that $R[y]\cap\varphi(s) = \emptyset$. An easy induction shows that for each $z \notin \varphi(s)$ with $R[z]\cap\varphi(s) = \emptyset$, we have $z \notin \varphi(\gamma)$ for each negation-free $\gamma \in B_s^\alpha$. To see this, let $\gamma = b_i$. Then as $\varphi(b_i) \subseteq \varphi(s)$, we have $z \notin \varphi(b_i)$. If $\gamma = \delta_1 \vee \delta_2$, then $\varphi(\gamma) = \varphi(\delta_1) \cup \varphi(\delta_2)$. By the induction hypothesis, $z \notin \varphi(\delta_1)$ and $z \notin \varphi(\delta_2)$. So $z \notin \varphi(\gamma)$. The case $\gamma = \delta_1 \wedge \delta_2$ is proved similarly. Finally, let $\gamma = \Diamond\delta$ and let zRu. As $R[z] \cap \varphi(s) = \emptyset$, we have $u \notin \varphi(s)$. Since R is weakly transitive and zRu, we have $R[u] \subseteq R^+[z]$. Therefore, $R[z] \cap \varphi(s) = \emptyset$ and $z \notin \varphi(s)$ imply $R[u] \cap \varphi(s) = \emptyset$. By the induction hypothesis, $u \notin \varphi(\delta)$. Thus, as u was an arbitrary successor of z, we have $z \notin \varphi(\Diamond\delta)$.

Now, as $R[y] \cap \varphi(s) = \emptyset$ and β is negation-free, we conclude that $y \notin \varphi(\beta)$. The obtained contradiction proves that $s\wedge\Diamond\beta = s\wedge\Diamond(s\wedge\beta)$. Thus, by induction we can conclude that $s\wedge\alpha(b_1, \dots, b_n) = \alpha_{B_s}(b_1, \dots, b_n)$. Finally, if $\alpha_{B_s}(b_1, \dots, b_n) \neq s$, then as $s \wedge \alpha(b_1, \dots, b_n) = \alpha_{B_s}(b_1, \dots, b_n) \neq s$, we obtain that $\alpha(b_1, \dots, b_n) \neq 1_B$. □

Consequently, we arrive at the following analogues for negation-free formulas of the corresponding results of Section 5.

Corollary 6.5. *If* **wK4** $\not\vdash \alpha(p_1, \dots, p_n)$*, where* $\alpha(p_1, \dots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each weakly transitive space* X*, we have* $X \not\models \alpha(p_1, \dots, p_n)$ *iff there exist* $i \leq m$*, a closed upset* Y *of* X*, and a transitive partial continuous p-morphism* f_i *from* Y *onto* $(A_i)_*$ *satisfying (CDC) for* $\mathfrak{D}_i = \{\varphi(a) : a \in D_i\}$*.*

Corollary 6.6.

(1) *If* **wK4** $\not\vdash \alpha(p_1, \dots, p_n)$*, where* $\alpha(p_1, \dots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each* **wK4***-algebra* B*, we have:*

$$B \models \alpha(p_1, \dots, p_n) \textit{ iff } B \models \bigwedge_{i=1}^{m} \beta(A_i, D_i).$$

(2) *If* **wK4** $\not\vdash \alpha(p_1, \dots, p_n)$*, where* $\alpha(p_1, \dots, p_n)$ *is negation-free, then there exist* $(A_1, D_1), \dots, (A_m, D_m)$ *such that each* A_i *is a finite subdirectly irreducible* **wK4***-algebra,* $D_i \subseteq A_i$*, and for each weakly transitive space* X*, we have:*

$$X \models \alpha(p_1, \dots, p_n) \textit{ iff } X \models \bigwedge_{i=1}^{m} \alpha(A_i, D_i).$$

Theorem 6.7. *Each logic* L *over* **wK4** *axiomatizable by negation-free formulas is axiomatizable by negation-free canonical formulas. Moreover, if* L *is axiomatizable by finitely many negation-free formulas, then* L *is axiomatizable by finitely many negation-free canonical formulas.*

7. Splitting and various kinds of subframe logics over **wK4**

In this section we study splitting and various kinds of subframe logics over **wK4**. As in the case of logics over **K4**, we show that splitting formulas for **wK4** are a particular case of canonical formulas for **wK4**. This results in an axiomatization of all splitting and join-splitting logics over **wK4** [23].

We already encountered four different notions of subframes of weakly transitive spaces; subframes, transitive subframes, cofinal subframes, and strongly cofinal subframes. Each of this notions yields the corresponding notion of a subframe logic over **wK4**. We will axiomatize all four kinds of subframe logics over **wK4**, thus generalizing the well-known results of Fine [17] and Zakharyaschev [29] for **K4**. Various examples of these kinds of subframe logics over **wK4** will be given in Section 8.

7.1. **Algebra-based formulas for wK4.** In axiomatizing splitting and various kinds of subframe logics over **wK4**, in addition to the technique of canonical formulas for **wK4**, we will also utilize the technique of frame-based formulas of [7] (see also [6, Sec. 3.4]). Although the frame-based formulas were developed for intuitionistic logic, they have a straightforward generalization to **wK4**, which we will sketch below.

In [7] and [6] the frame-based formulas were developed for intuitionistic frames, but using the standard duality between **wK4**-algebras and weakly transitive spaces, we can develop the corresponding algebra-based formulas for **wK4**-algebras. Since all the proofs of [7] and [6] transfer directly to **wK4**, we will only sketch the proofs and refer the interested reader to [7] and [6, Sec. 3.4].

Definition 7.1. Let $\leq$ be a reflexive and transitive relation on $\mathsf{wK4}$.[4] For $A, B \in \mathsf{wK4}$, we write $A < B$ if $A \leq B$ and $B \not\leq A$. We call $\leq$ an *algebra order* if the following two conditions are satisfied:

(1) If $A, B \in \mathsf{wK4}$ are finite, subdirectly irreducible, and $A < B$, then $|A| < |B|$.[5]
(2) If $A \in \mathsf{wK4}$ is finite and subdirectly irreducible, then there exists a formula $\alpha(A)$ such that for each $B \in \mathsf{wK4}$, we have $A \leq B$ iff $B \not\models \alpha(A)$.

The formula $\alpha(A)$ is called the *algebra-based formula* of A for $\leq$.

For a logic L over **wK4** and $A \in \mathsf{wK4}$, we say that A is an *L-algebra* if $A \models L$. Let V_L be the class of all L-algebras. (It is well known that V_L is a variety.) The following criterion of axiomatizability of logics by algebra-based formulas is a straightforward generalization of [7, Thm. 3.9] (see also [6, Thm. 3.4.12]) to logics over **wK4**:

Theorem 7.2. *Let L be a logic over* **wK4** *and let $\leq$ be an algebra order on* $\mathsf{wK4}$. *Then L is axiomatizable by algebra-based formulas for $\leq$ iff*

(a) *V_L is a $\leq$-downset of* $\mathsf{wK4}$.
(b) *For each $B \in \mathsf{wK4} - \mathsf{V}_L$, there exists a finite subdirectly irreducible $A \in \mathsf{wK4} - \mathsf{V}_L$ such that $A \leq B$.*

If (a) and (b) are satisfied, then the $\leq$-minimal elements in $\mathsf{wK4} - \mathsf{V}_L$ are finite and subdirectly irreducible, and L is axiomatizable by the algebra-based formulas of these $\leq$-minimal elements.

