CHAPTER 2
STRUCTURES

A structure is, roughly speaking, a set with some relations and operations over it. If there aren't any operations, we are dealing with a relational system; if there are no relations, with an algebra. We shall argue in chapter 4 that the mixed sort can be reduced, at least in theory, to algebras. For that reason, the general theory in later chapters will be restricted to algebras.

In this chapter we introduce a few kinds of structures by way of example. Several of these, in particular categories, lattices, and orders, will play an important part in the general theory of algebras that we shall develop later on.

§a Symbols and their interpretation

Assume given an inexhaustible class of *symbols*. What these symbols really are, matters very little. We assume only that every symbol is either a *relation symbol* or an *operation symbol*, and that it is associated with a unique natural number, its *arity*. A symbol of arity n is an *n-ary* symbol; nullary if $n = 0$, and so on. If S is any symbol, we denote its arity by n_S.

A *structure* is a pair $\mathbf{A} = \langle A, I \rangle$ of a set A (the *universe*) and a function I of symbols (the *interpretation*) that assigns to every relation symbol R in $\mathrm{Dom}\, I$ an n_R-ary relation over A, and to every operation symbol Q in $\mathrm{Dom}\, I$ an n_Q-ary operation over A. In particular I assigns to any nullary relation symbol p in its domain a subset of A^0; so because $|A^0| = 1, I(p) \in \{\emptyset, A^0\}$.

Actually, we shall also allow *large* structures, where the universe and the interpretation are proper classes. Even then the domain of the interpretation will always be a set. In Chapter 3 we shall briefly discuss how classes may be viewed as sets, from a higher standpoint so to say.[1]

We shall refer to structures by means of characters in bold type, and to their universes by the same characters in regular or italic type. If $\mathbf{A}$ is a structure, then $I_{\mathbf{A}}$ is its interpretation component. The *type* or *nominator of* $\mathbf{A}$, abbreviated $\mathrm{Nom}\,\mathbf{A}$, is $\mathrm{Dom}\, I_{\mathbf{A}}$. We shall write $\mathrm{Nom}_n \mathbf{A}$ to denote the subset of n-ary operation and relation symbols in $\mathrm{Nom}\,\mathbf{A}$. In general, by 'nominator' we refer to any set of symbols. If $\mathcal{T}$ is a nominator, then a *$\mathcal{T}$-structure* or *structure of type* $\mathcal{T}$ is a structure $\mathbf{A}$ such that $\mathrm{Nom}\,\mathbf{A} = \mathcal{T}$. Structures $\mathbf{A}$ and $\mathbf{B}$ are *similar* if they are of the same type, that is, if $\mathrm{Nom}\,\mathbf{A} = \mathrm{Nom}\,\mathbf{B}$; $\mathbf{A}$ is a *reduct* of $\mathbf{B}$, and $\mathbf{B}$ an *expansion* of $\mathbf{A}$, if $A = B$ and $I_{\mathbf{A}} = I_{\mathbf{B}} \restriction \mathrm{Nom}\,\mathbf{A}$. The *$\mathcal{T}$-reduct* of a structure $\mathbf{A}$ $= \langle A, I \rangle$ is the structure

$$\mathbf{A} \restriction \mathcal{T} := \langle A, I \restriction \mathcal{T} \rangle.$$

The relations and operations in the range of $I_{\mathbf{A}}$ are the *basic relations* and *basic operations* of $\mathbf{A}$. A nominator is *operational* if it consists entirely of operation

[1] That discussion is in no way meant to be the last word. For a summary of the attempts to construct a foundation for category theory the reader is referred to [Feferman 2013].

symbols, and *relational* if it consists entirely of relation symbols. If $\mathcal{T}$ is operational, $\mathcal{T}$-structures will also be called *$\mathcal{T}$-algebras*.

A set (class) may be viewed as a structure, with void interpretation.

Operations have a great notational advantage over relations. For example, suppose x, y, z, u are rational numbers: to express that $u = x{\cdot}y + z$ in terms of ternary relations '*a* is the sum of *b* and *c*' (abbreviated *Sabc*) and '*a* is the product of *b* and *c*' (*Pabc*) we would have to say something like

$$\text{there exists } v \text{ such that } Pvxy, \text{ and } Suvz. \tag{0}$$

However, when we are dealing with operations that are not defined for some of the objects that we want to consider, we have to exercise some care. Whereas

$$u + y = x{\cdot}y + z + y \tag{1}$$

is an unqualified consequence of our earlier equation,

$$\frac{u}{y} = x + \frac{z}{y} \tag{2}$$

is not. For we do not know what (2) means when y happens to be zero.

We could solve the perplexity by stipulating outcomes for division by zero. It turns out that

$$\frac{x}{0} = 0 \tag{3}$$

works fine [Bergstra & Tucker 2007]. There would, however, be an important difference between the justification of an ordinary division, such as ${}^{60}\!/_{30} = 2$ — if I distribute 60 apples fairly among 30 people, everyone gets 2 (and likewise for portions of apples) — and that of (3), which is rather that it fills a theoretical gap and does not imply anything that is manifestly false.

Alternatively, we could say that division by zero *does not make sense*, and therefore (2) does not make sense. This would be all right. It is common practice, in combination with avoiding situations in which a division by 0 would have to be evaluated. But it will not work for the degree of generality that we aim at. We would have to develop a complicated logic in which statements, apart from being true or false, could also not make sense. Instead we will interpret (2) in accordance with the translation of $u = x{\cdot}y + z$ in (0). We interpret it as saying, among other things, that there exists a rational number which is the result of division of u by y. If $y = 0$, this is false, and hence (2) is false. A fortiori (2) does not follow from $u = x{\cdot}y + z$. This is a pity, but we shall have to live with it.

In sum: if Q is an n-ary operation, and $x_0, \ldots, x_{n-1}$ are elements of the universe, then we can write

$$Q(x_0, \ldots, x_{n-1})$$

to denote the value of Q for $\langle x_0, \ldots, x_{n-1}\rangle$ in case it exists. If it does not exist then every statement that $Q(x_0, \ldots, x_{n-1})$ is equal to something, or stands in a logically simple relation to certain things, existent or not, is *false*. Moreover any larger expression M consisting of operation symbols and names of elements of the universe and containing the expression $Q(x_0, \ldots, x_{n-1})$ is undefined as well and likewise leads to false statements. In particular, for such an expression M, '$M = M$' is a formal way of saying that M exists.

The qualification 'logically simple' in the previous paragraph is essential. For instance, *France* is a country and *the king of* is an operation such that *the king of France* is undefined; for France is a republic. Then assuming that baldness is a logically simple property (unary relation),

The king of France is bald

is false. *Hence*, by the truth table for implication,

If the king of France is bald, the king of France is bald

is *true*.

We end with two relations that are *not* logically simple and that we shall often use. The first of these is *existence-implied equality* $\rightarrowtail$. By $M \rightarrowtail N$ we express that M is defined only if N is defined, *and* that $M = N$ if M is defined. The definition of existence-implied equality in terms of identity is

$$M \rightarrowtail N \text{ if and only if: } M = M \Rightarrow M = N.$$

The conjunction $M \rightarrowtail N$ & $N \rightarrowtail M$ was called *complete equality* by Kleene, and is symbolized by $\simeq$; $M \simeq N$ means that M is defined if and only if N is defined, *and* that $M = N$ if M and N are defined. The definition in terms of identity is

$$M \simeq N \text{ if and only if: } M = M \vee N = N \Rightarrow M = N.$$

§b Categories

To illustrate the preceding section, we introduce a notion that is of fundamental importance in the sequel.

A *category* is a class with a binary operation $\circ$ (*composition*) and a unary relation *Id* (the class of *identity* elements) satisfying the following axioms:

1. For all elements x, y and z, $(x \circ y) \circ z \simeq x \circ (y \circ z)$. (*Associative law*)
2. For all elements x, y and z, if $x \circ y$ and $y \circ z$ exist, then $(x \circ y) \circ z$ exists.
3. For all elements y and all $x \in Id$, $y \circ x \rightarrowtail y$ and $x \circ y \rightarrowtail y$.
4. For each element y, there are $x, z \in Id$ such that $x \circ y$ and $y \circ z$ exist.

Outer universal quantifiers, such as 'for all elements x, y and z' in axioms 1 and 2, are usually suppressed. They will be in the sequel, most of the time.

We call a category *small* if its universe is a set. If a category is not small, it is *large*.

Formally, we have structures $\langle A, I \rangle$ in which I assigns to a fixed binary operation symbol $\circ$ a binary operation over A, and to a unary relation symbol *Id* a subclass of A. If necessary, we could refer to these as $I(\circ)$ and $I(Id)$. A more pleasing notation, with almost the same meaning, will be introduced in §4a. All the same, in concrete cases we shall not hesitate to use identical characters to refer to a symbol and its interpretation.

Proposition. For every element x in a category, there is exactly one identity element u such that $x \circ u$ exists, and exactly one identity element v such that $v \circ x$ exists.

Proof. Let u be an identity element such that $x \circ u$ exists. By axiom 4, there must be such u. Now let u' be an arbitrary identity element for which $x \circ u'$ is defined. Then by axioms 3 and 1,

$$x \circ u' = (x \circ u) \circ u' = x \circ (u \circ u'),$$

so $u \circ u'$ exists. Since u and u' are identity elements, we have

$$u = u \circ u' = u'.$$

Similarly we prove that the identity element v for which $v \circ x$ is defined is unique. ☒

Examples

i. The class *Rel* of all triples $\langle Y, R, X\rangle$ consisting of sets Y and X and a relation $R \subseteq Y \times X$ is the universe of a large category **Rel** with composition defined by

$$\langle Z, S, Y\rangle \circ \langle Y, R, X\rangle = \langle Z, S \circ R, X\rangle$$

and identity elements $\langle X, \varDelta_X, X\rangle$. The composite of $\langle Y, S, Z\rangle$ and $\langle U, R, V\rangle$ is defined only if $Z = U$.

ii. The pairs $\langle Y, f\rangle$ of a set Y and a function f with range contained in Y form a large category **Set**, with composition defined by

$$\langle Z, g\rangle \circ \langle Y, f\rangle = \langle Z, g \circ f\rangle$$

provided that $\mathrm{Dom}(g) = Y$, and identity elements $\langle X, 1_X\rangle$.

Notation. It often happens that we already know, or that it is not important, how the symbols that a structure interprets are denoted. For example, if we present a structure as a category, we know that the interpretation of the composition symbol is a binary operation over the universe, and the interpretation of the symbol for the identity class a subclass of the universe. Hence we could present a category as a class C (the universe) with a binary operation $\circ$ and a subclass *Id*; or as a *triple* $\langle C, \circ, Id\rangle$. For this kind of sequence notation it is not necessary that the elements are of different type (relation or operation) or arity; it suffices that they are enumerated in a conventional order, or are denoted by conventional symbols.

Duality. The *opposite* or *dual* of a category $\mathbf{C} = \langle C, \circ, Id\rangle$ is the structure $\mathbf{C}^\partial = \langle C, \bullet, Id\rangle$, where $\bullet$ is the binary operation defined by $c \bullet d = d \circ c$. It is again a category. A consequence of this fact is the *duality principle* for categories: if we have a statement S that is valid for all categories, and we invert all the compositions in S, the resulting statement is still valid.

Categories are not the only kind of structures that have duals, though arguably they are the most general kind. If P and Q are notions definable in structures of some kind that have duals, then *Q is the dual of P* if for any structure $\mathbf{A}$ of the kind under consideration, the denotation of Q in $\mathbf{A}$ is exactly the denotation of P in $\mathbf{A}^\partial$. A notion that is its own dual is called *self-dual*. For example, in categories *Id* is self-dual.

Objects. Categories as in Example (a) above, consisting of triples that are composable if their outer elements match, or as in Example (b), where such triples can be easily constructed, are quite common. The outer elements are then called *objects*. Often such categories are named after their objects. (This can be confusing: **Rel** might have been called after sets just as well as **Set**, for example.) Objects stand in one-to-one correspondence with identity elements. Thus we may speak of objects in *any* category, and simply mean identity elements if we cannot think of anything better. We shall denote the class of objects of a category **C** by Ob**C**. The identity element belonging to an object x is denoted by 1_x. (This notation was introduced for the special case of the identity mapping on a set in §1h1 above.)

When we speak of objects, the elements proper of the category are called *arrows* or *morphisms*. An arrow goes from an object, its *domain*, to an object, its *codomain*. For example, for an arrow $\boldsymbol{f} = \langle Y, f \rangle \in Set$, Y would be its codomain, and Dom$f$ its domain; and if Dom$f = X$, we write

$$\boldsymbol{f}\colon Y \longleftarrow X$$

or $\boldsymbol{f}\colon X \longrightarrow Y$. Indeed, it is quite common to neglect the distinction between the pair and its second element, and write

$$f\colon Y \longleftarrow X$$

and $f\colon X \longrightarrow Y$ — notation we have seen before. We write $X = \mathrm{dom}(\boldsymbol{f})$, or $\mathrm{dom}(f)$, and $Y = \mathrm{cod}(\boldsymbol{f})$, or $\mathrm{cod}(f)$.

