CHAPTER 3
NUMERATION

We prove an induction principle for arbitrary well-founded relations, and use it to prove a general recursion theorem. We define the ordinals as transitive sets of transitive sets, and prove they represent the well-orders. Cardinal numbers are defined as initial ordinals. Inaccessible cardinals are introduced, and presented as a justification for treating classes as sets at a higher level in the hierarchy of sets.

§a Induction

The principle of mathematical induction generalizes to arbitrary well-founded relations:

a1 Theorem (*R-induction*). Let R be a well-founded relation on a class A. If $B \subseteq A$ has the property

$$\text{for all } a \in A\text{, if } a/R \subseteq B \text{ then } a \in B \tag{*}$$

then $B = A$.

Proof. Assume R, A and B as stated. If $A - B \neq \emptyset$, it has an element x such that $(A - B) \cap x/R = \emptyset$. Then $x/R \subseteq B$, so $x \in B$ by (*), which is a contradiction. ☒

To every induction principle corresponds a form of *recursive definition*. In the case at hand:

a2 Theorem (*R-recursion*). Let R be a well-founded relation on a class A, such that for every $a \in A$, a/R is a set. Let Q be a binary operation. Then there exists a unique operation F such that $\mathrm{Dom}(F) \subseteq A$ and for all $a \in A$,

$$F(a) \simeq Q(a, F\lceil(a/R)). \ (\dagger)$$

Proof (cf. Exercises 1-4).
(*Unicity*:) That there is at most one such F is proved by R-induction.
(*Existence*:) To show that there also is *at least* one, we first assume that R is transitive. We abbreviate

$$f \text{ is a function and } \mathrm{Dom} f \subseteq a/R \text{ and for all } x \in a/R, f(x) \simeq Q(x, f\lceil(x/R))$$

to $S(a, f)$. By R-induction we have:

$$\forall b, c \in A\ [S(b, f)\ \&\ S(c, g)\ \&\ x \in b/R \cap c/R \Rightarrow f(x) \simeq g(x)]. \tag{1}$$

Again by R-induction we prove

$$\text{for all } a \in A, \text{ there exists } f \text{ such that } S(a, f): \tag{2}$$

assume that for every $x \in a/R$ we have f such that $S(x, f)$. By Unicity, this f is unique; by the Replacement Axiom we get a function ϕ, defined on a/R, such that for all $x \in a/R$, $S(x, \phi_x)$. Define a function g with domain contained in a/R by

$$g(x) \simeq Q(x, \phi_x). \tag{3}$$

Now if yRx, by (3) and the transitivity of R,

$$\begin{aligned} g(y) \simeq Q(y, \phi_y) &\simeq Q(y, \phi_x \upharpoonright (y/R)) && \text{(by (1))} \\ &\simeq \phi_x(y) && \text{(since } S(x, \phi_x)\text{)}, \end{aligned}$$

so $\phi_x = g \upharpoonright (x/R)$.

Substituting in (3), we get: $g(x) \simeq Q(x, g \upharpoonright (x/R))$, i.e. $S(a, g)$; so by R-induction, we have (2).

By (2) and Unicity, we have a mapping G from A such that $S(a, G(a))$ for all $a \in A$. Now put $F(a) \simeq Q(a, G(a))$. This definition works just like (3), and by a similar calculation we prove that F satifies the recursion equation (†).

If R is not transitive, we apply what we proved so far to the transitive closure R^+ and the operation Q' defined by

$$Q'(a, f) \simeq Q(a, f \upharpoonright (a/R)).$$

It gives us a unique operation F such that

$$F(a) \simeq Q'(a, F \upharpoonright (a/R^+)) \simeq Q(a, F \upharpoonright (a/R^+) \upharpoonright (a/R)) \simeq Q(a, F \upharpoonright (a/R)). \qquad \boxtimes$$

Examples. (*a*) The successor relation $\prec\, = \{\langle n, n+1 \rangle \mid n \in \mathbb{N}\}$ and the strict ordering $<$ are well-founded on $\mathbb{N}$. The $\prec$-induction principle, also known as *incomplete induction*, is

> If B is a subset of $\mathbb{N}$ such that $0 \in B$ and for every $b \in B$, $b + 1 \in B$ as well, then $B = \mathbb{N}$.