Proof. (Sketch) First suppose that there exists a family $\{A_i : i \in I\}$ of finite subdirectly **wK4**-algebras such that $L = \mathbf{wK4} + \{\alpha(A_i) : i \in I\}$. Then, using Definition 7.1.2, it is easy to verify that conditions (a) and (b) are satisfied. Conversely, suppose that L satisfies conditions (a) and (b). By condition (b), each $\leq$-minimal element of $\mathsf{wK4} - \mathsf{V}_L$ is finite and subdirectly irreducible. We show that $L = \mathbf{wK4} + \{\alpha(A) : A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)\}$. Let $B \models L$. Then $B \in \mathsf{V}_L$. By condition (a), V_L is a downset. Therefore, $A \not\leq B$ for each $A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)$. As each such A is finite and subdirectly irreducible, $B \models \alpha(A)$ for each $A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)$. Thus, $B \models \mathbf{wK4} + \{\alpha(A) : A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)\}$. Conversely, let $B \not\models L$. Then $B \in \mathsf{wK4} - \mathsf{V}_L$. By

[4] In [7] and [6] the class is restricted to (the dual spaces of) finitely generated subdirectly irreducible algebras, but for our purposes this restriction is not essential.

[5] Here $|A|$ denotes the cardinality of A.

condition (b), there exists a finite subdirectly irreducible $A \in \mathsf{wK4} - \mathsf{V}_L$ such that $A \leq B$. As $\leq$ is transitive, by Definition 7.1.1, we may assume that $A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)$. Since A is finite and subdirectly irreducible, we have $B \not\models \alpha(A)$. Thus, $B \not\models \mathbf{wK4} + \{\alpha(A) : A \in \min_{\leq}(\mathsf{wK4} - \mathsf{V}_L)\}$, which concludes the proof. □

7.2. **Splitting and join-splitting logics over wK4.** Let A be a finite subdirectly irreducible **wK4**-algebra, $H = \Box^+(A)$, and t be the second largest element of H. For each $a \in A$, we introduce a new variable p_a and define the *Jankov-Rautenberg formula* $\chi(A)$ associated with A as

$$\begin{aligned}\chi(A) = \Box^+ \Big[& \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \wedge \\ & \bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in A\} \Big] \to p_t.\end{aligned}$$

Remark 7.3. The term Jankov-Rautenberg formula is not standard. Our reason for choosing it is that it was Rautenberg [23] who first developed these formulas for subdirectly irreducible n-transitive modal algebras as a direct generalization of the formulas for finite subdirectly irreducible Heyting algebras developed by Jankov [18]. The frame-theoretic analogues of these formulas for **K4** are Fine's frame formulas, which are sometimes called the Jankov-Fine formulas (see, e.g., [8, Sec. 3.4], [12, Sec. 9.8]).

Note that the formulas $\chi(A)$ are the Jankov-Rautenberg formulas for 1-transitive modal logics. It follows from [23, p. 157] that, given a finite subdirectly irreducible **wK4**-algebra A, a **wK4**-algebra B refutes $\chi(A)$ iff A is (isomorphic to) a subalgebra of a homomorphic image of B.

Let

$$\begin{aligned}\chi'(A) = \Box^+ \Big[& (\top \leftrightarrow p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge\{p_{\Diamond a} \leftrightarrow \Diamond p_a : a \in A\} \Big] \to p_t.\end{aligned}$$

The following lemma is a straightforward generalization of [2, Lem. 6.11], and we skip its proof.

Lemma 7.4. *Let A be a finite subdirectly irreducible* **wK4***-algebra and let B be a* **wK4***-algebra. The following three conditions are equivalent:*

(1) $B \models \chi(A)$,
(2) $B \models \chi'(A)$,
(3) $B \models \alpha(A, A)$.

As a direct consequence of Lemma 7.4 and the standard duality between MA and MS, we obtain:

Proposition 7.5. *Let A be a finite subdirectly irreducible* **wK4***-algebra.*

(1) *For each* **wK4***-algebra B, we have $B \not\models \alpha(A, A)$ iff A is a subalgebra of a homomorphic image of B.*
(2) *For each weakly transitive space X, we have $X \not\models \alpha(A, A)$ iff there exists a closed upset Y of X and a continuous p-morphism from Y onto A_*.*

Let $A, B \in \mathsf{wK4}$. We set $A \leq_{\mathsf{SH}} B$ if A is (isomorphic to) a subalgebra of a homomorphic image of B. It is easy to see that $\leq_{\mathsf{SH}}$ is an algebra order on $\mathsf{wK4}$ and for a finite subdirectly irreducible

wK4-algebra A, the formula $\alpha(A, A)$ is the algebra-based formula of A for $\leq_{\mathsf{SH}}$. As we will see, these algebra-based formulas axiomatize all splitting and join-splitting logics over **wK4**.

Let L be a logic over **wK4**. We recall that L is a *splitting logic* if there exists a logic S over **wK4** such that (L, S) splits the lattice of logics over **wK4**. That is, $L \not\subseteq S$ and for each logic L' over **wK4** we have $L \subseteq L'$ or $L' \subseteq S$. We also recall that L is a *join-splitting logic* if L is a join of splitting logics over **wK4**. For a **wK4**-algebra A, let $L(A)$ be the set of all formulas valid in A. It is well known that $L(A)$ is a logic over **wK4**.

The following theorem provides an axiomatization of all splitting and join-splitting logics over **wK4**. A version of it for n-transitive modal logics was first established by Rautenberg [23].

Theorem 7.6.

(1) *A logic L over* **wK4** *is a splitting logic iff $L = \mathbf{wK4} + \alpha(A, A)$ for some finite subdirectly irreducible* **wK4**-*algebra A.*

(2) *A logic L over* **wK4** *is a join-splitting logic iff $L = \mathbf{wK4} + \{\alpha(A_i, A_i) : i \in I\}$ for some family $\{A_i : i \in I\}$ of finite subdirectly irreducible* **wK4**-*algebras.*

Proof. (1) First suppose that L is a splitting logic over **wK4**. As $\mathsf{wK4}$ is congruence-distributive and has the FMP, using a well-known general result of McKenzie [22, Sec. 4], we can conclude that there exists a finite subdirectly irreducible **wK4**-algebra A such that $(L, L(A))$ is a splitting pair in the lattice of logics over **wK4**. Thus, for each **wK4**-algebra B, we have $B \models L$ iff A is not (isomorphic to) a subalgebra of a homomorphic image of B. This, by Proposition 7.5, means that $B \models L$ iff $B \models \alpha(A, A)$. Consequently, $L = \mathbf{wK4} + \alpha(A, A)$.