In any case objects may be used as a manner of speaking, to describe remarkable patterns in categories. In particular, an *initial object* in a category is an object u such that for any object a in the category there exists exactly one arrow $f\colon u \longrightarrow a$. In other words, we have $u \in Id$ such that for every $a \in Id$ there is precisely one x for which $a \circ x \circ u$ exists. With regard to the dual pattern we speak of *terminal objects*. An object that is at once initial *and* terminal is called a *zero* object.

The void set Ø is a zero object of **Rel**. The same set is initial in **Set**. The singletons are the terminal objects of **Set**.

§c Algebras

A structure is an *algebra* if its nominator is operational. A *total algebra* is an algebra with nonvoid universe in which every basic operation is total.

Terminology. Usage has been to reserve the word ‘algebra’ for what above has been termed *total algebras*, and to speak of ‘partial algebra’ in the general case. Burmeister and Reichel [1984/1986] followed this tradition. We choose to deviate.

Examples

i. We will show that categories may be considered as algebras

$$\langle C, \circ, \mathsf{d}, \mathsf{b} \rangle,$$

where instead of a class of identity elements two unary operations d (*domain identity*) and b (*codomain identity*) are specified. We replace the axioms 2-4 by

2′. $x \circ y$ exists if and only if $\mathsf{d}(x) = \mathsf{b}(y)$.

3′. $y \circ \mathsf{d}(y) = y = \mathsf{b}(y) \circ y$.

4′. If $x \circ y$ exists, then $\mathsf{d}(x \circ y) = \mathsf{d}(y)$ and $\mathsf{b}(x \circ y) = \mathsf{b}(x)$.

c1 Lemma. In an algebra that satisfies the axioms 2′-4′,

(i) the operations d and b are total;

and for all x, (ii) $\mathsf{d}(\mathsf{d}(x)) = \mathsf{b}(\mathsf{d}(x)) = \mathsf{d}(x)$, and

(iii) $\mathsf{b}(\mathsf{b}(x)) = \mathsf{d}(\mathsf{b}(x)) = \mathsf{b}(x)$.

Proof. (i) Take any element y. By axiom 3′, $y \circ \mathsf{d}(y)$ exists, and this can only be if d is defined for y. Likewise for b.
(ii) By axiom 3′, $x \circ \mathsf{d}(x)$ exists, hence by axiom 2′, $\mathsf{d}(x) = \mathsf{b}(\mathsf{d}(x))$. By axiom 3′, $\mathsf{b}(\mathsf{d}(x)) \circ \mathsf{d}(x)$ exists, hence $\mathsf{d}(x) \circ \mathsf{d}(x)$ exists. So by axiom 2′, $\mathsf{d}(\mathsf{d}(x)) = \mathsf{b}(\mathsf{d}(x))$.
(iii) Similar to (ii). ⊠

c2 Theorem. The two definitions of categories are equivalent. To be precise,
1° in a category defined as in §b, operations d and b can be defined in such a way that the axioms 2′-4′ are satisfied;
2° in a category defined as above, as an algebra, identity elements can be defined in such a way that the axioms 2-4 are satisfied.

Proof. 1° Suppose the axioms 1-4 hold for a structure $\mathbf{C}$. By the Proposition in §b, for every $c \in C$ there exists a unique identity element d such that $c \circ d$ exists. Define $\mathsf{d}(c)$ to be this identity element. Similarly define $\mathsf{b}(c)$ to be the identity element b such that $b \circ c$ exists. Then axiom 3′ follows immediately from axiom 3.

If $x \circ y$ exists, then so does $(x \circ \mathsf{d}(x)) \circ (\mathsf{b}(y) \circ y)$. So by axiom 1,

$$(x \circ (\mathsf{d}(x) \circ \mathsf{b}(y))) \circ y \text{ exists,}$$

which implies that $\mathsf{d}(x) \circ \mathsf{b}(y)$ exists. Since $\mathsf{d}(x)$ and $\mathsf{b}(y)$ are identity elements, $\mathsf{d}(x) = \mathsf{d}(x) \circ \mathsf{b}(y) = \mathsf{b}(y)$. This settles one direction of 2′. For the other direction, suppose $\mathsf{d}(x) = \mathsf{b}(y) = u$. Then by axiom 2, since $x \circ u$ and $u \circ y$ exist, $(x \circ u) \circ y$ exists, and since u is an identity element, $(x \circ u) \circ y = x \circ y$.

Axiom 4′ follows from the identities

$$(x \circ y) \circ \mathsf{d}(y) = x \circ y \quad \text{and} \quad \mathsf{b}(x) \circ (x \circ y) = x \circ y,$$

since by definition $\mathsf{d}(x \circ y)$ is the *unique* identity element u such that $(x \circ y) \circ u = x \circ y$, and $\mathsf{b}(x \circ y)$ the unique identity element v such that $v \circ (x \circ y) = x \circ y$.
2° Conversely, suppose the axioms 1, 2′, 3′ and 4′ hold for an algebra $\mathbf{D}$. If $x \circ y$ and $y \circ z$ exist, then by 4′ en 2′, $\mathsf{d}(x \circ y) = \mathsf{d}(y) = \mathsf{b}(z)$, so by 2′, $(x \circ y) \circ z$ exists. This proves axiom 2.

Define Id as the range of d. Suppose $u \in Id$, say $u = \mathsf{d}(x)$. If $y \circ u$ exists, then $\mathsf{d}(y) = \mathsf{b}(u) = \mathsf{b}(\mathsf{d}(x)) = \mathsf{d}(x) = u$ by the Lemma above. So $y \circ u = y$ by axiom 3′. Similarly $u \circ z = z$ if $u \circ z$ exists. This proves axiom 3.

To see that axiom 4 holds, observe that by the Lemma, for any x, both $\mathsf{d}(x)$ and $\mathsf{b}(x)$ are in Ran(d). ⊠

The dual of a category $\mathbf{C} = \langle C, \circ, \mathsf{d}, \mathsf{b}\rangle$ is $\mathbf{C}^\partial = \langle C, \bullet, \mathsf{b}, \mathsf{d}\rangle$, where $\bullet$ is defined as before (at the end of §b). Note that d and b have been interchanged. Accordingly, to dualize a statement about categories-as-algebras, we also have to replace all occurrences of d by b and vice versa.

ii. *Directed graphs*. A (directed) *graph* is an algebra $\mathbf{X} = \langle X, \mathsf{d}, \mathsf{b}\rangle$ where d and b are unary operations satisfying

> $\mathsf{d}(x)\!\downarrow$ if and only if $\mathsf{b}(x)\!\downarrow$;
> $\mathsf{dd}(x)\!\uparrow$ and $\mathsf{bb}(x)\!\uparrow$.

The elements of Dom(d) are the *edges*; the rest are the *nodes* of the graph. The *paths*, sequences of consecutive nodes and edges, form a category. (See Exercise 1.)

iii. A *groupoid* is a set S with a total binary operation $\cdot$ (product). A groupoid is a *semigroup* if its product is *associative*, that is, for all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. For example, the set $S = X^+$ of nonempty sequences of elements of a set X with concatenation for a product-operation is a semigroup. With composition, the relations (the ones that are sets) and the functions form large semigroups. A semigroup $\langle S, \cdot\rangle$ is *commutative* if for all $x, y \in S$, $x \cdot y = y \cdot x$.

The product symbol $\cdot$ is often omitted: we write xy instead of $x \cdot y$.

iv. A *monoid* is a semigroup $\langle M, \cdot\rangle$ with an *identity element*, i.e. an element e such that for all $x \in M$, $xe = ex = x$. The semigroup of sequences with concatenation becomes a monoid when we include the empty sequence ε. With composition, the binary relations on a set X form a monoid. Likewise for the mappings of X to itself.

v. A product-operation over a set X is *idempotent* if $xx = x$, for all $x \in X$. A *semilattice* is a commutative semigroup in which the product is idempotent. The operation of a semilattice is often denoted by $\vee$ or $\wedge$ instead of the multiplication sign $\cdot$.

vi. A *group* is a monoid $\langle G, \cdot, e\rangle$ in which every element has an *inverse*, i.e. for every element x there exists an element x^{-1} such that $x{\cdot}x^{-1} = x^{-1}{\cdot}x = e$. The *permutations* of a set X, i.e. the bijections of X onto itself, form a group with composition for multiplication, the *symmetric group* $\mathbf{S}_X$. A commutative group is also called *abelian*. The operations of an abelian group are often written *additively*: as $+$, 0 and $-$ instead of $\cdot$, e (or 1) and $^{-1}$.

vii. A *ring* is an algebra $\mathbf{R} = \langle R, +, 0, -, \cdot\rangle$ in which $\langle R, +, 0, -\rangle$ is an abelian group and $\langle R, \cdot\rangle$ a semigroup, and the following *distributive laws* hold:

$$\text{for all } x, y, z \in R,\ x{\cdot}(y + z) = xy + xz \text{ and } (x + y){\cdot}z = xz + yz.$$

The ring $\mathbf{R}$ is said to have an identity element if $\langle R, \cdot\rangle$ has one, and is called commutative if $\cdot$ is commutative.

viii. A ring $\mathbf{R}$ with identity element 1 is a *division ring* if $\langle R - \{0\}, \cdot, 1\rangle$ is a group. Observe that, if inversion is among the basic operations, a division ring is not a total algebra. A *field* is a commutative division ring.

ix. A *lattice* is an algebra $\langle L, \vee, \wedge\rangle$ such that $\langle L, \vee\rangle$ and $\langle L, \wedge\rangle$ are semilattices, and the two operations — the first is called *join* and the second *meet* — are connected by the *absorption laws*:

$$x \wedge (x \vee y) = x\ ; \qquad x \vee (x \wedge y) = x\ .$$

x. A lattice $\langle L, \vee, \wedge\rangle$ is *distributive* if for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (1),$$

or equivalently, for all $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad (2).$$

The subsets of a set K form a distributive lattice, with set union and intersection for operations.

xi. Let $\langle L, \vee, \wedge\rangle$ be a *bounded* distributive lattice, i.e. a distributive lattice with identity elements 1 for $\wedge$ (i.e. $x \wedge 1 = x$ for all $x \in L$) and 0 for $\vee$ (with $x \vee 0 = x$ for all $x \in L$). A *complement* of $x \in L$ is an element y of L such that $x \vee y = 1$ and $x \wedge y = 0$. Observe that the definition implies that x is the complement of its complement (if x *has* a complement).

c3 Proposition. An element of a bounded distributive lattice has at most one complement.

Proof. Let y and z be complements of x in a bounded distributive lattice $\mathbf{L} = \langle L, \vee, \wedge, 0, 1\rangle$. Then

$$y = y \wedge 1 = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z,$$

and similarly $z = z \wedge y$. ⊠

xii. A *boolean algebra* is a bounded distributive lattice in which every element x has a (unique) complement $\neg x$.

Let X be any set. Then $\mathcal{P}X := \langle \mathcal{P}X, \cup, \cap, \emptyset, X, \bar{\ }\rangle$ is a boolean algebra, with $\bar{A}$ defined as $X - A$. The truth values 0 (*false*) and 1 (*true*), with the operations $\vee$, $\wedge$ and $\neg$ defined by the Cayley tables

$\vee$	1	0
1	1	1
0	1	0

$\wedge$	1	0
1	1	0
0	0	0

x	¬x
1	0
0	1

form the standard Boolean algebra **2**.

§d Categories: Properties of arrows

Sections and retractions

Monoids are categories with a single object. Let us put this more precisely.

d1 Proposition. (a) Let $\mathbf{M} = \langle M, \cdot, e\rangle$ be a monoid. Then $\langle M, \cdot, \{e\}\rangle$ is a category.
(b) Let $\mathbf{K} = \langle K, \circ, Id\rangle$ be a nonvoid category. Then
1° the operation $\circ$ is total if and only if $|Id| = 1$;
2° if $Id = \{u\}$, then $\langle K, \circ, u\rangle$ is a monoid.

As soon as we know that categories are generalized monoids, we have a project: to generalize the concepts that help us understand monoids. In this section, we consider the notion of *inverse*. A left inverse of an element x of a monoid is an element y such that $yx = e$.

Notation. Two notational simplifications will increase the similarity to monoid theory. Instead of '$f \in Id$' we shall often write '$f = 1$'. The reader must keep in mind that in the context of categories different occurrences of 1 need not refer to the same thing. Moreover, in analogy with multiplication, we shall often omit the composition sign $\circ$.

d2 Definition. Let $\mathbf{C}$ be a category. An element $s \in C$ is a *section* if there exists $r \in C$ such that $rs = 1$.

Examples
i. An element $\langle Y, f\rangle$ of **Set** is a section if and only if f is injective and either Y is void or f is nonvoid.
ii. Let f: $Y \longleftarrow X$ be a morphism in **Set**. Then the embedding $x \longmapsto \langle f(x), x\rangle$ of X into $Y \times X$ is a section.
iii. If in a monoid **M** every element is a section, then **M** is a group. For suppose $rs = e$ and $qr = e$. Then $q = qrs = s$, so s has an inverse.

d3 Proposition. If f and g are sections, and gf exists, then gf is a section.