The special treatment of 0 reflects the circumstance that $0/\prec\, = \emptyset$. By the corresponding recursion principle, we define a function F on $\mathbb{N}$ by specifying (i) $F(0)$, and (ii) $F(n+1)$ in terms of $F(n)$. The $<$-induction principle, also known as *complete induction*, is

If B is a subset of $\mathbb{N}$ such that $n \in B$ whenever $\mathrm{Iv}(\leftarrow n) \subseteq B$, then $B = \mathbb{N}$.

By the corresponding recursion principle, we define a function F on $\mathbb{N}$ by specifying $F(n)$ in terms of $F(0), \ldots, F(n-1)$.

(*b*) As observed in §2E (Example V), elementhood is a well-founded relation. The $\in$-induction principle reads

If B is a subclass of a class A such that for all $a \in A$, $a \subseteq B$ implies $a \in B$, then $B = A$.

By $\in$-recursion, if A is a class and Q a binary operation, there exists a unique operation F such that $\mathrm{Dom}(F) \subseteq A$ and for all $a \in A$,

$$F(a) \simeq Q(a, F\lceil a).$$

§b Ordinals

A set X is *transitive* if every element of X is also a subset of X. An *ordinal number* (*ordinal* for short) is a transitive set of transitive sets.

b1 Lemma. Elements of ordinals are ordinals.

Proof. Let α be an ordinal, and $\beta \in \alpha$. Then β is transitive; and since α is transitive, every element of β is an element of α, and therefore a transitive set. ☒

The void set $\emptyset$ is an ordinal. If α is an ordinal, then so is $\alpha \cup \{\alpha\}$, its *successor*. The class *Ord* of all ordinals is — as any collection of sets — ordered by $\subseteq$. In this subset order, successor ordinals clearly have a greatest element. Ordinals other than $\emptyset$ that do not have a greatest element are called *limit ordinals*. Such ordinals exist. For example, recursively define a function f: $\mathbb{N} \longrightarrow$ *Ord* as follows:

$f(0) = \emptyset$;
$f(n+1) = f(n) \cup \{f(n)\}$.

By the replacement axiom, $\mathrm{Ran}(f)$ is a set. It is an ordinal, and it does not have a greatest element. This ordinal is usually denoted by ω.[1] Set theorists use ω as a substitute for $\mathbb{N}$. In accordance with this practice, from now on we shall often identify the number n and the set $f(n)$ we just defined. In particular, the void set qua ordinal may be indicated by 0.

b2 Theorem. Let α and β be ordinals. Then $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$.

Proof. By $\in$-induction (§A, Example (*b*)). Let α be any ordinal, and suppose that all $\gamma \in \alpha$ satisfy

$$\text{for all } \xi \in \mathit{Ord},\ \xi \in \gamma \text{ or } \xi = \gamma \text{ or } \gamma \in \xi. \tag{1}$$

Again, let β be any ordinal, and assume

$$\text{for all } \xi \in \beta,\ \xi \in \alpha \text{ or } \xi = \alpha \text{ or } \alpha \in \xi. \tag{2}$$

[1] By our assumption of sets of primitive elements, we get infinite sets such as ω more or less for free. Without suitable sets of primitives, we would have had to *postulate* the existence of infinite sets. Cf. §1H, Exercise 19.

If there exists $\xi \in \beta$ such that $\xi = \alpha$ or $\alpha \in \xi$, then $\alpha \in \beta$. Otherwise $\beta \subseteq \alpha$. Now for $\gamma \in \alpha$, by (1), $\beta \in \gamma$ or $\beta = \gamma$ or $\gamma \in \beta$. If there exists $\gamma \in \alpha$ such that $\beta \in \gamma$ or $\beta = \gamma$, then $\beta \in \alpha$. Otherwise $\alpha \subseteq \beta$, hence $\alpha = \beta$. So by $\in$-induction,

$$\text{for all } \beta \in \mathit{Ord}, \beta \in \alpha \text{ or } \beta = \alpha \text{ or } \alpha \in \beta \qquad (3),$$

and again by $\in$-induction,

$$\text{for all } \alpha \in \mathit{Ord}, \text{for all } \beta \in \mathit{Ord}, \beta \in \alpha \text{ or } \beta = \alpha \text{ or } \alpha \in \beta.$$ ⊠

For ordinals α and β, we often write $\alpha < \beta$ instead of $\alpha \in \beta$, and $\alpha \leq \beta$ instead of $\alpha \subseteq \beta$.