Conversely, let $L = \mathbf{wK4} + \alpha(A, A)$. We show that $(L, L(A))$ is a splitting pair. As $A \not\models \alpha(A, A)$, we have that $L \not\subseteq L(A)$. Now assume that L' is a logic over **wK4** such that $L \not\subseteq L'$. Then there is a **wK4**-algebra B such that $B \models L'$ and $B \not\models \alpha(A, A)$. Therefore, $A \leq_{\mathsf{SH}} B$, which means that $L' \subseteq L(B) \subseteq L(A)$. Thus, $(L, L(A))$ is a splitting pair.

(2) follows from (1) and the definition of join-splitting logics over **wK4**. □

7.3. **Various kinds of subframe logics over wK4.** Let A be a finite subdirectly irreducible **wK4**-algebra, $H = \Box^+(A)$, and t be the second largest element of H. Let

$$\begin{aligned}\alpha_{scs}(A) = \Box^+ \Big[&(\Diamond\Diamond p_1 \to \Diamond p_1) \wedge (\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ &\bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ &\bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ &\bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \Big] \to (p_1 \to p_t).\end{aligned}$$

We call $\alpha_{scs}(A)$ the *strongly cofinal subframe formula* of A. Note that $\alpha_{scs}(A) = \alpha(A, \emptyset)$.

Proposition 7.7. *Let A be a finite subdirectly irreducible* **wK4**-*algebra.*

(1) *For each* **wK4**-*algebra B, we have $B \not\models \alpha_{scs}(A)$ iff there exist a homomorphic image C of B and a 1-1 strongly cofinal relativized modal algebra homomorphism from A into C.*

(2) *For each weakly transitive space X, we have $X \not\models \alpha_{scs}(A)$ iff there exist a closed upset Y of X and a strongly cofinal partial continuous p-morphism from Y onto A_*.*

Proof. Apply Theorem 5.3 and Corollary 5.4. □

Transitive subframe formulas are obtained from strongly cofinal subframe formulas by deleting the conjunct $\top \leftrightarrow \Diamond^+ p_1$. Thus, the *transitive subframe formula* of a finite subdirectly irreducible

wK4-algebra A is

$$\begin{aligned}\alpha_{ts}(A) = \Box^+ \Big[& (\Diamond\Diamond p_1 \to \Diamond p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \Big] \to (p_1 \to p_t).\end{aligned}$$

Note that $\alpha_{ts}(A) = \beta(A, \emptyset)$.

Proposition 7.8. *Let A be a finite subdirectly irreducible* **wK4**-*algebra.*

(1) *For each* **wK4**-*algebra B, we have $B \not\models \alpha_{ts}(A)$ iff there exist a homomorphic image C of B and a 1-1 transitive relativized modal algebra homomorphism from A into C.*
(2) *For each weakly transitive space X, we have $X \not\models \alpha_{ts}(A)$ iff there exist a closed upset Y of X and a transitive partial continuous p-morphism from Y onto A_*.*

Proof. Apply Theorem 6.1 and Corollary 6.2. □

Next we generalize the algebraic account of subframe and cofinal subframe formulas for **K4** developed in [2, Sec. 6.3]. Let A be a finite subdirectly irreducible **wK4**-algebra.

$$\begin{aligned}\alpha_{cs}(A) = \Box^+ \Big[& (\top \leftrightarrow \Diamond^+ p_1) \wedge (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \Big] \to (p_1 \to p_t)\end{aligned}$$

We call $\alpha_{cs}(A)$ the *cofinal subframe formula of A*. The proof of the next proposition is similar to that of [2, Cor. 6.13] and we skip it.

Proposition 7.9. *Let A be a finite subdirectly irreducible* **wK4**-*algebra.*

(1) *For each* **wK4**-*algebra B, we have $B \not\models \alpha_{cs}(A)$ iff there exist a homomorphic image C of B and a 1-1 cofinal relativized modal algebra homomorphism from A into C.*
(2) *For each weakly transitive space X, we have $X \not\models \alpha_{cs}(A)$ iff there exist a closed upset Y of X and a cofinal partial continuous p-morphism from Y onto A_*.*

Recall from [2, Sec. 6.3] that subframe formulas are obtained from cofinal subframe formulas by deleting the conjunct $\top \leftrightarrow \Diamond^+ p_1$. Thus, the *subframe formula* of a finite subdirectly irreducible **wK4**-algebra A is

$$\begin{aligned}\alpha_{s}(A) = \Box^+ \Big[& (\bot \leftrightarrow p_0) \wedge \\ & \bigwedge \{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & \bigwedge \{p_{\Diamond a} \leftrightarrow \Diamond_{p_1} p_a : a \in A\} \Big] \to (p_1 \to p_t)\end{aligned}$$

The proof of the next proposition is similar to that of [2, Cor. 6.14] and we skip it.

Proposition 7.10. *Let A be a finite subdirectly irreducible* **wK4**-*algebra.*

(1) *For each* **wK4**-*algebra B, we have $B \not\models \alpha_s(A)$ iff there exist a homomorphic image C of B and a 1-1 relativized modal algebra homomorphism from A into C.*

(2) *For each weakly transitive space X, we have $X \not\models \alpha_s(A)$ iff there exist a closed upset Y of X and a partial continuous p-morphism from Y onto A_*.*

Definition 7.11. Let $A, B \in \mathsf{wK4}$. We set:

(1) $A \leq_{scs} B$ if there exists a 1-1 strongly cofinal relativized homomorphism from A into a homomorphic image of B.
(2) $A \leq_{ts} B$ if there exists a 1-1 transitive relativized homomorphism from A into a homomorphic image of B.
(3) $A \leq_{cs} B$ if there exists a 1-1 cofinal relativized homomorphism from A into a homomorphic image of B.
(4) $A \leq_s B$ if there exists a 1-1 relativized homomorphism from A into a homomorphic image of B.

In the terminology of algebra-based formulas, Propositions 7.7, 7.8, 7.9, and 7.10 mean that each of the four relations $\leq_{scs}, \leq_{ts}, \leq_{cs}$, and $\leq_s$ is an algebra order on $\mathsf{wK4}$; and for a finite subdirectly irreducible $A \in \mathsf{wK4}$, the formulas $\alpha_{scs}(A)$, $\alpha_{ts}(A)$, $\alpha_{cs}(A)$, and $\alpha_s(A)$ are the algebra-based formulas for these orders.

Definition 7.12. Let L be a logic over **wK4**.