Proof. If $qf, rg = 1$, then $\mathsf{b}(r) = \mathsf{d}(g) = \mathsf{b}(f) = \mathsf{d}(q)$, and $qr \circ gf = 1$. ☒

d4 Proposition. If gf is a section, then so is f.

Proof. If $r \circ gf = 1$, then $rg \circ f = 1$ as well. ☒

d5 Definition. Let **C** be a category. An element $r \in C$ is a *retraction* if there exists $s \in C$ such that $rs = 1$.

Example **iv**. An element $\langle Y, f\rangle$ of Set is a retraction if and only if f is surjective. (This is really just another formulation of the Axiom of Choice.)

d6 Proposition. *Section* and *retraction* are dual notions.

d7 Definition. Let **C** be a category. An element of C is an *isomorphism* if it is at once a section and a retraction.

d8 Corollary. *Isomorphism* is a self-dual notion.

d9 Proposition. Composites of isomorphisms are isomorphisms.

If $f \circ g = 1$, we say that f is a *left inverse* of g, and g a *right inverse* of f. We have defined sections as arrows with a left inverse, and retractions dually, as arrows with a right inverse.

d10 Proposition. Let **C** be a category, and $f \in C$. Then f is an isomorphism if and only if f has a unique right inverse h and a unique left inverse k, and $h = k$.

Proof. (⇒) Since f is a retraction, it has a right inverse h, and since f is a section, it has a left inverse k. If $fh' = 1$, then $h' = kfh' = k = kfh = h$, so h is unique and $k = h$; hence, k is unique as well.
(⇐) Trivial. ☒

We denote the (unique, two-sided) inverse of an isomorphism f by f^{-1}. We use the abbreviation *Sec* to denote the class of sections, *Ret* will denote the class of retractions, and *Iso* the class of isomorphisms. Observe that identity elements are isomorphisms: thus we have $Id \subseteq Iso = Sec \cap Ret$.

Example **v**. By Proposition 2 in 1§h1, the isomorphisms in **Set** are the bijections.

d11 Corollary. If f is an isomorphism, so is f^{-1}, and $(f^{-1})^{-1} = f$. Moreover $\mathsf{d}(f) = \mathsf{b}(f^{-1})$, and a fortiori $\mathsf{d}(f^{-1}) = \mathsf{b}(f)$.

Isomorphism

In another sense, isomorphism is a relation between arrows, and a fortiori between objects.

D12 Definition. Let **C** be a category. Arrows $f, g \in C$ are *isomorphic*, notation $f \cong g$, if there are isomorphisms h and k such that $hf = gk$.

In particular, identity arrows u and v are isomorphic if (and only if) ufv exists for some isomorphism f; and objects are said to be isomorphic when their identity arrows are.

D13 Proposition. An arrow f is an isomorphism if and only if f is isomorphic to an identity element.

Proof. ($\Rightarrow$) If f is an isomorphism, then $f^{-1} \circ f = 1 = 1 \circ 1$.
($\Leftarrow$) If there are $h, k \in Iso$ such that $hf = 1 \circ k$, then $f = h^{-1} \circ k \in Iso$. ☒

Monomorphisms and Epimorphisms

Exercise 1 of 1§h states two *cancellation* properties of arrows, one belonging to surjections and one belonging to injections, that are (as we shall see) more general than the invertibility properties of retractions and sections.

d15 Definition. Let **C** be a category. An arrow $m \in C$ is *left cancellable*, or *a monomorphism*, if

$$\text{for all } h, k \in C, mh = mk \text{ implies } h = k.$$

We use *Mon* to denote the class of monomorphisms.

Example **vi**. In **Set**, *Mon* consists of the injective functions — or to be precise, of the pairs $\langle Y, f\rangle \in Set$ in which f is injective. In particular, every pair $\langle Y, \emptyset\rangle$ is a monomorphism; so not all monomorphisms are sections.

d16 Proposition. (i) If $m, n \in Mon$, then $m \circ n \in Mon$.
(ii) If $f \circ g \in Mon$, then $g \in Mon$.
(iii) $Sec \subseteq Mon$.

Proof. (i) Let m and n be monomorphisms; suppose $mnh = mnk$. Then since m is a monomorphism, $nh = nk$; whence, since $n \in Mon$, $h = k$. Hence $mn \in Mon$.
(ii) Suppose fg is a monomorphism, and $gh = gk$. Then $fgh = fgk$, so since $fg \in Mon$, $h = k$. This shows that $g \in Mon$.
(iii) Suppose $m \in Sec$. Then there exists r such that $rm = 1$. Now suppose $mh = mk$. Then $rmh = rmk$, hence $h = k$, which proves that $m \in Mon$. ☒

d17 Proposition. $Iso = Mon \cap Ret$.

Proof. Suppose $f \in Mon \cap Ret$; say $fs = 1$. Then $fsf = f = f \circ 1$, therefore since $f \in Mon$, $sf = 1$. ☒

d18 Definition. Let **C** be a category. An arrow $e \in C$ is *right cancellable*, or *an epimorphism*, if

$$\text{for all } h, k \in C, he = ke \text{ implies } h = k.$$

We use *Epi* to denote the class of epimorphisms.

Example **vii**. An element $\langle Y, f\rangle$ of **Set** is a epimorphism if and only if f is surjective. To prove this, suppose f is not surjective, say $y_0 \in Y - \operatorname{Ran} f$. Let $A = \{a, b\}$ be a set with two elements, and define h, k: $A \longleftarrow Y$ by: $h(y) = a$ for all $y \in Y$, and $k(y) = a$ if $y \neq y_0$, $k(y_0) = b$. Then $hf = kf$, but $h \neq k$, so f is not an epimorphism. The converse is 1§h, Exercise 1(i). So in **Set**, all epimorphisms are retractions.

d19 Proposition. *Monomorphism* and *epimorphism* are dual concepts.

Corollary. (a) If $d, e \in Epi$, then $de \in Epi$.
(b) If $fg \in Epi$, then $f \in Epi$.
(c) $Ret \subseteq Epi$.
(d) $Iso = Epi \cap Sec$.

In the bargain we get a specimen of a category with epimorphisms that are not retractions: take $\mathbf{Set}^\partial$, and recycle the example of monomorphisms that are not sections.

§e Relational systems

A structure is a *relational system* if its nominator is relational.

Some classes of relational systems

I. A *sorted set* is a structure $\mathbf{A} = \langle A, I\rangle$ such that $\operatorname{Dom} I$ consists entirely of unary relation symbols and
(i) for all $P \in \operatorname{Dom} I$, $I(P) \neq \emptyset$,
(ii) for all distinct $P, Q \in \operatorname{Dom} I$, $I(P)$ and $I(Q)$ are disjoint, and
(iii) $\bigcup\{I(P) \mid P \in \operatorname{Dom} I\} = A$.
A sorted set may be presented as $\langle A, \mathcal{P}\rangle$, where $\mathcal{P}$ is a partition of A, if the exact nature of the symbols is not important; or as

$$\langle A, \langle X_i \mid i \in I\rangle\rangle,$$

or

$$\langle A, \{X_i \mid i \in I\}\rangle,$$

where $\langle X_i \mid i \in I\rangle$ is a family of pairwise disjoint nonvoid subsets of A whose union is A.

For the rest of this section we concentrate on the the particularly interesting case of binary relation symbols, considered one at the time.

II. A *quasi-order*, or *quasi-ordered set* (*qoset* for short), is a set X with a binary relation on X that is reflexive and transitive — a *quasi-ordering of X*.

A binary relation R is *antisymmetric* if

$$xRy \text{ and } yRx \text{ implies } x = y.$$

An *order*, or (*partially*) *ordered set*, *poset* for short, is a quasi-order $\langle X, R\rangle$ with R antisymmetric; the relation in this case is called an *ordering*.

Quasi-orderings will often be denoted by the symbol $\leq$. The inverse of a given quasi-ordering $\leq$ is denoted by $\geq$; $\geq$ is a quasi-ordering as well. In connection with a relation denoted by $\leq$, '$x < y$' means '$x \leq y$ and $x \ngeq y$'. The inverse of $<$ is denoted by $>$. Observe that *in an order*, $x > y$ if and only if $x \geq y$ and $x \neq y$.

Since the inverse of a (quasi-)ordering is again a (quasi-)ordering, any universally valid statement about (quasi-)orderings $\leq$ remains valid when every '$\leq$' in it is reversed. This is the *duality principle* for quasi-orderings. It may be construed as a special case of the duality principle for categories, cf. Exercise 2. It does not amount to much in the general case, but it can become useful for orderings with special properties, as we shall see in the next section. We denote the dual $\langle Q, \geq\rangle$ of a quasi-order $\mathbf{Q} = \langle Q, \leq\rangle$ by $\mathbf{Q}^\partial$, using the same notation as with categories.

Let X be a subset of the universe of a quasi-order $\mathbf{Q} = \langle Q, \leq\rangle$. An element x of X is a *least* element of X if for all $y \in X$, $x \leq y$; it is a *minimal* element if for all $y \in X$, *if* $y \leq x$, *then* $x \leq y$. If $\mathbf{Q}$ happens to be an order, a least element of X is unique. Dually we have *greatest* and *maximal* elements.

Intervals. An *interval* in a quasi-order is the set of all elements *between* two given elements, or *beyond* one given element. There are various possibilities. Let $\mathbf{Q} = \langle Q, \leq\rangle$ be a quasi-order, and $x, y \in Q$. Then

$\mathrm{Iv}_\mathbf{Q}[x, y] = \{q \in Q \,|\, x \leq q \leq y\}$;
$\mathrm{Iv}_\mathbf{Q}(x, y) = \{q \in Q \,|\, x < q < y\}$;
$\mathrm{Iv}_\mathbf{Q}(x, y] = \{q \in Q \,|\, x < q \leq y\}$;
$\mathrm{Iv}_\mathbf{Q}[x, y) = \{q \in Q \,|\, x \leq q < y\}$;
$\mathrm{Iv}_\mathbf{Q}(\leftarrow y] = \mathrm{Iv}_\mathbf{Q}(y] = \{q \in Q \,|\, q \leq y\}$;
$\mathrm{Iv}_\mathbf{Q}(\leftarrow y) = \{q \in Q \,|\, q < y\}$;
$\mathrm{Iv}_\mathbf{Q}[x \rightarrow) = \mathrm{Iv}_\mathbf{Q}[x) = \{q \in Q \,|\, x \leq q\}$;
$\mathrm{Iv}_\mathbf{Q}(x \rightarrow) = \{q \in Q \,|\, x < q\}$.

We sometimes omit the subscript, or even the label "Iv", in contexts where it is redundant.

Cofinal sets. Let $\mathbf{Q} = \langle Q, \leq\rangle$ be a quasi-order, and $X \subseteq Q$. Then X is *cofinal in* $\mathbf{Q}$ if for each $q \in Q$, $\mathrm{Iv}_\mathbf{Q}[q \rightarrow) \cap X \neq \emptyset$. For example, by Euclid's Theorem the prime numbers are cofinal in $\langle \mathbb{N}, \leq\rangle$.

For any set X, $\langle \mathcal{P}X, \subseteq\rangle$ is an order. Two elements x and y of a quasi-order $\langle X, \leq\rangle$ are *comparable* if $x \leq y$ or $y \leq x$.

III. A binary relation R is *irreflexive* if there is no x such that xRx. A *strict order* is a set X with a relation on X that is irreflexive and transitive (a *strict ordering*).

For any set X, $\langle \mathcal{P}X, \subset\rangle$ is a strict order.

Strict orderings are often denoted by the symbol $<$. In connection with a relation denoted by $<$, '$x \leq y$' means '$x < y$ or $x = y$'.

e1 Proposition. Let R be a binary relation on a set X. Then
(i) if R is an quasi-ordering of X, $R - R^{-1}$ is a strict ordering;
(ii) if R is a strict ordering, $R \cup \Delta_X$ is an ordering of X.

We call $R \cup \Delta_X$ the *lax* ordering associated with $\langle X, R\rangle$. Intervals in a strict order are defined by way of the associated lax ordering.

IV. A *chain* (also called *total* or *linear* order) is an order $\langle X, \leq \rangle$ in which any two elements are comparable. A *chain in* an order $\langle X, \leq \rangle$ is a subset C of X such that $\langle C, \leq_C \rangle$ is a chain. We use the same terminology with regard to strict orders: a strict order $\langle X, < \rangle$ is a chain if the corresponding order $\langle X, \leq \rangle$ is one, and so on.

V. Let R be a binary relation on a class A. We say that R is *well-founded* if every nonvoid subclass X of A contains an element x such that

$$X \cap x/R = \emptyset.$$

If R is a strict ordering, we call such an element x an *R-minimal* element of X.