Induction on ordinals may be formulated as an extension of incomplete induction.

b3 Proposition (Ordinal Induction). Let $A \subseteq \mathit{Ord}$ be a class. If
(i) $0 \in A$,
(ii) for any ordinal α, if $\alpha \in A$, then $\alpha \cup \{\alpha\} \in A$, and
(iii) for any limit ordinal λ, if A contains every ordinal less then λ, then $\lambda \in A$,
then $A = \mathit{Ord}$.

The operations of elementary arithmetic can be generalized to arbitrary ordinals. In the definition we use recursion on ordinals in accordance with the Ordinal Induction Principle.

b4 Definition (Ordinal Arithmetic). Let α and β be any ordinals.

(add) (i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta \cup \{\beta\}) = (\alpha + \beta) \cup \{\alpha + \beta\}$,
(iii) for any limit ordinal λ, $\alpha + \lambda = \bigcup_{\xi < \lambda} (\alpha + \xi)$.

(mult) (i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta \cup \{\beta\}) = (\alpha \cdot \beta) + \alpha$,
(iii) for any limit ordinal λ, $\alpha \cdot \lambda = \bigcup_{\xi < \lambda} (\alpha \cdot \xi)$.

(exp) (i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta \cup \{\beta\}} = \alpha^\beta \cdot \alpha$,
(iii) for any limit ordinal λ, $\alpha^\lambda = \bigcup_{\xi < \lambda} \alpha^\xi$.

§c Cumulative hierarchies

The ordinals serve as a scale in an orderly construction of all the sets. Let U be any given set of primitive elements — *urelements*, in dog German; they may be anything except classes.

Now we put:

$\mathcal{V}_0(U) = U$;
for all $\alpha \in Ord$, $\mathcal{V}_{\alpha+1}(U) = \mathcal{V}_\alpha(U) \cup \mathcal{P}\mathcal{V}_\alpha(U)$;
and for each limit ordinal λ, $\mathcal{V}_\lambda(U) = \bigcup_{\xi<\lambda} \mathcal{V}_\xi(U)$.

Finally, the universe over U is

$$\mathcal{V}(U) = \bigcup_{\alpha\in Ord} \mathcal{V}_\xi(U).$$

It can be proved that every set required by the axioms of set theory, as long as all the primitive elements it involves come from U, belongs to $\mathcal{V}(U)$. All the possible urelements together may be considered as forming a class $\mathcal{U}$. The global universe then, of all sets over $\mathcal{U}$, is the union of all universes $\mathcal{V}(U)$ for subsets U of $\mathcal{U}$.

As an application of this construction, we prove a generalization of the Axiom of Choice.

Theorem (Axiom of Choice for Classes). Let I be a set, and for every $i \in I$, $\mathcal{A}_i$ a nonvoid class. Assume that the class of urelements is the union of an ordinal-indexed chain $U_0 \subset U_1 \subset \ldots$ Then there exists a function f on I such that for every $i \in I, f(i) \in \mathcal{A}_i$.

Proof. Take any $i \in I$. An element of $\mathcal{A}_i$ must be primitive, or a set in some universe $\mathcal{V}(U_\xi)$. In any case, there must be a least ordinal α_i such that

$$\mathcal{A}_i \cap \mathcal{V}(U_{\alpha_i}) \neq \emptyset,$$

and then there must be a least ordinal β_i such that

$$A_i := \mathcal{A}_i \cap \mathcal{V}_{\beta_i}(U_{\alpha_i}) \neq \emptyset.$$

By the axiom of choice, there exists a function f on I such that for every $i \in I$, $f(i) \in A_i$; and since $A_i \subseteq \mathcal{A}_i$, we have made our point. ☒

§d. Well-Order

A *well-order* is a well-founded strict linear order. Every nonvoid set in a well-order has a *least* element.

d1 Theorem. $\in$ is a well-ordering of the ordinals.