(1) We call L a *subframe logic* if for each weakly transitive space X and a subframe S of X, from $X \models L$ it follows that $S \models L$.
(2) We call L a *transitive subframe logic* if for each weakly transitive space X and a transitive subframe S of X, from $X \models L$ it follows that $S \models L$.
(3) We call L a *cofinal subframe logic* if for each weakly transitive space X and a cofinal subframe S of X, from $X \models L$ it follows that $S \models L$.
(4) We call L a *strongly cofinal subframe logic* if for each weakly transitive space X and a strongly cofinal subframe S of X, from $X \models L$ it follows that $S \models L$.

Since subframes correspond to relativizations, transitive subframes correspond to transitive relativizations, cofinal subframes correspond to cofinal relativizations, and strongly cofinal subframes correspond to strongly cofinal relativizations, next proposition is obvious.

Proposition 7.13. *Let L be a logic over* **wK4** *and let V_L be its corresponding variety of* **wK4***-algebras.*

(1) *L is a subframe logic iff V_L is closed under relativizations.*
(2) *L is a transitive subframe logic iff V_L is closed under transitive relativizations.*
(3) *L is a cofinal subframe logic iff V_L is closed under cofinal relativizations.*
(4) *L is a strongly cofinal subframe logic iff V_L is closed under strongly cofinal relativizations.*

Let $\mathcal{SF}$, $\mathcal{TSF}$, $\mathcal{CSF}$, and $\mathcal{SCSF}$ denote the classes of subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe logics over **wK4**, respectively. Clearly $\mathcal{SF} \subseteq \mathcal{TSF}, \mathcal{CSF}$ and $\mathcal{TSF}, \mathcal{CSF} \subseteq \mathcal{SCSF}$.

Theorem 7.14. *Each strongly cofinal subframe logic over* **wK4** *has the FMP. Consequently, each subframe, transitive subframe, and cofinal subframe logic over* **wK4** *has the FMP.*

Proof. Let L be a strongly cofinal subframe logic and let $L \not\vdash \varphi$. Then there exists $A \in \mathsf{V}_L$ such that $A \not\models \varphi$. By [5, Main Lemma], there exist $s \in A$ and a finite subalgebra B_s of the relativization A_s of A such that $B_s \not\models \varphi$. As follows from the proof of Theorem 5.6, s is strongly cofinal in A. Since L is a strongly cofinal subframe logic, by Proposition 7.13.4, V_L is closed under strongly cofinal relativizations. Therefore, $A_s \in \mathsf{V}_L$. As V_L is closed under subalgebras, $B_s \in \mathsf{V}_L$. Thus, φ is refuted on a finite L-algebra, and hence L has the FMP. Since $\mathcal{SCSF}$ contains $\mathcal{SF}$, $\mathcal{TSF}$, and $\mathcal{CSF}$, it follows that each subframe, transitive subframe, and cofinal subframe logic over **wK4** has the FMP. □

Theorem 7.15. *Let L be a logic over* **wK4**. *Then:*

(1) $L \in \mathcal{SF}$ *iff* $L = \mathbf{wK4} + \{\alpha_s(A_i) : i \in I\}$ *for some family* $\{A_i : i \in I\}$ *of finite subdirectly irreducible* **wK4***-algebras.*
(2) $L \in \mathcal{TSF}$ *iff* $L = \mathbf{wK4} + \{\alpha_{ts}(A_i) : i \in I\}$ *for some family* $\{A_i : i \in I\}$ *of finite subdirectly irreducible* **wK4***-algebras.*
(3) $L \in \mathcal{CSF}$ *iff* $L = \mathbf{wK4} + \{\alpha_{cs}(A_i) : i \in I\}$ *for some family* $\{A_i : i \in I\}$ *of finite subdirectly irreducible* **wK4***-algebras.*
(4) $L \in \mathcal{SCSF}$ *iff* $L = \mathbf{wK4} + \{\alpha_{scs}(A_i) : i \in I\}$ *for some family* $\{A_i : i \in I\}$ *of finite subdirectly irreducible* **wK4***-algebras.*

Proof. (1) First suppose that $L \in \mathcal{SF}$. As V_L is closed under homomorphic images, by Proposition 7.13.1, it is obvious that V_L is a $\leq_s$-downset. Therefore, $\leq_s$ satisfies condition (a) of Theorem 7.2. To see that $\leq_s$ also satisfies condition (b), let $B \in \mathsf{wK4} - \mathsf{V}_L$. It follows from the proof of Theorem 5.6 that there exists a finite subdirectly irreducible $A \in \mathsf{wK4} - \mathsf{V}_L$ such that $A \leq_s B$. Therefore, $\leq_s$ satisfies condition (b) of Theorem 7.2. Thus, by Theorem 7.2, there exists a family $\{A_i : i \in I\}$ of finite subdirectly irreducible **wK4**-algebras such that $A_i \not\models L$ for each $i \in I$ and $L = \mathbf{wK4} + \{\alpha_s(A_i) : i \in I\}$.

Next suppose that $L = \mathbf{wK4} + \{\alpha_s(A_i) : i \in I\}$. Let $A \in \mathsf{V}_L$ and let A_s be a relativization of A. If $A_s \notin \mathsf{V}_L$, then there exists $i \in I$ such that $A_s \not\models \alpha_s(A_i)$. By Proposition 7.10.1, there exist a homomorphic image B of A_s and a 1-1 relativized modal algebra homomorphism from A_i into B. By Lemma 5.5, B is isomorphic to a relativization of a homomorphic image of A. Therefore, there is a 1-1 relativized modal algebra homomorphism from A_i into a homomorphic image of A. This, by Proposition 7.10.1, means that $A \not\models \alpha_s(A_i)$. Thus, $A \notin \mathsf{V}_L$, which is a contradiction. Consequently, $A_s \in \mathsf{V}_L$, so V_L is closed under relativizations, and so by Proposition 7.13, $L \in \mathcal{SF}$.

The proofs of (2), (3), and (4) are similar. For the if direction, as follows from the proof of Theorem 5.6, we in fact have $A \leq_{ts} B$ (resp. $A \leq_{cs} B$, $A \leq_{scs} B$), and so by Theorem 7.2, there exists a family $\{A_i : i \in I\}$ of finite subdirectly irreducible **wK4**-algebras such that $A_i \not\models L$ for each $i \in I$ and $L = \mathbf{wK4} + \{\alpha_{ts}(A_i) : i \in I\}$ (resp. $L = \mathbf{wK4} + \{\alpha_{cs}(A_i) : i \in I\}$, $L = \mathbf{wK4} + \{\alpha_{scs}(A_i) : i \in I\}$). For the only if direction, if $A_s \not\models \alpha_{ts}(A_i)$ (resp. $A_s \not\models \alpha_{cs}(A_i)$, $A_s \not\models \alpha_{scs}(A_i)$), by Proposition 7.8.1 (resp. Proposition 7.9.1, Proposition 7.7.1), there exists a homomorphic image B of A_s and a 1-1 transitive (resp. cofinal, strongly cofinal) relativized modal algebra homomorphism from A_i into B. But then, by Lemma 5.5, there is a 1-1 transitive (resp. cofinal, strongly cofinal) relativized modal algebra homomorphism from A_i into a homomorphic image of A. The obtained contradiction proves that V_L is closed under transitive (resp. cofinal, strongly cofinal) relativizations, and so by Proposition 7.13, $L \in \mathcal{TSF}$ (resp. $L \in \mathcal{CSF}$, $L \in \mathcal{SCSF}$). □

As a result, we obtain that subframe logics over **wK4** are axiomatized by algebra-based formulas for $\leq_s$, transitive subframe logics over **wK4** are axiomatized by algebra-based formulas for $\leq_{ts}$, cofinal subframe logics over **wK4** are axiomatized by algebra-based formulas for $\leq_{cs}$, and strongly cofinal subframe logics over **wK4** are axiomatized by algebra-based formulas for $\leq_{scs}$.