For example, let A be a class, the elements of which may be sets, and may have elements in common with A. Then the relation $\in_A$ is well-founded. For suppose X is a nonvoid subset of A. By the Regularity Axiom, X has an element x from which it is disjoint. That is to say: $X \cap x/\in = \emptyset$; from which $X \cap x/\in_A = \emptyset$ immediately follows.

Let $\mathbf{A} = \langle A, R \rangle$ and $\mathbf{B} = \langle B, S \rangle$ be structures with one binary relation. Then $\mathbf{A}$ is a *weak substructure* of $\mathbf{B}$ if $A \subseteq B$ and $R \subseteq S$; a *closed substructure*, or simply *substructure*, if moreover $R = S_A$.

Example **i**. $\langle \mathbb{Z}, < \rangle$ and $\langle \mathbb{Z}_+, | \rangle$ (where $x | y$ means x is a divisor of y, that is, there exists z such that $x \cdot z = y$) are weak, but not closed, substructures of $\langle \mathbb{Z}, \leq \rangle$. If $\mathbf{A}$ is a substructure of $\mathbf{B} = \langle B, S \rangle$, we sometimes present $\mathbf{A}$ as $\langle A, S \rangle$ instead of $\langle A, S_A \rangle$; for example, we say that $\langle \mathbb{N}, \leq \rangle$ is a substructure of $\langle \mathbb{Z}, \leq \rangle$.

Again, let $\mathbf{A} = \langle A, R \rangle$ and $\mathbf{B} = \langle B, S \rangle$ be structures with one binary relation. A *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $f: A \longrightarrow B$ such that

$$\text{for all } x, y \in A, Rxy \text{ implies } S(f(x), f(y)). \qquad (*)$$

A homomorphism that is injective is called an *embedding*. If in (*) instead of an implication we have an equivalence, we say f is *closed*.

Examples

ii. If $\mathbf{A} = \langle A, R \rangle$ is a weak substructure of $\mathbf{B} = \langle B, S \rangle$, 1_B^A is an embedding of $\mathbf{A}$ into $\mathbf{B}$. If $\mathbf{A}$ is a closed substructure, the embedding is closed.

iii. Let $\mathbf{A} = \langle A, R \rangle$ be a quasi-order, and α the equivalence relation $R \cap R^{-1}$. Put $B = A/\alpha$, $S = (\ni \circ R \circ \in)_B$, and $\mathbf{B} = \langle B, S \rangle$. Then $a \longmapsto a/\alpha$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$; observe that $\mathbf{B}$ is an order.

The homomorphisms that we defined above form a — large — category. Its objects are (or better perhaps, 'may be taken to be') the structures with a single binary relation. Composition is function composition, the composite of $f: \mathbf{A} \longrightarrow \mathbf{B}$ and $g: \mathbf{B} \longrightarrow \mathbf{C}$ being

$$g \circ f: \mathbf{A} \longrightarrow \mathbf{C}.$$

The identity arrow belonging to an object $\langle A, R \rangle$ is 1_A.

e2 Proposition. Let $\mathbf{A} = \langle A, R \rangle$ and $\mathbf{B} = \langle B, S \rangle$ be structures with one binary relation. A homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism if and only if it is closed and f is a bijection from A onto B.

Proof. ($\Rightarrow$) Suppose g is the inverse of f. Then $\langle A, g\rangle$ is the inverse of $\langle B, f\rangle$ in **Set**, so by Proposition 2 in 1§h1, f is a bijection. And, if $S(f(x), f(y))$, then since g is a homomorphism, $R(gf(x), gf(y))$, whence, since $gf = 1$, Rxy.

($\Leftarrow$) Let f: $\mathbf{A} \longrightarrow \mathbf{B}$ be a closed bijective homomorphism. We need only show that f^{-1} is a homomorphism. Suppose Suv. Since f is surjective, there are x, y $\in A$ such that $f(x) = u$ and $f(y) = v$. Since f is closed, Rxy. But

$$x = f^{-1}(u) \qquad \text{and} \qquad y = f^{-1}(v):$$

so f^{-1} preserves the relation. ⊠

These ideas can be generalized to structures with an arbitrary number of relations of various arities, associated with fixed relation symbols; how to do this, will appear from Chapter 4.

§f Lattices and closed set systems

This section is about lattice orders, and their connection with lattices as introduced in §c. The discussion leads naturally to complete lattices.

f1 Lattice orders

Let $\mathbf{P} = \langle P, \leq\rangle$ be a quasi-order, and $X \subseteq P$. An *upper bound* of X is an element p of P such that $x \leq p$ for all $x \in X$. A set is *upward bounded* if it has an upper bound. A *least upper bound* or *supremum* of X is an upper bound u such that for every upper bound p of X, $u \leq p$. In an order the supremum, if it exists, is unique. In this case we denote the supremum of X by $\bigvee X$, or, if X is a pair $\{x, y\}$, $x \vee y$.

Examples

i. Consider the order $\mathbf{N} = \langle \mathbb{N}, \leq\rangle$ of the natural numbers. Since $\mathbf{N}$ is linear, every finite subset of its universe has a least upper bound. The entire universe $\mathbb{N}$ has no upper bounds, and hence no least upper bound.

ii. Let $\mathbf{Q} = \langle \mathbb{Q}, \leq\rangle$ be the order of the rationals, and $X = \{x \in \mathbb{Q} \,|\, x^2 < 2\}$. Then X has upper bounds, but there is no least one among them.

iii. The same set X *does* have a least upper bound in the order $\mathbf{R} = \langle \mathbb{R}, \leq\rangle$ of the real numbers: it is $\sqrt{2}$. Indeed, *every* upward bounded set in $\mathbf{R}$ has a supremum.

iv. Let $\mathbf{D} = \langle \mathbb{N}, |\rangle$ be the order of the natural numbers under divisibility: $x|y$ if and only if for some $z \in \mathbb{N}$, $y = x \cdot z$.[2] Every set X in this order has a least upper bound. In particular, if X is infinite, then $\bigvee X = 0$.

Dually, we have *lower bounds* and *greatest lower bounds* or *infima*. The infimum of X is denoted by $\bigwedge X$, or by $x \wedge y$ if $X = \{x, y\}$.

Example **v**. Every set in the order $\mathbf{N}$ of Example **i** has a greatest lower bound.

A quasi-order $\mathbf{P} = \langle P, \leq\rangle$ is *directed* if every finite set of elements of P has an upper bound in $\mathbf{P}$. We call a set $X \subseteq P$ directed if the sub-quasi-order $\langle X, \leq\rangle$ is directed. In particular, the void set has an upper bound, so directed sets are nonvoid. An order $\mathbf{P}$ is an *upper semilattice order* if for all $p, q \in P$, $p \vee q$ ex-

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[2] We deviate here from our bold/nonbold convention for structures and their universes. We will do so again in similar circumstances.

ists, and a *lower semilattice order* if for all $p, q \in P$, $p \wedge q$ exist. A *lattice order* is an order that is both an upper and a lower semilattice order. Observe that the dual $\mathbf{P}^\partial = \langle P, \geq\rangle$ of a lattice order $\mathbf{P} = \langle P, \leq\rangle$ is again a lattice order.

We met with lattices before, in §c, as algebras. The relation with lattice orders is as follows.

f1.0 Theorem. (a) Let $\mathbf{L} = \langle L, \leq\rangle$ be an upper semilattice order. Define an operation $\vee$ on L by: $x \vee y = \bigvee\{x, y\}$. Then $\mathbf{L}^{\mathrm{a}} := \langle L, \vee\rangle$ is a semilattice.
(b) Let $\mathbf{L} = \langle L, \vee\rangle$ be a semilattice. Define a relation $\leq$ on L by: $x \leq y$ if and only if $x \vee y = y$. Then $\mathbf{L}^{\mathrm{r}} := \langle L, \leq\rangle$ is an upper semilattice order.
(c) Let $\mathbf{L}$ be an upper semilattice order. Then $\mathbf{L}^{\mathrm{ar}} = \mathbf{L}$.
(d) Let $\mathbf{L}$ be a semilattice. Then $\mathbf{L}^{\mathrm{ra}} = \mathbf{L}$.

Proof. Exercise. ⊠

f1.0.1 Corollary. (a) Let $\mathbf{L} = \langle L, \leq\rangle$ be a lower semilattice order. Define an operation $\wedge$ on L by: $x \wedge y = \bigwedge\{x, y\}$. Then $\mathbf{L}^{\mathrm{a}'} := \langle L, \wedge\rangle$ is a semilattice.
(b) Let $\mathbf{L} = \langle L, \wedge\rangle$ be a semilattice. Define a relation $\leq$ on L by: $x \leq y$ if and only if $x \wedge y = x$. Then $\mathbf{L}^{\mathrm{r}'} := \langle L, \leq\rangle$ is a lower semilattice order.
(c) Let $\mathbf{L}$ be a lower semilattice order. Then $\mathbf{L}^{\mathrm{a}'\mathrm{r}'} = \mathbf{L}$.
(d) Let $\mathbf{L}$ be a semilattice. Then $\mathbf{L}^{\mathrm{r}'\mathrm{a}'} = \mathbf{L}$.

Proof. Use the theorem: if $\langle L, \leq\rangle$ is a lower semilattice order, then $\langle L, \geq\rangle$ is an upper semilattice order. ⊠

f1.1 Theorem. (a) Let $\mathbf{L} = \langle L, \leq\rangle$ be a lattice order. Define operations $\vee$ and $\wedge$ on L by: $x \vee y = \bigvee\{x, y\}$; $x \wedge y = \bigwedge\{x, y\}$. Then $\mathbf{L}^{@} := \langle L, \vee, \wedge\rangle$ is a lattice.
(b) Let $\mathbf{L} = \langle L, \vee, \wedge\rangle$ be a lattice. Define a relation $\leq$ on L by: $x \leq y$ if and only if $x \vee y = y$. Then $\mathbf{L}^{®} := \langle L, \leq\rangle$ is a lattice order.
(c) Let $\mathbf{L}$ be a lattice order. Then $\mathbf{L}^{@®} = \mathbf{L}$.
(d) Let $\mathbf{L}$ be a lattice. Then $\mathbf{L}^{®@} = \mathbf{L}$.

Proof. Exercise. ⊠

Now that we know the one-to-one correlation between (semi-)lattice orders and (semi-)lattice algebras, we can make light of the difference. A semilattice may be called an *upper semilattice*, or $\vee$-semilattice, if its operation is denoted by $\vee$, and a *lower semilattice* ($\wedge$-semilattice) if it is $\wedge$. Observe that we have established a duality principle for lattices: interchanging $\vee$ and $\wedge$ in a universally valid statement gives a dual universally valid statement.

A lattice $\mathbf{L}$ is *complete* if every subset of L has a supremum.

f1.2 Proposition. (a) In a complete lattice, every subset of the universe has an infimum.
(b) Every complete lattice is bounded.

Proof. Let $\mathbf{L}$ be a complete lattice.
(a) Let X be a subset of L. Define Y to be the set of all lower bounds of X. Then $\bigvee Y$ is a lower bound of X: for every element of X is an upper bound of Y, and $\bigvee Y$ is the *least* of the upper bounds. But $\bigvee Y$ is an upper bound of Y, so it must be the *greatest* lower bound of X.
(b) We have $1 = \bigvee L$ and $0 = \bigwedge L$. ⊠

By duality, (a) implies that a lattice in which every set has an infimum is complete.

f1.3 Definition. Let $\mathbf{Q} = \langle Q, \leq\rangle$ be a quasi-order. A set $D \subseteq Q$ is *downwards closed* if $q \leq d \in D$ implies $q \in D$. A downwards closed set D is an *ideal* if $\langle D, \leq_D\rangle$ is directed. In particular, an interval of the form $\mathrm{Iv}_{\mathbf{Q}}(\leftarrow y]$ is a *principal ideal*. The collection of all ideals of $\mathbf{Q}$ will be denoted by $\mathrm{Idl}\,\mathbf{Q}$.

f1.4 Theorem. Let $\mathbf{Q} = \langle Q, \leq\rangle$ be a quasi-order in which every pair of elements has a least upper bound. Then $\mathbf{Idl\,Q} = \langle \mathrm{Idl}\,\mathbf{Q}, \subseteq\rangle$ is a complete lattice.

Proof. Observe that the intersection of any collection $\mathcal{I}$ of ideals is again an ideal: in particular, if $p, q \in \bigcap\mathcal{I}$, then every $I \in \mathcal{I}$ contains an upper bound of p and q, and if r is an upper bound of p and q in I, and r_0 is a least upper bound of p and q, then since I is downwards closed, and $r_0 \leq r$, also $r_0 \in I$, hence $r_0 \in \bigcap\mathcal{I}$. So every collection in $\mathrm{Idl}\,\mathbf{Q}$ has an infimum, and by the dual of Proposition 2 this is sufficient. ⊠

The *dual* ideals, or *filters*, of a quasi-order $\mathbf{Q}$ are the ideals of $\mathbf{Q}^\partial$.

A complete lattice with complementation is a *complete boolean algebra*.