Proof. Since ordinals are transitive, $\in_{Ord}$ is a transitive relation. By Regularity it is well-founded, and by b2 linear. ☒

A fortiori, every ordinal (or to be precise, every structure $\boldsymbol{\alpha} = \langle \alpha, \in_\alpha \rangle$ with $\alpha \in Ord$) is a well-order.

We say a relational structure $\mathbf{A} = \langle A, R\rangle$ is *uniquely isomorphic* to an element of a class C if there is just one $\mathbf{C} \in C$ such that $\mathbf{A} \cong \mathbf{C}$, and moreover just one isomorphism from $\mathbf{A}$ to $\mathbf{C}$.

d2 Theorem. Every well-order is uniquely isomorphic to an ordinal.

Proof. Let $\langle X, <\rangle$ be a well-order, and define a function f on X recursively by

$$f(x) = \{f(y) \mid y < x\}. \quad (1)$$

Then $f[X]$ is an ordinal, and $f(y) \in f(x)$ if and only if $f(y) < f(x)$. A fortiori f is injective. Finally, observe that the definition (1) is forced upon us if we want to have an isomorphism from $\langle X, <\rangle$ to an ordinal structure $\boldsymbol{\alpha} = \langle \alpha, \in_\alpha\rangle$. ☒

Let $\mathbf{A} = \langle A, \leq\rangle$ be a quasi-order. An *initial segment* of $\mathbf{A}$ is an order $\langle B, \leq_B\rangle$ with $B \subseteq A$ downwards closed. Likewise an initial segment of a strict order $\langle A, <\rangle$ is an order $\langle B, <_B\rangle$ with $B \subseteq A$ downwards closed.

Corollary. Let $\mathbf{X}$ and $\mathbf{Y}$ be well-orders. Then either $\mathbf{X}$ is uniquely isomorphic to an initial segment of $\mathbf{Y}$, or $\mathbf{Y}$ is uniquely isomorphic to an initial segment of $\mathbf{X}$.

Proof. Suppose $\mathbf{X} \cong \boldsymbol{\alpha}$, and $\mathbf{Y} \cong \boldsymbol{\beta}$; then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. ☒

Observe that $\langle \alpha + \beta, \in\rangle$ may be constructed by putting the well-order $\boldsymbol{\beta}$ *on top of* the well-order $\boldsymbol{\alpha}$. Indeed, for any two disjoint quasi-orders $\mathbf{A} = \langle A, R\rangle$ and $\mathbf{B} = \langle B, S\rangle$ (*disjoint* meaning that the universes are disjoint) we can define

$$\mathbf{A} \oplus \mathbf{B} = \langle A \cup B, R \cup S \cup (A \times B)\rangle. \quad (2)$$

In general we define $\mathbf{A} \oplus \mathbf{B}$ as any order obtained by applying the construction (2) to disjoint isomorphic copies of $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ are strict, we construct the sum $\mathbf{A} \oplus \mathbf{B}$ by first switching to the associated lax orders $\widetilde{\mathbf{A}} = \langle A, R \cup \Delta_A\rangle$ and $\widetilde{\mathbf{B}} = \langle B, S \cup \Delta_B\rangle$, then forming $\widetilde{\mathbf{A}} \oplus \widetilde{\mathbf{B}}$, and finally switching back to the associated strict order. Then

$$\langle \alpha + \beta, \in\rangle \cong \boldsymbol{\alpha} \oplus \boldsymbol{\beta}.$$

Similarly, multiplication may be defined for arbitrary relations:

$$\mathbf{A} \odot \mathbf{B} := \langle A \times B, \{\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \mid \langle b_1, b_2\rangle \in S \text{ or } [b_1 = b_2 \ \&\ \langle a_1, a_2\rangle \in R\}]\rangle.$$

Intuitively, every element of $\mathbf{B}$ is replaced by a copy of $\mathbf{A}$.

For infinite α and $n \in \omega$, $\mathbf{n} \oplus \boldsymbol{\alpha} \cong \boldsymbol{\alpha}$. For a limit λ, $\mathbf{n} \odot \boldsymbol{\lambda} \cong \boldsymbol{\lambda}$.