Theorem 7.16.

(1) $\mathcal{SF} \cap \mathbf{K4} = \mathcal{TSF} \cap \mathbf{K4} \subsetneq \mathcal{SCSF} \cap \mathbf{K4} = \mathcal{CSF} \cap \mathbf{K4}$.
(2) $\mathcal{SF} \subsetneq \mathcal{TSF} \subsetneq \mathcal{SCSF}$.
(3) $\mathcal{SF} \subsetneq \mathcal{CSF} \subsetneq \mathcal{SCSF}$.
(4) $\mathcal{TSF} \not\subseteq \mathcal{CSF}$ *and* $\mathcal{CSF} \not\subseteq \mathcal{TSF}$.

Proof. (1) Since each subframe of a transitive space is a transitive subframe, we have $\mathcal{SF} \cap \mathbf{K4} = \mathcal{TSF} \cap \mathbf{K4}$. Similarly since a subframe of a transitive space is cofinal iff it is strongly cofinal, we have $\mathcal{SCSF} \cap \mathbf{K4} = \mathcal{CSF} \cap \mathbf{K4}$. That $\mathcal{SF} \cap \mathbf{K4} \subsetneq \mathcal{CSF} \cap \mathbf{K4}$ is well known (see, e.g., [12, Cor. 11.23]). Thus, $\mathcal{SF} \cap \mathbf{K4} = \mathcal{TSF} \cap \mathbf{K4} \subsetneq \mathcal{SCSF} \cap \mathbf{K4} = \mathcal{CSF} \cap \mathbf{K4}$.

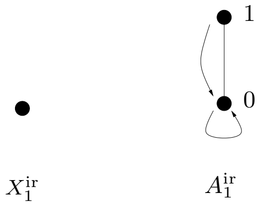


FIGURE 2. The weakly transitive space X_1^{ir} and the **wK4**-algebra A_1^{ir}

(2) It is obvious that $\mathcal{SF} \subseteq \mathcal{TSF} \subseteq \mathcal{SCSF}$. It follows from (1) that $\mathcal{TSF} \subsetneq \mathcal{SCSF}$. To see that $\mathcal{SF} \subsetneq \mathcal{TSF}$, let X_1^{ir} be the space consisting of one irreflexive point and let A_1^{ir} be its dual **wK4**-algebra. The pictures of X_1^{ir} and A_1^{ir} are shown in Figure 2, where the arrows indicate the action of $\Diamond$ on each element of the algebra. By Theorem 7.15.2, $L = \mathbf{wK4} + \alpha_{ts}(A_1^{\text{ir}})$ is a transitive subframe logic. We show that L is not a subframe logic. Let X_2^{ir} be the weakly transitive space of Example 3.2. Then the only nonempty upset of X_2^{ir} is X_2^{ir}, and the only transitive subframe of X_2^{ir} is X_2^{ir}. Obviously X_1^{ir} is not a p-morphic image of X_2^{ir}. So $X_2^{\text{ir}} \models \alpha_{ts}(A_1^{\text{ir}})$, and so $X_2^{\text{ir}} \models L$. On the other hand, X_1^{ir} is a subframe of X_2^{ir} and $X_1^{\text{ir}} \not\models \alpha_s(A_1^{\text{ir}})$. Thus, the class of weakly transitive spaces validating L is not closed under taking subframes, and so $L \in \mathcal{TSF} - \mathcal{SF}$.

(3) It is obvious that $\mathcal{SF} \subseteq \mathcal{CSF} \subseteq \mathcal{SCSF}$. That $\mathcal{SF} \subsetneq \mathcal{CSF}$ follows from (1). The proof of $\mathcal{CSF} \subsetneq \mathcal{SCSF}$ is similar to that of $\mathcal{SF} \subsetneq \mathcal{TSF}$. We again use the space X_2^{ir} of Example 3.2, but this time for the logic $L' = \mathbf{wK4} + \alpha_{scs}(A_1^{\text{ir}})$. The argument is based on the fact that X_1^{ir} is a cofinal subframe of X_2^{ir}, but that it is not a strongly cofinal subframe of X_2^{ir}.

(4) That $\mathcal{CSF} \not\subseteq \mathcal{TSF}$ follows from (1). To see that $\mathcal{TSF} \not\subseteq \mathcal{CSF}$ let $L = \mathbf{wK4} + \alpha_{ts}(A_1^{\text{ir}})$ be the logic constructed in (2). Then $L \in \mathcal{TSF}$, but $L \notin \mathcal{CSF}$ because X_1^{ir} is a cofinal subframe of X_2^{ir}. $\square$

Let Λ be either of $\mathcal{SF}$, $\mathcal{TSF}$, $\mathcal{CSF}$, $\mathcal{SCSF}$. As for logics over **K4** (see [12, Sec. 11.3]), we have that Λ is a complete sublattice of the lattice of all logics over **wK4**. Following Wolter [26], we call $L \in \Lambda$ a *splitting logic* in Λ if there exists $S \in \Lambda$ such that (L, S) splits Λ. The next theorem characterizes splitting logics in $\mathcal{SF}$, $\mathcal{TSF}$, $\mathcal{CSF}$, and $\mathcal{SCSF}$. A version of it for subframe logics is due to Wolter [26, Sec. 4].

Theorem 7.17.
(1) *A logic L over* **wK4** *is a splitting logic in $\mathcal{SCSF}$ iff $L = \mathbf{wK4} + \alpha_{scs}(A)$ for some finite subdirectly irreducible* **wK4***-algebra A.*
(2) *A logic L over* **wK4** *is a splitting logic in $\mathcal{CSF}$ iff $L = \mathbf{wK4} + \alpha_{cs}(A)$ for some finite subdirectly irreducible* **wK4***-algebra A.*
(3) *A logic L over* **wK4** *is a splitting logic in $\mathcal{TSF}$ iff $L = \mathbf{wK4} + \alpha_{ts}(A)$ for some finite subdirectly irreducible* **wK4***-algebra A.*
(4) *A logic L over* **wK4** *is a splitting logic in $\mathcal{SF}$ iff $L = \mathbf{wK4} + \alpha_s(A)$ for some finite subdirectly irreducible* **wK4***-algebra A.*

Proof. (1) First suppose that L is a splitting logic in $\mathcal{SCSF}$. By Theorem 7.15.4, there exists a family $\{A_i : i \in I\}$ of finite subdirectly irreducible **wK4**-algebras such that $L = \mathbf{wK4} + \{\alpha_{scs}(A_i) : i \in I\}$. As L is a splitting logic in $\mathcal{SCSF}$, this implies that there exists $i \in I$ such that $L = \mathbf{wK4} + \alpha_{scs}(A_i)$.