Examples

vi. Finite lattices and finite boolean algebras are complete.

vii. Let $\mathbf{R}$ and $\mathbf{Q}$ be the ordered sets of, respectively, the reals and the rationals. The interval $\langle \mathrm{Iv}_{\mathbf{R}}[0, 1], \leq\rangle$ is complete, whereas $\langle \mathrm{Iv}_{\mathbf{Q}}[0, 1], \leq\rangle$ is *not* complete.

viii. The divisibility order $\mathbf{D} = \langle \mathbb{N}, |\rangle$ is a complete lattice.

ix. For any set X, $\langle \mathcal{P}X, \subseteq\rangle$ is a complete boolean algebra.

f1.5 Definition. An ideal I of a lattice $\mathbf{L}$ is a *complete ideal* if, for any $X \subseteq I$ that has a supremum in $\mathbf{L}$, $\bigvee X \in I$. Dually, a filter F is a *complete filter* if for any $X \subseteq F$ that has an infimum in $\mathbf{L}$, $\bigwedge X \in F$.

In a complete lattice $\mathbf{L}$, an ideal or filter is complete if and only if it is principal, that is, has the form $(x]_{\mathbf{L}}$ or $[x)_{\mathbf{L}}$, respectively, for some $x \in L$.

Example **x**. The complete ideals of a powerset lattice $\langle \mathcal{P}(X), \subseteq\rangle$ are the powersets $\mathcal{P}(Y)$ with $Y \subseteq X$; the complete filters, the sets $\{Z \subseteq X \mid Y \subseteq Z\}$ for $Y \subseteq X$.

f2 Closure

f2.1 Definition. Let A be any set. A *closed set system* (or *closure system*) on A is a collection $C \subseteq \mathcal{P}A$ that is closed under arbitrary intersections: if $\mathcal{B} \subseteq C$, then $\bigcap\mathcal{B} \in C$.

In particular, $A = \bigcap\emptyset \in C$. If C is a closed set system, then by f1.2 and duality, $\langle C, \subseteq\rangle$ is a complete lattice.

Examples

xi. The ideals in an $\vee$-semilattice form a closed set system.

xii. The collection of all transitive relations on a set X is seen to be a closed set system as follows. Let C consist of all the subsets of $X \times X$ that are transitive relations. Suppose $\langle R_i \mid i \in I\rangle$ is a family of elements of C. Then $\bigcap_i R_i$ is a transitive relation. For suppose $\langle x, y\rangle$ and $\langle y, z\rangle$ belong to $\bigcap_i R_i$. Then for every $i \in I$, $\langle x, y\rangle$ and $\langle y, z\rangle$ belong to R_i, hence by transitivity of R_i, $\langle x, z\rangle \in R_i$. So $\langle x, z\rangle \in \bigcap_i R_i$. Note that in particular $\bigcap\emptyset = X \times X$ is a transitive relation.

Since a quasi-ordering of X is a transitive relation that includes Δ_X, it immediately follows that the quasi-orderings of X form a closed set system.

xiii. The closed sets of a topological space form a closed set system.

xiv. The convex sets in a real vector space form a closed set system.

xv. The collection of all reflexive relations on subsets of a given set X is a closed set system.

xvi. Likewise for the symmetric relations on a given set.

f2.1.1 Lemma. Let $\langle C_i \mid i \in I\rangle$ be a family of closed set systems on the same set A. Then $\bigcap\limits_{i\in I} C_i$ is a closed set system.

Proof. If $\mathcal{B} \subseteq \bigcap_i C_i$, then $\mathcal{B} \subseteq C_i$ for every $i \in I$, so $\bigcap\mathcal{B} \in C_i$. Hence $\bigcap\mathcal{B} \in \bigcap_i C_i$. ☒

Examples

xvii. By xii, xvi and the lemma, the equivalence relations on a set A form a closed set system. Adding xv, we see that the same goes for the full equivalence relations.

xviii. Let A be a set, and f an n-ary operation over A. A subset B of A is *closed under f* if $f[B^n] \subseteq B$. (Observe that if $n = 0$, this simply means that the object denoted by f, if f is defined, belongs to B.) The subsets of A closed under f form a closed set system on A. By the lemma we may also consider, instead of just one operation, any *set* $\mathcal{F}$ of operations over A. A set is *closed under* $\mathcal{F}$ if it is closed under every element of $\mathcal{F}$.

f2.2 Definition. Let A be a set.
(a) An *operator on A* is a total operation on $\mathcal{P}A$.
(b) Let F be an operator on A.
1. F is *nondecreasing*, or *an extension operator*, if for all $X \subseteq A$, $X \subseteq F(X)$.
2. F is *idempotent* if for all $X \subseteq A$, $F(F(X)) = F(X)$.
3. F is *isotone* if for all $X, Y \subseteq A$, if $X \subseteq Y$, then $F(X) \subseteq F(Y)$.
(c) An extension operator is a *closure operator* if it is idempotent and isotone.

Examples

xix. The closure operator of a topological space: $\mathbf{C}(X)$ is the least closed set that contains X.

xx. The convex hull operator of a real vector space: $\mathbf{C}(X)$ is

$$\{x + r(y - x) \mid x, y \in X \text{ and } 0 \le r \le 1\}.$$

f2.3 Lemma. Let $\mathbf{C}$ and $\mathbf{D}$ be closure operators on a set A. If

$$\mathbf{DCD}(X) \subseteq \mathbf{CD}(X)$$

for all $X \subseteq A$, then the composite $\mathbf{C} \circ \mathbf{D}$ is a closure operator on A as well.

Proof. Exercise. ☒

Like lattices and lattice orders, closed set systems and closure operators are different ways of looking at the same thing.

f2.4 Definition. Let S be a closed set system on a set A. Define an operator $\mathbf{C}_S$ on A by: $\mathbf{C}_S(X) = \bigcap\{G \in S \mid X \subseteq G\}$.

f2.5 Theorem. Let A be a set.
(a) Let S be a closed set system on A. Then $\mathbf{C}_S$ is a closure operator on A.
(b) Let $\mathbf{C}$ be a closure operator on A. Then $\mathrm{Ran}(\mathbf{C})$ is a closed set system on A.
(c) Let S be a closed set system. Then $\mathrm{Ran}(\mathbf{C}_S) = S$.
(d) Let $\mathbf{C}$ be a closure operator. Then $\mathbf{C}_{\mathrm{Ran}(\mathbf{C})} = \mathbf{C}$.

Proof. Exercise. ☒

With reference to a closure operator $\mathbf{C}$, a set Y is *closed* if it can be represented as $\mathbf{C}(X)$ for some set X; in this case we say Y is *generated by* X, and X is a *generating set* of Y. If X is sufficiently fixed, its elements may be called *generating elements* or *generators* of Y. The set Y is *finitely generated* if X may be taken finite. Note that if Y is closed, $\mathbf{C}(Y) = Y$.

Examples

xxi. Let R be a binary relation on a set X. The *transitive closure* of R, which we denote by R^+, is defined by

$$R^+ = \bigcap\{S \subseteq X \times X \mid R \subseteq S \text{ and } S \text{ is transitive}\}.$$

A certain parallel is to be observed — or at least, confusion is to be avoided — with the X^+-notation for finite sequences. Pursuing this further, we define the n-th power of R, for $n \geq 1$, by

$$R^1 = R, \qquad R^{n+1} = R^n \circ R.$$

Then clearly, for all $m, n \geq 1$, $R^m \circ R^n = R^{m+n}$, and

$$R^+ = \bigcup_{n \geq 1} R^n. \tag{1}$$

xxii. Similarly the *transitive-reflexive closure* of R, notation R^*, is defined by

$$R^* = \bigcap\{S \subseteq X \times X \mid R \subseteq S \text{ and } S \text{ is a quasi-ordering}\}.$$

Putting $R^0 = \Delta_X$, we have

$$R^* = \bigcup_{n \geq 0} R^n. \tag{2}$$

xxiii. Let R be a binary relation on a set A. For $X \subseteq A$, $R^*[X]$ is the *closure of* X *under* R. In fact, the collection $C_R :=$

$$\{Y \subseteq A \mid R[Y] \subseteq Y\}$$

is a closed set system on A, and $R^*[X] = \bigcap\{Y \in C_R \mid X \subseteq Y\}$. — Cf. Example xviii: the closure of a set X under a unary operation f may be denoted by $f^*[X]$.

xxiv. The closure of a set B in a topological space is the intersection of all the closed sets that contain B.

xxv. The convex hull of a set B in a real vector space is the intersection of all the convex sets that contain B.

xxvi. The reflexive closure of a relation R on a set X is $R \cup \Delta_X$; the symmetric closure is $R \cup R^{-1}$.

xxvii. Let S be the successor relation on $\mathbb{N}$: xSy if and only if $x = y + 1$. Then S^+ is the strict ordering $>$, and S^* the lax ordering $\geq$.

xxviii. We have only considered, thus far, closures of sets, and of relations that are sets. Indeed, allowing *closed class systems* in general is somewhat tricky. Still there are cases in which the closure of a proper class is not a problem. In particular, the characterization (1) in example xxi may be applied to relations that are proper classes, hence any such relation has a transitive closure. For example, the transitive closure of the membership relation $\in$ is defined by

$x \in^+ y$ if and only if there is a sequence $\langle z_0, \ldots, z_n \rangle$ such that for all $i < n$, $z_i \in z_{i+1}$, and $x = z_0$, and $y = z_n$.

A similar case that we shall come across is that of the closure $R^*[C]$ of a class C under a relation R.

xxix. The closure of a set $X \subseteq A$ under a collection F of operations on A is the intersection of the subsets of A that include X and are closed under F.

f2.6 Proposition. Let $\mathbf{C}$ be a closure operator on a set A. In the complete lattice $\langle \operatorname{Ran} \mathbf{C}, \subseteq \rangle$ we have, for all $\mathcal{B} \subseteq \operatorname{Ran} \mathbf{C}$,

$$\bigwedge \mathcal{B} = \bigcap \mathcal{B},$$
$$\bigvee \mathcal{B} = \mathbf{C}(\bigcup \mathcal{B}).$$

Proof. Suppose $\mathcal{B} \subseteq \operatorname{Ran} \mathbf{C}$.

If $X \in \operatorname{Ran} \mathbf{C}$ and $X \subseteq B$ for all $B \in \mathcal{B}$, then $X \subseteq \bigcap \mathcal{B}$. Since by Theorem 5(b), $\bigcap \mathcal{B} \in \operatorname{Ran} \mathbf{C}$, $\bigwedge \mathcal{B} = \bigcap \mathcal{B}$.

If $X \in \operatorname{Ran} \mathbf{C}$ and $X \supseteq B$ for all $B \in \mathcal{B}$, then $X \supseteq \bigcup \mathcal{B}$, hence

$$X = \mathbf{C}(X) \supseteq \mathbf{C}(\bigcup \mathcal{B}).$$

Moreover $\mathbf{C}(\bigcup \mathcal{B}) \supseteq \bigcup \mathcal{B}$, hence $\bigvee \mathcal{B} = \mathbf{C}(\bigcup \mathcal{B})$. ⊠

As remarked under f2.1, closed set systems are complete lattices. The converse is also true, *up to isomorphism*. Let us call lattices $\langle C, \subseteq \rangle$ where C is a closed set system, *closed set lattices*.

f2.7 Theorem. Every complete lattice is isomorphic to a closed set lattice.

Proof. Let $\mathbf{L} = \langle L, \leq \rangle$ be a complete lattice. Let $\mathcal{P} \subseteq \operatorname{Idl} \mathbf{L}$ be the collection of all principal ideals. Then obviously $\mathbf{L} \cong \langle \mathcal{P}, \subseteq \rangle$, and $\mathcal{P}$ is a closed set system since

$$\bigcap_{i \in I} (\leftarrow y_i] = (\leftarrow \bigwedge_{i \in I} y_i].$$ ⊠

f3 Algebraic closure operators and algebraic lattices

Some of the examples in the previous subsection — in particular, xviii, xxi-xxiii — suggest a general construction of closure operators. Let $F\colon \mathcal{P}A \longrightarrow \mathcal{P}A$ be an operator on a set A. Define for $X \subseteq A$:

$$F^0(X) = X,$$
$$F^{n+1}(X) = F(F^n(X)).$$

Then for a relation R as in example xxiii, we can take $F(X) := R[X]$; and the closure of X under R will be

$$R^*[X] = \bigcup_{n\geq 0} F^n(X).$$

However, infinitely repeated application of an operator does not *necessarily* lead to a closure operator.

Example **xxx**. Consider the divisibility lattice **D** of example iv. For a set $X \subseteq \mathbb{N}$, define $F(X) = \{\bigvee\{x+1 \,|\, x \in X\}\}$, and

$$\mathbf{C}(X) = \bigcup_{n\geq 0} F^n(X).$$

Then **C** is not a closure operator: $F^n\{2\} = \{2+n\}$, so $\mathbf{C}\{2\} = [2)$ in the standard order of the natural numbers; but $F[2) = \{0\}$, so $\mathbf{CC}\{2\} \neq \mathbf{C}\{2\}$.

f3.1 Definition. An operator F on a set A is *algebraic* if for all $X \subseteq A$,

$$F(X) = \bigcup_{U \in p(X)} F(U).$$

f3.2 Lemma. Algebraic operators are isotone.