Finite strict linear orders are well-orders. The set of natural numbers is naturally well-ordered by <. The standard strict ordering of $\mathbb{Z}$ is not a well-ordering, but it can be used to well-order $\mathbb{Z}$: for example, we could place the negative numbers, ordered by >, after the natural numbers with their familiar ordering. By an extension of this method a well-ordering of $\mathbb{Q}$ may be devised. But could we well-order $\mathbb{R}$? The answer is positive; in fact, any set can be made the universe of a well-order, though we may not be able to say how.

d3 Well-Ordering Theorem. Every set can be well-ordered.

Proof. Let X be an arbitrary set. It suffices to construct a bijection from an ordinal onto X: then we can use the ordering of the ordinal to order X.

By the axiom of choice, there exists a choice function

$$f\colon \mathcal{P}X - \{\emptyset\} \longrightarrow X.$$

Define an operation ϕ from the ordinals into X recursively, by

$$\phi(\beta) \simeq f(X - \{\phi(\xi) \mid \xi \in \beta\}).$$

Let α be the least ordinal for which ϕ is not defined. Then ϕ is a bijection from α onto X. ☒

§e Cardinals

Any finite set is equipollent with a finite ordinal; this finite ordinal is the number of its elements, its *cardinality*. We want to generalize this to infinite sets. By the Well-Ordering Theorem, for any set X we can find an ordinal that is equipollent with X. But for infinite X this ordinal is not unique. If we use the proof of the Well-Ordering Theorem to well-order $\mathbb{N}$, for example, and we happen on a choice function f that maps every segment $[n, \infty)$ to its least element, we find $\alpha = \omega$; but if

$$f(\{0\} \cup [n+1, \infty)) = n + 1$$

for all n, we get $\alpha = \omega + 1$. However, luckily, every nonvoid class of ordinals contains a least element; and we define the cardinality $|X|$ as the *least* ordinal equipollent with X. The ordinals that are least elements in their equipollence classes we call *cardinals*.

When we employ infinite ordinals for counting, as opposed to enumerating, we use different notation. We denote ω by $\aleph_0$, and write Fraktur letters such as $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{k}$, for arbitrary cardinals.

The arithmetical operations of the ordinals are bound up with the ordering, and for this reason not immediately suitable for cardinals. We define the *sum* of cardinals $|X|$ and $|Y|$ as $|X \cup Y|$, provided that X and Y are disjoint. In general,

$$|X| + |Y| = |(\{0\} \times X) \cup (\{1\} \times Y)|.$$

The *product* is defined by $|X| \cdot |Y| = |X \times Y|$.

In fact, thus far the definitions still tally, in the sense that, with cardinal operations on the left and ordinal operations on the right, we have $|\alpha| + |\beta| = |\alpha + \beta|$, and $|\alpha| \cdot |\beta| = |\alpha \cdot \beta|$. Exponentiation is really different, however. Ordinal exponentiation is repeated multiplication, and hence, for example, $|2^{\omega}| = |\omega^{\omega}| = \aleph_0$. In contrast, $|X|^{|Y|}$ is defined as $|X^Y|$.

Proposition (Cardinal arithmetic). Let $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{n}$ and n be cardinals, with $n \in \omega - \{0\}$.
(a) If $\mathfrak{k} \geq \mathfrak{m} \geq 1$, and $\mathfrak{k}$ is infinite, then $\mathfrak{k} + \mathfrak{m} = \mathfrak{k} \cdot \mathfrak{m} = \mathfrak{k}$. Hence $\mathfrak{k}^n = \mathfrak{k}$.
(b) $(\mathfrak{k}^{\mathfrak{m}})^{\mathfrak{n}} = \mathfrak{k}^{\mathfrak{m}\mathfrak{n}}$.
(c) Let $\mathfrak{k}$ be infinite, and $2 \leq \mathfrak{m} \leq 2^{\mathfrak{k}}$. Then $\mathfrak{m}^{\mathfrak{k}} = 2^{\mathfrak{k}}$.

Proof. Exercise. ⊠

Fix a set X, and let U be a subset of X. Then χ_X, the *characteristic function* of U, is the mapping from X into $2 = \{0, 1\}$ defined by

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Cantor). If $\mathfrak{m} > 1$, then $2^{\mathfrak{m}} > \mathfrak{m}$.