Conversely, let $L = \mathbf{wK4} + \alpha_{scs}(A)$ for some finite subdirectly irreducible **wK4**-algebra A. Let S be the meet (in $\mathcal{SCSF}$) of all logics in $\mathcal{SCSF}$ containing $L(A)$. We show that (L, S) is a splitting pair in $\mathcal{SCSF}$. As $A \not\models L$ and $A \models S$, we have that $L \not\subseteq S$. Now assume that $L' \in \mathcal{SCSF}$ is such that $L \not\subseteq L'$. Then there is a **wK4**-algebra B such that $B \models L'$ and $B \not\models \alpha_{scs}(A)$. By

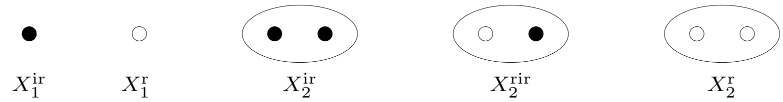


FIGURE 3. The weakly transitive spaces X_1^{ir}, X_1^{r}, X_2^{ir}, X_2^{rir}, X_2^{r}

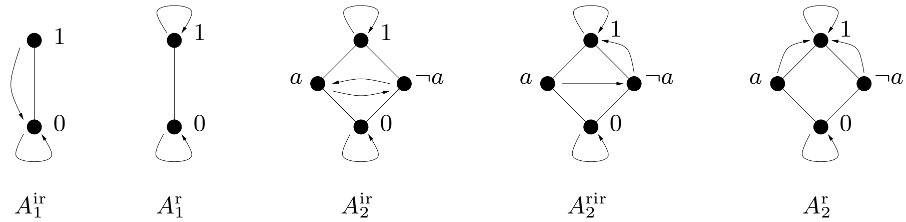


FIGURE 4. The **wK4**-algebras A_1^{ir}, A_1^{r}, A_2^{ir}, A_2^{rir}, A_2^{r}

Proposition 7.7.1, $A \leq_{scs} B$. As $L' \in \mathcal{SCSF}$ and $B \models L'$, we have $A \models L'$. Thus, $L' \subseteq L(A) \subseteq S$, and so (L, S) is a splitting pair in $\mathcal{SCSF}$.

The proofs of (2), (3), and (4) are similar. □

An obvious generalization of Theorem 7.17 provides an axiomatization of finite join-splitting logics in $\mathcal{SF}$, $\mathcal{TSF}$, $\mathcal{CSF}$, and $\mathcal{SCSF}$.

8. EXAMPLES

In this section we give examples of various logics over **wK4** that are axiomatizable by subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe formulas. We start by introducing some notation. We denote the frame consisting of a single irreflexive point by X_1^{ir}, the frame consisting of a single reflexive point by X_1^{r}, the two-point cluster consisting of two irreflexive points by X_2^{ir}, the two-point cluster consisting of one reflexive and one irreflexive point by X_2^{rir}, and the two-point cluster consisting of two reflexive points by X_2^{r}. We denote the corresponding dual **wK4**-algebras by A_1^{ir}, A_1^{r}, A_2^{ir}, A_2^{rir}, and A_2^{r}, respectively. The weakly transitive spaces X_1^{ir}, X_1^{r}, X_2^{ir}, X_2^{rir}, X_2^{r} are shown in Figure 3, where bullets indicate irreflexive points, circles indicate reflexive points, and ellipses indicate clusters. The **wK4**-algebras A_1^{ir}, A_1^{r}, A_2^{ir}, A_2^{rir}, A_2^{r} are shown in Figure 4, where the arrows indicate the action of $\Diamond$ on each element of the algebra. We already encountered X_2^{ir}, X_1^{ir}, and A_1^{ir} in Example 3.2 and Theorem 7.16.

It is well known (see, e.g., [12, Sec. 9.4]) that some of the most utilized modal logics such as **GL**, **S4**, **S4.Grz**, and **K4.Grz** are all subframe logics over **K4**. As we will see shortly, **K4** is a subframe logic over **wK4**. Hence, each of these logics is a subframe logic over **wK4**.

It is well known that **K4** is the logic of all finite transitive spaces; that is, **K4** is the logic of all finite weakly transitive spaces such that each proper cluster is reflexive. On the other hand, as follows from [3], $\mathbf{wK4T_0} = \mathbf{wK4} + \mathbf{T_0}$ (where $\mathbf{T_0} = p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$) is the logic of all finite weakly transitive spaces such that each proper cluster has at most one irreflexive point.

Proposition 8.1.

(1) $\mathbf{wK4T_0} = \mathbf{wK4} + \alpha_s(A_2^{\mathrm{ir}})$.
(2) $\mathbf{K4} = \mathbf{wK4} + \alpha_s(A_2^{\mathrm{ir}}) + \alpha_s(A_2^{\mathrm{rir}}) = \mathbf{wK4T_0} + \alpha_s(A_2^{\mathrm{rir}})$.

Proof. It is well known that both logics on the left-hand side have the FMP. That both logics on the right-hand side have the FMP follows from Theorem 7.14. Thus, in order to prove that both pairs of logics of the theorem are equal to each other, it is sufficient to show that the classes of their finite weakly transitive spaces coincide. Let X be a finite weakly transitive space.

(1) It is sufficient to show that $X \not\models \mathbf{T_0}$ iff $X_2^{\rm ir}$ is a partial p-morphic image of a (closed) upset of X. First suppose that $X \not\models \mathbf{T_0}$. Then there is a proper cluster of X that contains at least two irreflexive points, x and y. The subframe of X based on $\{x, y\}$ is isomorphic to $X_2^{\rm ir}$, and is clearly a partial p-morphic image of X. Therefore, X refutes $\alpha_s(A_2^{\rm ir})$. Next suppose that there exists a (closed) upset Y of X and a partial onto p-morphism $f : Y \to X_2^{\rm ir}$. We denote the points of $X_2^{\rm ir}$ by x' and y'. Since f is onto, there exists $x \in \mathrm{dom}(f)$ such that $f(x) = x'$. As $x'Ry'$, there exists $y \in \mathrm{dom}(f)$ such that xRy and $f(y) = y'$. As $y'Rx'$, there exists $u \in \mathrm{dom}(f)$ such that yRu and $f(u) = x'$. If $u \neq x$, then as R is weakly transitive, xRu, so $f(x)Rf(u)$, and so $x'Rx'$, a contradiction. Therefore, $u = x$, which implies that x and y belong to the same cluster. Since $f(x) = x'$, $f(y) = y'$, and x' and y' are irreflexive, so are x and y. Thus, $X \not\models \mathbf{T_0}$, and so we can conclude that $\mathbf{wK4T_0} = \mathbf{wK4} + \alpha_s(A_2^{\rm ir})$.