Proof. Suppose $X \subseteq Y$, and $z \in F(X)$. Then for some finite $U \subseteq X$, $z \in F(U)$. But $U \subseteq Y$ as well, so $z \in F(Y)$. ☒

f3.3 Lemma. Let A be a set. If F and G are algebraic operators on A, then so is $G \circ F$.

Proof. If $y \in G(F(X))$, then $z \in G(U)$ for some finite $U \subseteq F(X)$. Likewise for every $u \in U$, there is a finite set $V_u \subseteq X$ such that $u \in F(V_u)$. Now take $V = \bigcup_u V_u$: V is finite, and $y \in G(F(V))$. ☒

f3.4 Proposition. Let A be a set. If F is an algebraic operator on A, then so is the operation $X \longmapsto X \cup F(X)$.

f3.5 Theorem. Let A be a set, and F an algebraic extension operator on A. Define $\mathbf{C}$: $\mathcal{P}A \longrightarrow \mathcal{P}A$ by

$$\mathbf{C}(X) = \bigcup_{n=0}^{\infty} F^n(X).$$

Then **C** is an algebraic closure operator on A.

Proof. (i) $X = F^0(X) \subseteq \mathbf{C}(X)$.
(ii) By (i), $\mathbf{C}(X) \subseteq \mathbf{C}(\mathbf{C}(X))$. For the converse, it will suffice to show that $F(\mathbf{C}(X)) \subseteq \mathbf{C}(X)$. Suppose $y \in F(\mathbf{C}(X))$. Then there is some finite $U \subseteq \mathbf{C}(X)$ with $y \in F(U)$. Since U is finite, there must be some n such that $U \subseteq F^n(X)$. Then $y \in F^{n+1}(X) \subseteq \mathbf{C}(X)$.
(iii) Suppose $X \subseteq Y \subseteq A$. By a simple induction on n one shows, using the isotonicity of F, that $F^n(X) \subseteq F^n(Y)$. Then a fortiori $\mathbf{C}(X) \subseteq \mathbf{C}(Y)$.
(iv) It remains to be shown that **C** is algebraic. Take any $X \subseteq A$. By (iii), if $U \subseteq X$ is finite, $\mathbf{C}(U) \subseteq \mathbf{C}(X)$. Conversely, if $y \in \mathbf{C}(X)$, there must be some n such that $y \in F^n(X)$; by Lemma 3 and an easy induction, F^n is algebraic. ☒

Not all closure operators are algebraic. For example, if X is a set of points of the real line, and $y \in \mathbb{R}$ an arbitrary point, no finite number of elements of X

will be sufficient evidence to establish that y belongs to the closure of X, unless $y \in X$.

f3.6 Theorem. A closure operator is algebraic if and only if for every collection C of closed sets, if $\langle C, \subseteq\rangle$ is directed, then $\bigcup C$ is closed.

Proof. Let $\mathbf{C}$ be a closure operator.
($\Rightarrow$) Suppose $\langle C, \subseteq\rangle$ is a directed collection of closed sets, and $x \in \mathbf{C}(\bigcup C)$. Since $\mathbf{C}$ is algebraic, there is a finite set $U \subseteq \bigcup C$ such that $x \in \mathbf{C}(U)$. For each $u \in U$, let C_u be an element of C that u belongs to. Since $\langle C, \subseteq\rangle$ is directed, there exists $D \in C$ that includes all sets C_u. Since D is closed, $x \in D$, and hence $x \in \bigcup C$.
($\Leftarrow$) Let X be a set in the domain of $\mathbf{C}$. Define

$$C := \{\mathbf{C}(U) \mid U \in \wp(X)\}.$$

Then $\langle C, \subseteq\rangle$ is a directed system of closed sets, so $\bigcup C$ is closed. Suppose $y \in \mathbf{C}(X)$. Since $X \subseteq \bigcup C$, $\mathbf{C}(X) \subseteq \mathbf{C}(\bigcup C) = \bigcup C$; so for some $U \in \wp(X)$, $y \in \mathbf{C}(U)$. ⊠

We call a closure system S *algebraic* if for any directed $C \subseteq S$, $\bigcup C \in S$.

f3.7 Corollary. Let A be a set, $\mathcal{U}_i$, for $i \in I$, I nonvoid, algebraic closure systems on A, and

$$\mathcal{V} = \bigcap_{i \in I} \mathcal{U}_i.$$

(a) If the systems $\mathcal{U}_i$ are algebraic, then so is $\mathcal{V}$.
(b) Let F_i, for $i \in I$, be algebraic extension operators such that for all $X \subseteq A$,

$$\mathbf{C}_{\mathcal{U}_i}(X) = \bigcup_{n=0}^{\infty} F_i^n(X).$$

Define $G(X) := \bigcup_i F_i(X)$. Then G is an algebraic extension operator, and

$$\mathbf{C}_{\mathcal{V}}(X) = \bigcup_{n=0}^{\infty} G^n(X).$$

Proof. (a) If $C \subseteq \mathcal{V}$ is directed, then, since $\mathcal{V} \subseteq \mathcal{U}_i$, $\bigcup C \in \mathcal{U}_i$.
(b) If $a \in G(X)$, there must be some i with $a \in F_i(X)$. Then if U is a finite subset of X such that $a \in F_i(U)$, $a \in G(U)$. Since $X \subseteq F_i(X) \subseteq G(X)$, G is an extension operator.

Put $W := \bigcup_n G^n(X)$. We must show that $\bigcap\{V \in \mathcal{V} \mid X \subseteq V\} = W$.
($\supseteq$) If $Y \subseteq V \in \mathcal{V}$, then $F_i(Y) \subseteq V$. Hence $G(X) \subseteq V$, and by induction $G^n(X) \subseteq V$.
($\subseteq$) If $a \in F_i(W)$, there is a finite $U \subseteq W$ such that $a \in F_i(U)$. For some n, $U \subseteq G^n(X)$; so $a \in F_i(G^n(X)) \subseteq W$. Hence $W \in \mathcal{U}_i$, and since i was arbitrary, $W \in \mathcal{V}$. ⊠

f3.8 Definition. Let $\mathbf{L} = \langle L, \le\rangle$ be a complete lattice. An element a of L is *compact* if for all $X \subseteq L$ such that $a \le \bigvee X$, there exists $U \in \wp X$ with $a \le \bigvee U$. A complete lattice $\mathbf{L}$ is *algebraic* if for every element x of L there exists a set Y of compact elements such that $x = \bigvee Y$.

In particular, the least element of **L** is compact, since the flnite set U can be void. If a and b are compact, so is $a \vee b$; thus the compact elements form a sub-sup-semilattice of **L**.

f3.9 Theorem. If **C** is an algebraic closure operator, then $\langle \operatorname{Ran}\mathbf{C}, \subseteq\rangle$ is an algebraic lattice, and its compact elements are the finitely generated closed sets. Conversely, for every algebraic lattice **L** there exists an algebraic closure operator **C** such that $\mathbf{L} \cong \langle \operatorname{Ran}\mathbf{C}, \subseteq\rangle$.

Proof. (I) Let **C** be a closure operator.

Suppose X is a compact element of the closed set lattice $\langle \operatorname{Ran}\mathbf{C}, \subseteq\rangle$. Since $X \subseteq \bigvee(\mathbf{C}\{x\} \mid x \in X)$, there exists $U \in \wp X$ such that

$$X \subseteq \bigvee(\mathbf{C}\{x\} \mid x \in U).$$

Now $U \subseteq \bigvee(\mathbf{C}\{x\} \mid x \in U)$, and $\mathbf{C}\{x\} \subseteq \mathbf{C}(U)$ for all $x \in U$, so

$$\bigvee(\mathbf{C}\{x\} \mid x \in U) = \mathbf{C}(U);$$

and $\mathbf{C}(U) \subseteq \mathbf{C}(X) = X$, since X is closed, so $X = \mathbf{C}(U)$.

Conversely, suppose U is finite, **C** is algebraic, and $\mathbf{C}(U) \subseteq \bigvee C$ for some collection C of closed sets. Since **C** is algebraic,

$$\bigvee C = \mathbf{C}(\bigcup C) = \bigcup(\mathbf{C}(V) \mid V \in \wp(\bigcup C)).$$

It follows that for every $u \in U$, there is some $V_u \in \wp(\bigcup C)$ such that $u \in \mathbf{C}(V_u)$. Again, for every V_u there exists $\mathcal{D}_u \in \wp C$ with $V_u \subseteq \bigcup \mathcal{D}_u$; hence $u \in \mathbf{C}(\bigcup \mathcal{D}_u) = \bigvee \mathcal{D}_u$. So for the finite collection $C_0 = \bigcup(\mathcal{D}_u \mid u \in U)$, $U \subseteq \mathbf{C}(\bigcup C_0)$, therefore $\mathbf{C}(U) \subseteq \mathbf{C}(\bigcup C_0) = \bigvee C_0$.

Finally, if **C** is algebraic, by definition, for any $X \in \operatorname{Dom}\mathbf{C}$,

$$\mathbf{C}(X) = \mathbf{C}(\mathbf{C}(X)) = \mathbf{C}(\bigcup_{U \in \wp X} \mathbf{C}(U)) = \bigvee_{U \in \wp X} \mathbf{C}(U).$$

(II) Let **L** be an algebraic lattice, and let A be the set of compact elements of **L**. Define a function D with domain L by

$$D(x) := \{a \in A \mid a \leq x\}.$$

Then $\operatorname{Ran} D$ is a closed set system (cf. the proof of Theorem 2.7), and D is an isomorphism from **L** to the lattice $\langle \operatorname{Ran} D, \subseteq\rangle$. To show that the associated closure operator on L is algebraic, we use Theorem 6:

Suppose $\{D(x) \mid x \in X\}$ is directed by inclusion. Then for $a \in A$,

$a \leq \bigvee X$ if and only if there exists $U \in \wp X$ such that $a \leq \bigvee U$

if and only if there exists $x \in X$ such that $a \leq x$,

since there must be some $D(x) \supseteq \bigcup_{u \in U} D(u)$

if and only if $a \in \bigcup_{x \in X} D(x)$.

That is, $\bigcup\{D(x) \mid x \in X\}$ is $D(\bigvee X)$, a closed set. ⊠

Corollary. If S is an algebraic closure system, then $\langle S, \subseteq\rangle$ is an algebraic lattice; conversely, for every algebraic lattice **L** there exists an algebraic closure system S such that $\mathbf{L} \cong \langle S, \subseteq\rangle$.

Examples

xxxi. Let $\mathbf{S} = \langle S, \vee\rangle$ be a semilattice. For any $X \subseteq S$ let $\Delta^{\mathbf{S}}(X)$ be the ideal generated by X. Then $\Delta^{\mathbf{S}}$ is an algebraic closure operator on S. In fact, define

$$\mathsf{V}(X) = \bigcup\{(x \vee y] \mid x, y \in X\};$$

then V is an algebraic extension operator, and $\Delta^{\mathbf{S}}(X) = \bigcup_n \mathsf{V}^n(X)$.

xxxii. The operation $(.)^+$ that takes a binary relation R to its transitive closure R^+ is an algebraic closure operator. That is to say, for any set X, the restriction of $(.)^+$ to subsets of $X \times X$ is an algebraic closure operator on $X \times X$. Likewise for the transitive-reflexive closure R^*. (See examples xxi and xxii.)

xxxiii. We already mentioned that topological closure operators need not be algebraic. For example, consider the real line, with the Euclidean topology. The system

$$\{\{1,\dots, 1/n\} \mid n \in \mathbb{N}\}$$

of closed sets is directed by inclusion, but its union is not closed.

xxxiv. The convex hull operator of a real vector space is algebraic.

xxxv. The symmetric closure operator, for relations on a given set X, is algebraic.

xxxvi. Let Eq be the operation that takes any binary relation R to the least equivalence that includes R. For any set A, the limitation of Eq to $\mathcal{P}(A \times A)$ is an algebraic closure operator on $A \times A$ (cf. xxxii). In fact, if R_1 and R_2 are symmetric relations, then $R_1 \circ R_2$ is symmetric as well; hence $\mathrm{Eq}^A R$, the equivalence on A generated by R, is $(R \cup R^{-1})^+$.

xxxvii. For a given set A, let Eg^A be the operation that takes any $R \subseteq A \times A$ to the equivalence of A generated by R. Then Eg^A is an algebraic closure operator on $A \times A$, and $\mathrm{Eg}^A R = (\Delta_A \cup R \cup R^{-1})^+$. For a given set A, let Eg^A be the operation that takes any $R \subseteq A \times A$ to the equivalence of A generated by R. Then Eg^A is an algebraic closure operator on $A \times A$, and $\mathrm{Eg}^A R = (\Delta_A \cup R \cup R^{-1})^+$.

xxxviii. Let A be a set, and f an n-ary operation over A. The closure system of subsets of A closed under f is algebraic. Likewise, by Corollary 7 above, for the sets closed under a set $\mathcal{F}$ of operations over A.

f4 Lattices of transitive relations

By Example xxxii above, transitive closure is an algebraic closure operator. Since it will be of particular importance to us, we pause to take a closer look.