Proof. Let X be a set such that $\mathfrak{m} = |X|$. Observe that 2^X is pecisely the set of characteristic functions of subsets of X; so $2^{\mathfrak{m}} = |2^X| = |\mathcal{P}X|$. Hence if $2^{\mathfrak{m}} \leq \mathfrak{m}$, there exists a surjection $f \colon X \twoheadrightarrow \mathcal{P}X$. Define U as

$$\{x \in X \,|\, x \notin f(x)\}.$$

Since f is surjective, there exists $u \in X$ such that $U = f(u)$. Now if $u \in U$, by the definition of U, $u \notin U$. So $u \notin U$. But then, again by the definition of U, $u \in U$. So $2^{\mathfrak{m}} \nleq \mathfrak{m}$, and $2^{\mathfrak{m}} > \mathfrak{m}$ by Theorem b2. ⊠

Arbitrary sums and products are defined as follows. Let $\langle \mathfrak{m}_i \,|\, i \in I \rangle$ be a family of cardinals. If $\langle X_i \,|\, i \in I \rangle$ is a family of sets such that $|X_i| = \mathfrak{m}_i$, then

$$\sum_{i \in I} \mathfrak{m}_i = \Big| \bigcup_{i \in I} (\{i\} \times X_i) \Big| \quad \text{and} \quad \prod_{i \in I} \mathfrak{m}_i = \Big| \prod_{i \in I} X_i \Big|.$$

A cardinal $\mathfrak{n}$ is *regular* if for every family $\langle \mathfrak{m}_i \,|\, i \in I \rangle$ of cardinals such that $\mathfrak{m}_i < \mathfrak{n}$ for all $i \in I$ and moreover $|I| < \mathfrak{n}$,

$$\sum_{i \in I} \mathfrak{m}_i < \mathfrak{n}.$$

A cardinal $\mathfrak{n}$ is a *strong limit cardinal* if $2^{\mathfrak{m}} < \mathfrak{n}$ for all $\mathfrak{m} < \mathfrak{n}$. A cardinal greater than $\aleph_0$ that is both regular and a strong limit cardinal, is called *inaccessible*.

Inaccessible cardinals form a justification, in a sense, for dealing with classes as if they were sets. For suppose $\mathfrak{v}$ is an inaccessible cardinal, and that any given sets of primitive elements have cardinality less than $\mathfrak{v}$. Then the stage $\mathcal{V}_{\mathfrak{v}}$ in the cumulative hierarchy satifies the axioms of set theory. If moreover we have not more than $\mathfrak{v}$ primitive elements, then every subclass of $\mathcal{V}_{\mathfrak{v}}$ becomes a set in a higher stage of the hierarchy. So we could choose to do our serious mathematics within $\mathcal{V}_{\mathfrak{v}}$, and if we want to do something with proper classes, take a little excursion to higher stages.

Do inaccessible cardinals exist? Russell's Paradox proves that it is foolish to maintain that all classes are sets. In a precise sense, it is more foolish to believe that inaccessible cardinals exist than to believe that the axioms of set theory are true. But inaccessible cardinals have been around almost as long as Zermelo's axioms. So if there is anything wrong with them, it must be quite profound.

Exercises

§a

1. (a) Prove, for any set a, that $a/{\in^+}$ is a set as well.
(b) Prove that, if R is well-founded, then R^+ is also well-founded.

2. Let R, A and Q be as in the statement of the R-Recursion Theorem. Let F and G be operations such that $\mathrm{Dom}(F) \cup \mathrm{Dom}(G) \subseteq A$ and for all $a \in A$, $F(a) \simeq Q(a, F{\restriction}(a/R))$ and $G(a) \simeq Q(a, G{\restriction}(a/R))$. Prove that $F = G$.

3. Prove claim (1) in the proof of the R-Recursion Theorem.

4. Let X be a set, and S a relation such that for every $x \in X$, there is exactly one y such that Sxy. Construct a function f with domain X such that for every $x \in X$, $S(x, f(x))$.

5. Prove that incomplete induction (called mathematical induction in §1B) is equivalent to complete induction.

§b

1. Let $f\colon \mathbb{N} \longrightarrow \mathit{Ord}$ be as in the text.
(a) Prove, for all $m, n \in \mathbb{N}$: $f(m) + f(n) = f(m + n)$.
(b) Analogously for multiplication and exponentiation.

