II

ALGEBRAS

Relations and operations

Thus far, we have assumed symbols for relations as well as operations. However, in the context of partial operations, there is a relatively natural way of reducing relations to operations. Instead of saying 'x is blue' we can say 'x is the blue x', at least in theory; 'the blue x' being defined exactly when x is blue, so that 'x is the blue x' is true exactly if x is blue. Likewise we might interpret 'x is less than y' as 'x is the lesser of x and y', and forgo the less-than-relation as a primitive of our structures.

A relation does not require anything to exist beyond the objects that it is predicated of. Hence if we operationalize a relation R, we should choose one of these objects as the value. For the sake of uniformity, and to enforce that relations do not contain unusual information, we stipulate that an n-ary operation R is a *predicate*, in a certain domain D, if in D the statement

$$\text{if } Rx_1\dots x_n \text{ exists, then } x_1 = Rx_1\dots x_n \qquad (*)$$

universally holds. So on second thoughts 'the lesser of x and y' is not a predicate, since it might be x just as well as y; but 'the lower bound x of $\{y\}$' *is* a predicate.

Example 1. An order $\langle X, \leq\rangle$ may be construed as an algebra $\langle X, R\rangle$, where $R = \{\langle x, \langle x, y\rangle\rangle \mid x \leq y\}$.

This approach does not work for nullary relations. We might represent them by constants, but this would violate our principle that we do not give extra information. Moreover, in a void universe all nullary relations would of necessity not hold. But, since we are not interested in nullary relations, this problem will not stop us. It does not counterbalance the considerable gain we expect to make in uniformity.

So from this point onwards we forego relation symbols, at least in theory. Every nominator will be operational. When we need relations in a structure, we transform them into operations satisfying the condition (*); which turns the structure into an algebra.

Example 2 (Burmeister). A *Mealy machine* is usually given by a sixtuple

$$\langle S, s_0, \Sigma, \Lambda, t, \lambda \rangle,$$

where S is a finite set, the set of *states*; s_0 is a designated element of S, the *initial state*; Σ and Λ are finite sets, the *input alphabet* and *output alphabet* respectively; t: $S \times \Sigma \longrightarrow S$ is the *transition function*, specifying the next state from a given state and an input symbol; and λ: $S \times \Sigma \longrightarrow \Lambda$ is the *output function*. The corresponding algebra would be

$$\mathbf{S} = \langle S \cup \Sigma \cup \Lambda, S, \Sigma, \Lambda, s_0, T, \lambda \rangle,$$

where S, Σ and Λ are unary predicates, with $Sx\!\downarrow$ if and only if $x \in S$, and in that case $Sx = x$, and so on. The binary operations T and λ satisfy the conditions

(t) if $Sx\!\downarrow$ and $\Sigma y\!\downarrow$, then $S(T(x, y))\!\downarrow$,

(λ) if $Sx\!\downarrow$ and $\Sigma y\!\downarrow$, then $\Lambda(\lambda(x, y))\!\downarrow$.

The algebra **S** is *sorted*: its universe consists of three *sorts* of elements, S, Σ, and Λ, the domains of the operations are cartesian products of sorts, and the range of each operation is contained in a single sort. The sequence of argument and range sorts of an operation we call its *type*.

Sorted algebras are common in computer science. They embody a pleasant kind of partiality, in which the sort structure completely controls where an operation is defined. Not all partiality is of this type though, not even in computer science, as the next example shows.

Example 3 (Burmeister). *Stacks* are a simple kind of data structures, in which the following components play a part:
— There are items to be stacked, the elements of some set D.
— The stacks, forming a set S.
— An *empty* stack $e \in S$.
— An item d may be *pushed* onto a stack s; the result is a higher stack $P(d, s)$.
— From a stack s you may *pop* the *top* item. This item is $T(s)$, the remaining stack is $p(s)$.
Together these ingredients make up an algebra

$$\mathbf{S} = \langle S \cup D, S, D, e, P, T, p \rangle.$$

The intended typing is P: $D \times S \longrightarrow S$, T: $S \longrightarrow D$, and p: $S \longrightarrow S$. More explicitly,

(P) if $Sx\!\downarrow$ and $Dy\!\downarrow$, then $S(P(x, y))\!\downarrow$,

and so on.