Notation. We denote the set of transitive relations on a given set A by $\mathrm{Tr}\, A$.

By Subsection 2, $\mathbf{Tr}\, A := \langle \mathrm{Tr}\, A, \subseteq\rangle$ is a complete lattice.

Proposition. Suppose $\mathcal{R} \subseteq \mathrm{Tr}\, A$. Then

(*) $\qquad \bigvee\mathcal{R} = \bigcup\{R_0 \circ \dots \circ R_n \mid n \in \mathbb{N}, R_0,\dots, R_n \in \mathcal{R}\}.$

Proof. By Proposition 2.6, $\bigvee\mathcal{R} = (\bigcup\mathcal{R})^+$; and by example xxi,

$$(\bigcup\mathcal{R})^+ = \bigcup((\bigcup\mathcal{R})^i \mid i \in \mathbb{Z}_+).$$

Now (*) follows by the observation that $\langle a, b\rangle \in \bigcup\mathcal{R}$ if and only if $\langle a, b\rangle \in R$ for some $R \in \mathcal{R}$. ☒

We may assume, for $i \leq n$ in (*), that R_i and R_{i+1} are incomparable under inclusion, since otherwise $R_i \circ R_{i+1} \in \{R_i, R_{i+1}\}$. Hence

Corollary I. Suppose $R, S \in \mathrm{Tr}\, A$. Then

$$R \vee S = \bigcup\{S \circ (R \circ S)^i, R \circ (S \circ R)^i, S \circ (R \circ S)^i \circ R, R \circ (S \circ R)^i \circ R \mid i \in \mathbb{N}\}.$$

In some cases this description of $R \vee S$ may be simplified.

Corollary II. If R and S are reflexive, $R \vee S = \bigcup\{(R \circ S)^i \mid i \in \mathbb{N}\}$.

Two relations R and S are said to *permute* if $R \circ S = S \circ R$.

Corollary III. (i) If R and S permute, $R \vee S = R \cup S \cup (R \circ S)$;
(ii) if they also are reflexive, $R \vee S = R \circ S$.

Proof. For example, $R \circ S \circ R = R \circ R \circ S \subseteq R \circ S$. ☒

Definition. Let $\mathbf{A} = \langle A, \leq_1\rangle$ and $\mathbf{B} = \langle B, \leq_2\rangle$ be lattices, with $A \subseteq B$. Then
(i) $\mathbf{A}$ is a *sublattice* of $\mathbf{B}$ if for all $x, y \in A$, the supremum and infimum of $\{x, y\}$ are the same in $\mathbf{A}$ and $\mathbf{B}$;
(ii) $\mathbf{A}$ is a *bounded sublattice* of $\mathbf{B}$ if $\mathbf{A}$ is a sublattice of $\mathbf{B}$, both $\mathbf{A}$ and $\mathbf{B}$ are bounded, and the identity elements of $\mathbf{B}$ belong to $\mathbf{A}$;
(iii) $\mathbf{A}$ is a *complete sublattice* of $\mathbf{B}$ if whenever a subset of $\mathbf{A}$ has a supremum or infimum in $\mathbf{B}$, this supremum or infimum also belongs to $\mathbf{A}$.

We denote the set of quasi-orderings of a set A, that is, the set of $R \in \mathrm{Tr}\, A$ that are reflexive, by $\mathrm{Qo}\, A$; the set of equivalences of A by $\mathrm{Eq}\, A$; the set of full equivalences of A by $\mathrm{Eqv}\, A$; and

$$\mathbf{Qo}\, A = \langle \mathrm{Qo}\, A, \subseteq\rangle, \mathbf{Eq}\, A = \langle \mathrm{Eq}\, A, \subseteq\rangle, \mathbf{Eqv}\, A = \langle \mathrm{Eqv}\, A, \subseteq\rangle.$$

One easily checks that $\mathrm{Qo}\, A$ and $\mathrm{Eqv}\, A$ are complete filters of $\mathbf{Tr}\, A$ and $\mathbf{Eq}\, A$ respectively, both generated by Δ_A. It follows that $\mathbf{Qo}\, A$ and $\mathbf{Eqv}\, A$ are *almost* complete sublattices: they have the same infima, and except for $\bigvee\emptyset$ the same suprema, as, respectively, $\mathbf{Tr}\, A$ and $\mathbf{Eq}\, A$. Such sublattices are called *complete filter sublattices*. The dual notion is that of a *complete ideal sublattice*.

Corollary IV. Suppose $\mathcal{R} \subseteq \mathrm{Eqv}\, A$ is nonvoid. Then

(*) $\quad \bigvee\mathcal{R} = \bigcup\{R_0 \circ \ldots \circ R_n \mid n \in \mathbb{N}, R_0, \ldots, R_n \in \mathcal{R}\}$.

Further observe that $\mathbf{Eq}\, A$ is a complete sublattice of $\mathbf{Tr}\, A$, and hence $\mathbf{Eqv}\, A$ a complete sublattice of $\mathbf{Qo}\, A$.

f5 Polarities

Fix a binary relation R. We define two operations on classes, the *polarities* of R:

$$X^{\rightarrow} = \{y \mid xRy \text{ for all } x \in X\};$$
$$Y^{\leftarrow} = \{x \mid xRy \text{ for all } y \in Y\}.$$

We speak of X *right-polar* and Y *left-polar*, respectively.

Theorem. Let A and B be classes, and $R \subseteq A \times B$; let $^{\rightarrow}$ and $^{\leftarrow}$ be the polarities of R. Then
(1) For all $X \subseteq A$ and $Y \subseteq B$, $X \subseteq X^{\rightarrow\leftarrow}$ and $Y \subseteq Y^{\leftarrow\rightarrow}$.

(2a) If $W \subseteq X \subseteq A$, then $X^{\rightarrow} \subseteq W^{\rightarrow}$.
(2b) If $Y \subseteq Z \subseteq B$, then $Z^{\leftarrow} \subseteq Y^{\leftarrow}$.
(3) For all $X \subseteq A$ and $Y \subseteq B$, $X^{\rightarrow} = X^{\rightarrow\leftarrow\rightarrow}$ and $Y^{\leftarrow} = Y^{\leftarrow\rightarrow\leftarrow}$.
(4a) The operation ${}^{\rightarrow\leftarrow}$ is a closure operator on A. Its closed classes are precisely the polars of subclasses of B.
(4b) The operation ${}^{\leftarrow\rightarrow}$ is a closure operator on B. Its closed classes are precisely the polars of subclasses of A.
(5) Let $\mathcal{L}(A)$ be the lattice of closed subclasses of A, and $\mathcal{L}(B)$ the lattice of closed subclasses of B. Then the polarities of R are isomorphisms between $\mathcal{L}(A)$ and $\mathcal{L}(B)^{\partial}$.

The relationship that holds between $\mathcal{L}(A)$ and $\mathcal{L}(B)^{\partial}$ by the last part of the theorem is what is called a *Galois connection*. The most famous connection of this kind is that between certain subfields of a field **F** and certain groups of automorphisms of **F**, discovered by Évariste Galois.

§15 Abstract Reduction Systems

Of particular interest are binary relations that represent some kind of simplification. For example, transitions like

$$(16 \times 15)/12 \longrightarrow 240/12 \longrightarrow 20$$

are typically considered to be *reductions* of complex expressions to simpler forms.

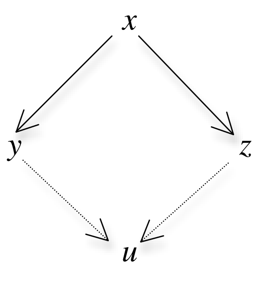
Typical questions arising in this context concern the existence of objects that cannot be simplified, and whether two different simplifications of the same thing must have some further simplification in common.

Let R be a binary relation on a set X. We use arrow-notation, as follows: $x \longrightarrow_R y$ for Rxy; $x \twoheadrightarrow_R y$ for $\langle x, y\rangle \in R^*$. If R is sufficiently clear from the context, we omit the index. In particular, $x \longrightarrow_{R^+} y$ then becomes $x \longrightarrow^+ y$. We write $x \sim_R y$ to express that $\langle x, y\rangle \in \mathrm{Eq}^X R$. An *R-normal form* is an $n \in X$ for which no $x \in X$ exists with Rnx, and such an n is a *normal form of* y if $n \sim_R y$.

15.1 Definition. (i) A binary relation R has the *diamond property* if

$$Rxy \;\&\; Rxz \text{ implies there exists } u \text{ such that } Ryu \;\&\; Rzu.$$

(ii) A binary relation R is *confluent* if R^* has the diamond property.



The diamond property.

15.2 Theorem. If R is confluent, then $x \sim_R y$ implies that there exists z such that $x \twoheadrightarrow_R z$ and $y \twoheadrightarrow_R z$.

Such an element z is called a *common reduct* of x and y.

Proof. Show that $\{\langle x, y\rangle \mid x$ and y have a common reduct$\}$ contains R, and is an equivalence relation. ⊠

15.3 Corollary. Suppose R is confluent. Then
(i) if n is an R-normal form of y, then $y \twoheadrightarrow_R n$;
(ii) any element has at most one R-normal form.

Exercises

§b

0. Let $\langle X, \cdot\rangle$ be a groupoid, and $\equiv$ a full equivalence relation of X that satisfies the condition

$$\text{if } x \equiv y \text{ and } u \equiv z, \text{ then } x \cdot u \equiv y \cdot z. \qquad (*)$$

(We shall call such a relation a full *congruence*.) Define products of sequences of elements of X inductively as follows:
x is the product of the one-element sequence $\langle x\rangle$;
assume $1 \le n \le m$: if s is a product of $\langle x_1,\ldots, x_n\rangle$, and t a product of $\langle x_{n+1},\ldots, x_m\rangle$, then $s \cdot t$ is a product of $\langle x_1,\ldots, x_m\rangle$.
Given that $x \cdot (y \cdot z) \equiv (x \cdot y) \cdot z$ for all $x, y, z \in X$, prove that $s \equiv t$ whenever s and t are products of the same sequence.

1. Let X and Y be sets. A *partial function* from X to Y is a function included in $Y \times X$. Verify that the partial functions, with sets as objects and composition based on function composition, form a category. What is its relation to **Rel** and **Set**?

Analogous tot the notation $f: A \longrightarrow B$ for mappings, the notation $f: X \circ\!\!\longrightarrow Y$ is used for "f is a partial function from X to Y".

2. (a) Prove that the surjections, with sets as objects and composition based on function composition, form a category.
(b) Likewise for the injections.

3. Prove that an arrow u in a category is an identity element if and only if whenever $u \circ f$ exists, it equals f.

§c

1. Let $\mathbf{X} = \langle X, \mathsf{d}, \mathsf{b}\rangle$ be a graph. Define an algebra $\mathbf{C}(\mathbf{X}) = \langle C(\mathbf{X}), \circ, \mathbf{d}, \mathbf{b}\rangle$ as follows.
The universe $C(\mathbf{X})$ consists of the *paths* of $\mathbf{X}$, i.e. the sequences $\langle x_0,\ldots, x_{2k}\rangle$ such that
x_{2i} $(i \le k)$ is a node,
x_{2i+1} $(i < k)$ is an edge,
and for all $i < k$, $\mathsf{b}(x_{2i+1}) = x_{2i}$ and $\mathsf{d}(x_{2i+1}) = x_{2i+2}$.
Let us write sequences as words, i.e. without brackets, commas or blanks. For a path $x_0\ldots x_{2k}$, $\mathbf{b}(x_0\ldots x_{2k}) = x_0$ and $\mathbf{d}(x_0\ldots x_{2k}) = x_{2k}$;
for paths $x_0\ldots x_{2k}$ and $y_0\ldots y_{2m}$, $x_0\ldots x_{2k} \circ y_0\ldots y_{2m}$ exists if and only if $x_{2k} = y_0$, and then it is $y_0\ldots y_{2m}$ in case $k = 0$, and $x_0\ldots x_{2k-1}y_0\ldots y_{2m}$ otherwise.
Prove that $\mathbf{C}(\mathbf{X})$ is a category.

2. Show that in any ring, $0{\cdot}x = x{\cdot}0 = 0$.

3. Let X be a nonvoid set; define S as X^+, and let $\mathbf{S} = \langle S, *\rangle$, where $*$ is the operation of concatenation (cf.§1H2). By Exercise 11 of §1H, $\mathbf{S}$ is a semigroup. Can there be a ring of which $\mathbf{S}$ is the multiplicative semigroup?

4. Let X be a nonvoid set; define M as $\mathcal{P}(X \times X)$, and let $\mathbf{M} = \langle M, \circ\rangle$, where $\circ$ is the operation of composition. Show that $\mathbf{M}$ is a monoid.