(2) The argument is similar to (1) and rests on the fact that X is transitive iff each proper cluster C of X is reflexive, which happens iff neither $X_2^{\rm ir}$ nor $X_2^{\rm rir}$ is a subframe of X. □

Next we examine the subframe, transitive subframe, cofinal subframe, and strongly cofinal subframe formulas of $A_1^{\rm ir}$. As we will see, these four kinds of subframe formulas axiomatize four different logics over **wK4**, with quite a sensitive difference between them. We will show that $\alpha_s(A_1^{\rm ir})$ axiomatizes **S4**, while the other three subframe formulas axiomatize three weakenings of **S4**. We recall that **S4** is the logic of all finite reflexive and transitive spaces. Let **wS4** (*weak* **S4**) be the logic of all finite weakly transitive spaces with no degenerate clusters, let **mS4** be the logic of all finite weakly transitive spaces whose maximal clusters are reflexive, and let **qS4** be the logic of all finite weakly transitive spaces with no degenerate maximal clusters. Clearly each of **qS4**, **mS4**, and **wS4** is properly contained in **S4**; moreover, **qS4** is properly contained in both **mS4** and **wS4**, while **mS4** and **wS4** are incomparable. It is also evident that **S4** is the only logic among the four that contains **K4**.

Proposition 8.2.

(1) $\mathbf{S4} = \mathbf{wK4} + \alpha_s(A_1^{\rm ir}) = \mathbf{K4} + \alpha_s(A_1^{\rm ir})$.
(2) $\mathbf{wS4} = \mathbf{wK4} + \alpha_{ts}(A_1^{\rm ir})$.
(3) $\mathbf{mS4} = \mathbf{wK4} + \alpha_{cs}(A_1^{\rm ir})$.
(4) $\mathbf{qS4} = \mathbf{wK4} + \alpha_{scs}(A_1^{\rm ir})$.

Proof. The proof follows the same path as the proof of Proposition 8.1. Let X be a finite weakly transitive space.

(1) That $\mathbf{S4} = \mathbf{K4} + \alpha_s(A_1^{\rm ir})$ is well known (see, e.g. [12, Sec. 9.4], where a frame-theoretic version of subframe formulas is used). That $\mathbf{S4} = \mathbf{wK4} + \alpha_s(A_1^{\rm ir})$ follows from the fact that X is reflexive and transitive iff each cluster C of X is reflexive, which happens iff $X_1^{\rm ir}$ is not a subframe of X.

(2) If X contains a degenerate cluster, then by Corollary 4.3.3, $X_1^{\rm ir}$ is a transitive partial p-morphic image of X. Conversely, suppose that Y is an upset of X and $f : Y \to X_1^{\rm ir}$ is an onto transitive partial p-morphism. Then $\mathrm{dom}(f) = f^{-1}(X_1^{\rm ir})$ is a nonempty antichain of Y, so $\mathrm{dom}(f)$ cannot intersect any proper cluster in more than one point. But since $\mathrm{dom}(f)$ is transitive, by Theorem 4.2, if $\mathrm{dom}(f)$ intersects a proper cluster in only one point, then that point is reflexive. This means that $\mathrm{dom}(f)$ cannot intersect any proper cluster, and so $\mathrm{dom}(f)$ consists of only degenerate clusters. Therefore, Y and hence X contains at least one degenerate cluster. Thus, $\mathbf{wS4} = \mathbf{wK4} + \alpha_{ts}(A_1^{\rm ir})$.

(3) If the maximum of X contains an irreflexive point x, then $X_1^{\rm ir}$ is a cofinal partial p-morphic image of the upset $R^+(x)$ of X generated by x. Conversely, if there is a cofinal partial p-morphism f from an upset Y of X onto $X_1^{\rm ir}$, then $\mathrm{dom}(f) = f^{-1}(X_1^{\rm ir})$ is a nonempty antichain of irreflexive points contained in the maximum of Y. Therefore, there exists at least one irreflexive point in the maximum of Y, and hence in the maximum of X. Thus, $\mathbf{mS4} = \mathbf{wK4} + \alpha_{cs}(A_1^{\rm ir})$.

(4) If the maximum of X contains a degenerate cluster, then this cluster is an upset of X isomorphic to X_1^{ir}. Clearly this cluster is cofinal, and it is transitive by Corollary 4.3.3. Thus, X_1^{ir} is a strongly cofinal partial p-morphic image of an upset of X. Conversely, if Y is an upset of X and $f : Y \to X_1^{\mathrm{ir}}$ is an onto strongly cofinal partial p-morphism, then $\mathrm{dom}(f) = f^{-1}(X_1^{\mathrm{ir}})$ is a nonempty antichain in the maximum of Y, and the same argument as in (2) shows that $\mathrm{dom}(f)$ does not intersect any proper cluster. Therefore, $\mathrm{dom}(f)$ consists of only degenerate clusters. So the maximum of Y and hence the maximum of X contains at least one degenerate cluster. Thus, $\mathbf{qS4} = \mathbf{wK4} + \alpha_{scs}(A_1^{\mathrm{ir}})$. □

Next we examine the four kinds of subframe formulas of A_1^{r}. The situation is different here. As we will see, the subframe and transitive subframe formulas of A_1^{r} are equivalent and axiomatize the well-known modal logic **GL**. Also, the cofinal subframe and strongly cofinal subframe formulas of A_1^{r} are equivalent and axiomatize a weakening of **GL**. We recall that **GL** is the logic of all finite irreflexive transitive spaces. Let **mGL** be the logic of all finite weakly transitive spaces with no reflexive points in the maximum. Evidently **mGL** is properly contained in **mGL**, and **mGL** is incomparable with **K4**.

Proposition 8.3.

(1) $\mathbf{GL} = \mathbf{K4} + \alpha_s(A_1^{\mathrm{r}}) = \mathbf{wK4} + \alpha_s(A_1^{\mathrm{r}}) = \mathbf{wK4} + \alpha_{ts}(A_1^{\mathrm{r}})$.
(2) $\mathbf{mGL} = \mathbf{wK4} + \alpha_{cs}(A_1^{\mathrm{r}}) = \mathbf{wK4} + \alpha_{scs}(A_1^{\mathrm{r}})$.

Proof. The proof follows the same path as the proofs of Propositions 8.1 and 8.2. Let X be a finite weakly transitive space.