2. Prove Proposition 3.

3. Prove:
(a) $1 + \omega = \omega$;
(b) $2{\cdot}\omega = \omega$.

4. Prove that ordinal addition and multiplication are associative.

5. Prove, for all ordinals α, β, γ:
(a) $\alpha^{\beta+\gamma} = \alpha^{\beta}\cdot\alpha^{\gamma}$;
(b) $\alpha^{\beta\cdot\gamma} = (\alpha^{\beta})^{\gamma}$.

§c

Assume that the natural numbers are the only primitive elements, and that $\mathbb{N}$ is the only set of primitive elements that is given.

(a) Assume that the integers are constructed as equivalence classes of pairs of natural numbers, as in the Example in §7. At which level $\mathcal{V}_\alpha$ of the hierarchy of §18 do we first meet the set $\mathbb{Z}$ of the integers?
(b) Define the ordering of the integers in (a) in terms of the ordering of $\mathbb{N}$.
(c) Next construct the rationals as equivalence classes of pairs consisting of an integer and a positive integer. At which level $\mathcal{V}_\alpha$ do we first meet $\mathbb{Q}$?
(d) Define the ordering of these rationals in terms of the ordering of $\mathbb{Z}$.
(e) Let the reals be defined as *the upper classes of Dedekind cuts of the first kind*, that is, as upwards closed sets of rationals without a least element. Then, at which level $\mathcal{V}_\alpha$ do we first meet $\mathbb{R}$? How would you define the ordering of these reals?

§d

1. A *maximal element* in an order $\langle X, \leq \rangle$ is an $x \in X$ with the property that if $y \in X$ and $x \leq y$, then $y \leq x$. Prove *Zorn's Lemma*: if in an order $\mathbf{X}$ every chain has an upper bound, then $\mathbf{X}$ has a maximal element. (Use some form of the Axiom of Choice — e.g. the Well-ordering Theorem — to define recursively a mapping F from a set of ordinals into X so that if $\alpha < \beta$ and $F(\beta){\downarrow}$, then $F(\alpha) < F(\beta)$.)
2. Let $\mathbf{X} = \langle X, R \rangle$ be a structure with one binary relation. Prove: if every nonvoid subset of X has an R-least element, then $\mathbf{X}$ is a well-order.

3. Construct a well-ordering of $\mathbb{Q}$.

§e

1. Prove that every finite ordinal is a cardinal.

2. Prove that on the finite ordinals, ordinal exponentiation and cardinal exponentiation coincide.

3 Prove the Proposition.

4. Prove: if there exists a function $f\colon X \twoheadrightarrow Y$, then $|X| \geq |Y|$.

Let X be a set, and $\mathfrak{k}$ a cardinal. Then $\mathcal{P}^{<\mathfrak{k}}(X) := \{Y \subseteq \mathcal{P}(X) \mid |Y| < \mathfrak{k}\}$. — In particular, $\mathcal{P}^{<\aleph_0}(X) = \mathcal{p}(X)$.

5. Assume that $\mathfrak{k} > 0$, and for all $\mathfrak{m} < \mathfrak{k}$, $2^{\mathfrak{m}} \leq \mathfrak{n}$. Prove that $\sum_{\mathfrak{m}<\mathfrak{k}} \mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$. Conclude that if $|X| = \mathfrak{n}$, $|\mathcal{P}^{<\mathfrak{k}}(X)| = |X|$.

Let $\mathbf{Q} = \langle Q, \leq \rangle$ be a quasi-order. The *cofinality* of $\mathbf{Q}$ is the least cardinality of a cofinal set in $\mathbf{Q}$:

$$\mathrm{cf}\,\mathbf{Q} = \bigcap\{|X| \mid X \text{ is cofinal in } \mathbf{Q}\}.$$

For $\alpha \in \mathit{Ord}$, we simplify $\mathrm{cf}\langle \alpha, \subseteq \rangle$ to $\mathrm{cf}\,\alpha$. cardinal. For example, $\mathrm{cf}\,\omega = \mathrm{cf}(\omega + \omega) = \omega$.

6. (a) Assume α is infinite. Prove that $\mathrm{cf}\,\alpha = \alpha$ if and only if α is a regular cardinal.
(b) Which finite cardinals are regular, and which satisfy $\mathrm{cf}\,n = n$?