5. Let $\langle L, \vee, \wedge\rangle$ be a lattice in which for all x, y, z: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Prove that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ as well. Conclude by duality that the two forms of the distributive law are equivalent.

6. Formulate duality principles for distributive lattices, bounded lattices, and boolean algebras.

7. Prove that for every element x of a bounded lattice, $x \vee 1 = 1$ and $x \wedge 0 = 0$.

8. In a bounded distributive lattice, let x' be the complement of x, and y' the complement of y. Show that $x' \wedge y'$ is the complement of $x \vee y$. Conclude that $x' \vee y'$ is the complement of $x \wedge y$.

9. A *boolean ring* is a ring $\mathbf{R} = \langle R, +, 0, -, \cdot\rangle$ with idempotent multiplication.
(a) Show that in boolean rings the law $x + x = 0$ holds (so that $x = -x$), and multiplication is commutative.
(b) Let $\langle B, \vee, \wedge, 0, 1, \neg\rangle$ be a boolean algebra. Define: $b + c := (b \wedge \neg c) \vee (\neg b \wedge c)$; $b \cdot c := b \wedge c$; and $-b := b$. Verify that $\langle B, +, 0, 1, -, \cdot\rangle$ is a Boolean ring with multiplicative identity element 1.
(c) Let $\langle R, +, 0, 1, -, \cdot\rangle$ be a Boolean ring with multiplicative identity element. Define operations $\vee$, $\wedge$ and $\neg$ in such a way that $\langle R, \vee, \wedge, 0, 1, \neg\rangle$ is a boolean algebra.
10. (Generalisation of the distributive laws). How may $(X \cup Y) \cap (Z \cup W)$ be written as a union of intersections of X, Y, Z and W? How must we go about the general case, where we have an intersection

$$\bigcap_{i\in I} \bigcup_{j\in J_i} X_{ij}\ ?$$

(By duality, a similar law holds for the distribution of union over intersection.)

§D

1. Prove Proposition 1.

2. A *left identity element* in a semigroup $\langle X, \cdot\rangle$ is an element e such that for all $x \in X$, $ex = x$. A *left inverse of* x in a semigroup $\langle X, \cdot\rangle$ with respect to an element e is an element x^{-1} such $x^{-1}x = e$. Prove: a semigroup with a left identity element with respect to which every element of the semigroup has a left inverse is a group.

3. Verify Example **i**.

4. Verify Example **iv**.

5. Verify Example **v**.

6. Let f and g be isomorphic arrows with the same domain. It it generally true that there exists an isomorphism h such that $hf = g$?

§e

1. Prove that Δ_X is the only relation on X that is a full equivalence of X as well as an ordering.

2. Let $\langle X, R\rangle$ be a quasi-order. Define a composition $\circ$ on R by

$$\langle x, y\rangle \circ \langle u, v\rangle = \langle x, v\rangle \text{ if } y = u,$$
$$\text{and is undefined otherwise.}$$

Show that $\langle R, \circ, \Delta_X\rangle$ is a category.

3. Let $\mathbf{C} = \langle C, \circ, Id\rangle$ be a category. Show that

$$\langle Id, \{\langle u, v\rangle \in Id^2 \mid \text{for some } c \in C, u \circ c \circ v \text{ exists}\}\rangle$$

is a quasi-order.

4M. For a category $\mathbf{C}$, let $\mathbf{C}^{\mathrm{R}}$ be the relational system defined in the previous exercise; and for a quasi-order $\mathbf{X}$, $\mathbf{X}^{\mathrm{K}}$ the category defined in Exercise 2. Let $\mathbf{X}$ be any quasi-order.
(a) Show that $(\mathbf{X}^{\mathrm{K}})^{\mathrm{R}} \cong \mathbf{X}$.
(b) Verify that $(\mathbf{X}^{\partial})^{\mathrm{K}} \cong (\mathbf{X}^{\mathrm{K}})^{\partial}$, and that, for any category $\mathbf{C}$, $(\mathbf{C}^{\partial})^{\mathrm{R}} = (\mathbf{C}^{\mathrm{R}})^{\partial}$.
5. For any structure $\mathbf{Q} = \langle Q, R\rangle$ with one binary relation, define

$$L(\mathbf{Q}) = \langle Q, R \cup \Delta_Q \rangle,$$
$$S(\mathbf{Q}) = \langle Q, R - R^{-1} \rangle.$$

Prove that a quasi-order $\mathbf{Q}$ is an order if and only if $LS(\mathbf{Q}) = \mathbf{Q}$.

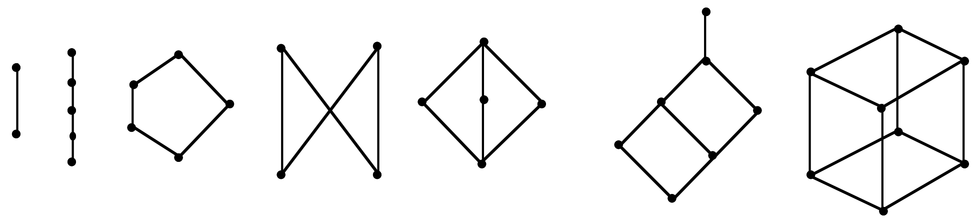
6. A *proper divisor* of an integer x is an integer $y \neq x$ that is a divisor of x. Consider the relation $\{\langle m, n\rangle \in \mathbb{N} \mid m$ is a proper divisor of $n\}$. Is it well-founded?

§f

Let $\langle X, \leq \rangle$ be an order, and $x \in X$; a *cover* of x is an element y of X such that $x < y$ and there is no $z \in X$ such that $x < z$ and $z < y$. Notation: $x \prec y$.

Finite orders are represented graphically by *Hasse-diagrams*, consisting of nodes corresponding to the elements of the order, and edges connecting the nodes representing covering pairs $x \prec y$, in such a way that the cover y is always *higher* on the page than the element x that it covers.

1. Consider the diagrams below. Which ones represent lattices? Which of the lattices are distributive?



2. Prove by induction on $n > 0$ that any set of n elements of a lattice order has an upper bound.

3a. Prove Theorem 1.0.
b. Prove Theorem 1.1.

4. Let $\langle L, \vee, \wedge \rangle$ be a bounded distributive lattice, and $x \in L$ with complement x'. Show that $x' = \bigvee\{y \in L \mid x \wedge y = 0\}$. Conclude that $x' = \bigwedge\{y \in L \mid x \vee y = 1\}$.

5. Let R be any binary relation. Show that the transitive closure of $R \cup R^{-1}$ is a classification.

6a. Prove Lemma 2.3.
b. Prove Theorem 2.5.

7. Prove Proposition 14.3.4.

8. Let $\mathcal{R}$ be a set of binary relations. Prove that the least transitive relation including all the relations in $\mathcal{R}$ is

$$\bigcup\{R_0 \circ \ldots \circ R_n \mid n \in \mathbb{N}, R_0, \ldots, R_n \in \mathcal{R}\}.$$

9. Let A be a set, $Q\varrho(A)$ the set of quasi-orderings of subsets of A, and $\mathbf{Q\varrho}(A) = \langle Q\varrho(A), \subseteq \rangle$. Show that $\mathbf{Q\varrho}(A)$ is a complete lattice.

10. Let A be a set, $Ord(A)$ the set of orderings of subsets of A, and $\mathbf{Ord}(A) = \langle Ord(A), \subseteq \rangle$. Prove, for any $\mathcal{R} \subseteq Ord(A)$:

(a) $\mathcal{R}$ has an upper bound if and only if for all $R, S \in \mathcal{R}$ and $a, b \in A$,

$$\langle a, b\rangle \in R \text{ and } \langle b, a\rangle \in S \text{ only if } a = b.$$

(b) The infimum of $\mathcal{R}$ exists if and only if either $\mathcal{R}$ is nonvoid or $|A| \leq 1$, and if it exists, it is the infimum of $\mathcal{R}$ in $\mathbf{Q\varrho}(A)$.

(c) If $\mathcal{R}$ has an upper bound, or $\mathcal{R}$ is directed, it has a supremum, which is the supremum of $\mathcal{R}$ in $\mathbf{Q\varrho}(A)$.

11. Let $\mathbf{Ord}(A)$ be as in the previous exercise. Prove that the maximal elements of $\mathbf{Ord}(A)$ are the total orderings of A.

12. Prove:

(i) if $\mathbf{A}$ is a sublattice of $\mathbf{B}$, and $\mathbf{B}$ is a sublattice of $\mathbf{C}$, then $\mathbf{A}$ is a sublattice of $\mathbf{C}$.

(ii) Likewise for complete sublattices and complete filter sublattices.

13. Let $\mathbf{L} = \langle L, \leq \rangle$ and $\mathbf{L}' = \langle L', \leq' \rangle$ be complete lattices. Mappings f: $L \longrightarrow L'$ and g: $L' \longrightarrow L$ form a *Galois connection* if

(*) for all $x \in L$ and $y \in L'$, $f(x) \leq' y$ if and only if $x \leq g(y)$.

(a) Show that for any g: $L' \longrightarrow L$ there exists at most one f: $L \longrightarrow L'$ that forms a Galois connection with g.

If f and g are as in (*), then g is the *upper adjoint* of f, $g = f^{\#}$; and f the *lower adjoint* of g, $f = g^{b}$.

(b) Let f and g be as in (*). Show that for all $x \in L$, $x \leq g(f(x))$, and for all $y \in L'$, $f(g(y)) \leq' y$.

(c) Show that (*) implies that f and g are isotone, i.e. $x \leq y$ implies $f(x) \leq' f(y)$ and $u \leq' v$ implies $g(u) \leq g(v)$.

(d) Show that a function g: $L' \longrightarrow L$ is an upper adjoint if and only if g preserves infima, that is, for all $X \subseteq L'$, $g(\bigwedge^{\mathbf{L}'} X) = \bigwedge^{\mathbf{L}} \boldsymbol{g}[X]$. (Dually, lower adjoints preserve suprema.)

14. Let f: $L \longrightarrow L'$ and g: $L' \longrightarrow L$ form a Galois connection between complete lattices $\mathbf{L} = \langle L, \leq \rangle$ and $\mathbf{L}' = \langle L', \leq' \rangle$. Let $M := g[L']$ and $M' := f[M]$.

(a) Show that $\mathbf{M} = \langle M, \leq_M \rangle$ and $\mathbf{M}' = \langle M', \leq'_{M'} \rangle$ are complete lattices. (They need not be complete sublattices of $\mathbf{L}$ and $\mathbf{L}'$: see Exercise 18 below.)

(b) Show that $\mathbf{M} \cong \mathbf{M}'$.

15. Prove the Theorem in 14.5.

16 (A. Tarski). Let $\mathbf{L}$ be a complete lattice, and f: $\mathbf{L} \longrightarrow \mathbf{L}$ an isotone operation, i.e. such that $x \leq y$ implies $f(x) \leq f(y)$. Show that for some $a \in L$, $f(a) = a$. (Such an element a is called a *fixed point* of f.) (Hint: consider $\bigwedge(x \mid fx \leq x)$.)

17 Let $\mathbf{C}$ be an algebraic closure operator on a set A. A subset X of A is $\mathbf{C}$-*independent* if

$$\forall x \in X.\ x \notin \mathbf{C}(X - \{x\}).$$

(a) Prove that the following statements are equivalent:

(i) for all $X \subseteq A$ and $u, v \in A$, if $u \in \mathbf{C}(X \cup \{v\})$ and $u \notin \mathbf{C}(X)$, then $v \in \mathbf{C}(X \cup \{u\})$;

(ii) for all $X \subseteq A$ and $u \in A$, if X is $\mathbf{C}$-independent and $u \notin \mathbf{C}(X)$, then $X \cup \{u\}$ is $\mathbf{C}$-independent;

(iii) for all $X \subseteq A$, if Y is a maximal $\mathbf{C}$-independent subset of X, then $\mathbf{C}(Y) = \mathbf{C}(X)$;

(iv) if $Y \subseteq X \subseteq A$, and Y is $\mathbf{C}$-independent, then there is a $\mathbf{C}$-independent set Z such that $Y \subseteq Z \subseteq X$ and $\mathbf{C}(Z) = \mathbf{C}(X)$.

(b) Suppose that (i)-(iv) hold. Prove: if X and Y are $\mathbf{C}$-independent, and $\mathbf{C}(X) = \mathbf{C}(Y)$, then $|X| = |Y|$. (That is, X and Y have the same cardinality; a concept explained in the next chapter.)

18. Let $A = B = \mathbb{N}$; let $R \subseteq A \times B$ be the divisibility relation (cf. Example 1(*d*)). Show that for $X \subseteq \mathbb{N}$, $X^{\rightarrow} = \{nd \mid n \in \mathbb{N}\}$, where d is the greatest common divisor of all elements of X. Conclude that the union of two closed classes need not be closed.

19. Prove that a quasi-order $\mathbf{Q}$ contains a cofinal chain if and only if $\mathbf{Q}$ is directed.

Ex. substructures

Ex. homomorphisms

Check every single example.