(1) That $\mathbf{GL} = \mathbf{K4} + \alpha_s(A_1^{\mathrm{r}})$ is well known (see, e.g., [12, Sec. 9.4]). Moreover, $X \not\models \alpha_s(A_1^{\mathrm{r}})$ iff either a proper or a simple cluster is a subframe of X, which happens iff $X \not\models \mathbf{GL}$. By Corollary 4.3.3, each of these is a transitive subframe of X. Thus, $\mathbf{GL} = \mathbf{wK4} + \alpha_s(A_1^{\mathrm{r}}) = \mathbf{wK4} + \alpha_{ts}(A_1^{\mathrm{r}})$.

(2) It is sufficient to observe that X refutes $\alpha_{cs}(A_1^{\mathrm{r}})$ iff either a proper or a simple cluster is contained in the maximum of X. By Corollary 4.3.3, each of these is a transitive and hence strongly cofinal subframe of X. Thus, we obtain that $\mathbf{mGL} = \mathbf{wK4} + \alpha_{cs}(A_1^{\mathrm{r}}) = \mathbf{wK4} + \alpha_{scs}(A_1^{\mathrm{r}})$. □

It is well known that **S4.Grz** is the logic of all finite partially ordered spaces and that **K4.Grz** is the logic of all finite transitive spaces that are obtained from finite partially ordered spaces by deleting any number of reflexivities. We recall (see, e.g., [12, Sec. 5.3]) that **K4.1** is the logic of all finite transitive spaces whose maximal clusters are simple. We also let **wK4.1** be the logic of all finite weakly transitive spaces whose maximal clusters are simple. In other words, **wK4.1** is the **wK4**-version of **K4.1**. Clearly **wK4.1** is properly contained in **K4.1** and is incomparable with **K4**.

Proposition 8.4.

(1) $\mathbf{S4.Grz} = \mathbf{S4} + \alpha_s(A_2^{\mathrm{r}}) = \mathbf{wK4} + \alpha_s(A_1^{\mathrm{ir}}) + \alpha_s(A_2^{\mathrm{r}})$.
(2) $\mathbf{K4.Grz} = \mathbf{K4} + \alpha_s(A_2^{\mathrm{r}}) = \mathbf{wK4} + \alpha_s(A_2^{\mathrm{ir}}) + \alpha_s(A_2^{\mathrm{rir}}) + \alpha_s(A_2^{\mathrm{r}})$.
(3) $\mathbf{K4.1} = \mathbf{K4} + \alpha_{cs}(A_1^{\mathrm{ir}}) + \alpha_{cs}(A_2^{\mathrm{r}}) = \mathbf{wK4} + \alpha_s(A_2^{\mathrm{ir}}) + \alpha_s(A_2^{\mathrm{rir}}) + \alpha_{cs}(A_1^{\mathrm{ir}}) + \alpha_{cs}(A_2^{\mathrm{r}})$.
(4) $\mathbf{wK4.1} = \mathbf{wK4} + \alpha_{cs}(A_1^{\mathrm{ir}}) + \alpha_{cs}(A_2^{\mathrm{r}})$.

Proof. The proof follows the same path as the proofs of Propositions 8.1, 8.2, and 8.3. Let X be a finite weakly transitive space.

(1) For $\mathbf{S4.Grz} = \mathbf{S4} + \alpha_s(A_2^{\mathrm{r}})$ see [12, Sec. 9.4]. Now apply Proposition 8.2.1.

(2) That $\mathbf{K4.Grz} = \mathbf{K4} + \alpha_s(A_2^{\mathrm{r}})$ is proved similarly to (1). Now apply Proposition 8.1.2.

(3) For the first equation see [12, Sec. 9.4]. The second equation follows from Proposition 8.1.2.

(4) We recall from Proposition 8.2.3 that $X \models \alpha_{cs}(A_1^{\mathrm{ir}})$ iff each maximal cluster of X is reflexive. Now, as in (3), we have that $X \models \alpha_{cs}(A_1^{\mathrm{ir}}), \alpha_{cs}(A_2^{\mathrm{r}})$ iff each maximal cluster of X is simple. □

These examples underline once again the similarities and differences between various kinds of subframe logics over **wK4** and **K4**. Many other classes of weakly transitive spaces are also axiomatizable by subframe, transitive subframe, cofinal subframe, or strongly cofinal subframe formulas over **wK4**. We invite the reader to find axiomatizations of other interesting classes of weakly transitive spaces by means of these four kinds of subframe formulas.

9. Conclusions

In this paper we developed the theory of canonical formulas for logics over **wK4**, and proved that each logic over **wK4** is axiomatizable by canonical formulas, thus generalizing Zakharyaschev's theorem for logics over **K4**. Our approach followed the same lines as [2], where an algebraic approach to canonical formulas for logics over **K4** was developed. The key new ingredients include the concepts of transitive and strongly cofinal subframes of weakly transitive spaces. This yielded, along with the standard notions of subframe and cofinal subframe logics, the new notions of transitive subframe and strongly cofinal subframe logics over **wK4**. We obtained axiomatizations of all four kinds of subframe logics over **wK4**, along with axiomatizations of splitting and join-splitting logics over **wK4**. We also gave a number of examples of different kinds of subframe logics over **wK4**.

We conclude by pointing out several venues for further research in this area. Firstly, the developed technique of canonical formulas for logics over **wK4** may provide a useful tool for studying the lattice of logics over **wK4**. In particular, it may help answering the questions of completeness, finite axiomatizability, the FMP, and/or decidability for large families of logics over **wK4** such as logics of finite depth, finite width, etc. It may also help to generalize the result of Zakharyaschev and Alekseev [30] that all finitely axiomatizable logics over **K4.3** are decidable to logics over the weak transitive version of **K4.3**.

Secondly, it appears plausible that the proposed approach may be generalized to n-transitive modal logics. On the positive side, Rautenberg's generalization of Jankov's formulas works for all n-transitive modal logics [23]. The problem, however, lies in finding the appropriate notions of subframes for n-transitive frames, and more importantly, in proving the FMP for these n-transitive subframe logics. Thus, this task is by no means straightforward.

As for the boundaries of the proposed approach, it appears unlikely that it can be generalized to all normal modal logics since on the one hand, there exist subframe logics over **K** without the FMP [25], and on the other hand, there is no obvious way to define canonical formulas (nor Jankov-Rautenberg formulas) in this general setting. One possible way to overcome the second difficulty might be in enriching the modal language with additional modalities, such as say the universal modality or the fixed point operators.

Lastly, the key ingredient of the algebraic approach of [1] to canonical formulas for superintuitionistic logics was in restricting the signature of Heyting algebras to the locally finite $(\wedge, \to, 0)$-reduct and treating $\vee$ as an additional operation. An alternative approach, which will be discussed elsewhere, is in restricting the signature of Heyting algebras to the locally finite $(\wedge, \vee, 0, 1)$-reduct and treating $\to$ as an additional operation. This yields a new notion of canonical formulas for superintuitionistic logics, and with the appropriate adjustment, a new notion of canonical formulas for modal logics over **wK4** and **K4**. It also suggests possible generalizations to other non-classical logics such as substructral logics, which constitutes yet another interesting direction for future research.

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