Unfortunately, the description of the situation does not entirely warrant this typing. We have a clear conception of the stack that we get by pushing item d onto stack s; and we know what is the *top* of $P(d, s)$, and which stack will result if we pop it. But how about $p(e)$ and $T(e)$? Experience shows a common preference for $p(e) = e$; the item $T(e)$, on the other hand, would be special only by being $T(e)$ — as far as we know. And then, in practice, popping the empty stack may result in an error condition, or a *very full* stack. So that if **S** is to embody something like the *minimal requirements* for stacks, $p(e)$ and $T(e)$ have to remain undefined.

Example 4 (Burmeister). Let k be a negative integer, and l a positive. On the interval $[k, l]$, the unary successor operation S is defined everywhere except in l, and symmetrically the predecessor operation P is defined everywhere except in k. On the structure $\langle [k, l], S, P, 0\rangle$ we can further specify addition and subtraction by

$$x + 0 = x, \qquad x + Sy \simeq S(x + y), \qquad x + Py \simeq P(x + y);$$
$$x - 0 = x, \qquad x - Sy \simeq P(x - y), \qquad x - Py \simeq S(x - y).$$

We shall denote the expanded structure $\langle [k, l], S, P, 0, +, -\rangle$ by $\mathbf{Z}_{k\text{-}l}$.

Further adding multiplication, specified by

$$x \cdot 0 = 0, \qquad x \cdot Sy \simeq (x \cdot y) + x, \qquad x \cdot Py \simeq (x \cdot y) - x$$

we obtain $\mathbf{Z}^*_{k\text{-}l}$.

Example 5. Let $\mathbf{Q} = \langle \mathbb{Q}, <\rangle$ be the strict order of rational numbers, and define $I_\mathbb{Q}$ to be the set of *intervals* $[q_1, q_2]$ with $q_1 < q_2$. On $I_\mathbb{Q}$ we specify the operations of

addition: $[x_1, x_2] + [y_1, y_2] = [x_1 + y_1,\ x_2 + y_2]$;

subtraction: $[x_1, x_2] - [y_1, y_2] = [x_1 - y_2,\ x_2 - y_1]$;

multiplication: $[x_1, x_2] \cdot [y_1, y_2] = [a, b]$, where a is the infimum of $\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$, and b the supremum;

division: $[x_1, x_2] \div [y_1, y_2] = [a, b]$, where a is the infimum of $\{x_1/y_1, x_1/y_2, x_2/y_1, x_2/y_2\}$, and b the supremum, *provided* $0 \notin [y_1, y_2]$; if $0 \in [y_1, y_2]$, the quotient is not defined.

We shall denote the structure $\langle I_\mathbb{Q}, +, -, \cdot, \div\rangle$ by $\mathbf{I}_\mathbb{Q}$.

Example 6 (Burmeister). Let $\mathbf{R} = \langle R, +, 0, -, \cdot\rangle$ be a ring, and k a natural number greater than 1. Let $k \times k$ be the set of pairs $\langle i, j\rangle$ of natural numbers less than k. Define

$$A := R \cup R^k \cup R^{k\times k};$$

assume the three components R, R^k and $R^{k\times k}$ are disjoint. (They will be if the elements of R are primitive.) We call the elements of the first component *scalars*, those of the second component *vectors*, and the rest *matrices*. We denote scalars by italic letters x, y, z; vectors by bold letters $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$, usually writing r_i instead of $\mathbf{r}(i)$; and matrices by italic capitals M, N, usually writing $m_{i,j}$ or m_{ij} instead of $M(i, j)$. The following extensions of the addition and multiplication of $\mathbf{R}$ to A may be considered reasonable:

addition: $\mathbf{r} + \mathbf{s} = \langle r_0 + s_0, \ldots, r_{k-1} + s_{k-1}\rangle$; $M + N = L$, where $l_{ij} = m_{ij} + n_{ij}$, or in a more direct notation: $\langle m_{ij}\rangle_{ij} + \langle n_{ij}\rangle_{ij} = \langle m_{ij} + n_{ij}\rangle_{ij}$;

multiplication: $x \cdot \mathbf{r} = \langle x \cdot r_0, \ldots, x \cdot r_{k-1}\rangle$,
$\mathbf{r} \cdot x = \langle r_0 \cdot x, \ldots, r_{k-1} \cdot x\rangle$,
$\mathbf{r} \cdot \mathbf{s} = r_0 \cdot s_0 + \ldots + r_{k-1} \cdot s_{k-1}$,
$x \cdot M = \langle x \cdot m_{ij}\rangle_{ij}$,
$M \cdot x = \langle m_{ij} \cdot x\rangle_{ij}$,

$\mathbf{r} \cdot M = \mathbf{s}$, where $s_i = r_0 \cdot m_{0,i} + \ldots + r_{k-1} \cdot m_{k-1,i}$,
$M \cdot \mathbf{r} = \mathbf{t}$, where $t_i = m_{i,0} \cdot r_0 + \ldots + m_{i,k-1} \cdot r_{k-1}$, and
$M \cdot N = \langle m_{i,0} \cdot n_{0,j} + \ldots + m_{i,k-1} \cdot n_{k-1,i} \rangle_{ij}$.

CHAPTER 4

FIRST CONCEPTS

§a Default interpretation

We chose, in 2§a, to consider statements like

The king of France is bald (1)

false, at a time when France is a republic. This is of course a simplification for the sake of theory, comparable to, but less stringent than, assuming that every operation is defined for every possible argument in the universe. The normal attitude would be that *the king of France* fails to refer to anything, and hence statement (1) is nonsense.[1]

There is a slight difference between (1) and

Charlemagne has a big nose (2)

— if we step over the fact that in reality we would assume that the speaker used the wrong tense. The symbols 'the king of', 'France' and 'is bald' in (1) undeniably have meaning. There is a king of Sweden, for example. But 'Charlemagne' does not apply to anyone. In our theory, there are two ways in which this can come about: 'Charlemagne' may be in our nominator, and have the void interpretation; or it might not even belong to the nominator. (We are assuming that we have some sort of present-day frame of reference that has the characteristics of a structure.) Which of the two is actually the case is rather a vacuous issue. We shall not bother to settle it, and consider (2) *false*, just like (1), whether there is an empty interpretation or none at all.

We extend this treatment to all symbols that are not in the nominator of the structure at hand. They get the default interpretation, which is *void*. Formally, this involves defining an extension of the interpretation.

Definition. Let $\mathbf{A} = \langle A, I \rangle$ be an algebra. Then $\mathfrak{I}_{\mathbf{A}}$ is the extension of I to the class of all possible symbols, defined, for $S \notin \mathrm{Dom}(I)$, by $\mathfrak{I}_{\mathbf{A}}(S) = \emptyset$.

Instead of $\mathfrak{I}_{\mathbf{A}}(S)$ we write $\mathbf{A}(S)$ or $S^{\mathbf{A}}$. If every $S^{\mathbf{A}}$ is void, $\mathbf{A}$ is called *discrete*. In particular, a set, considered as an algebra, is discrete.

Observe that $\mathfrak{I}_{\mathbf{A}}$ is not a function; but there is a function, the interpretation I, which tells us all there is to know about it.

§b Subalgebras

Let $\mathbf{A}$ be an algebra. A *subuniverse* of $\mathbf{A}$ is a subset of the universe A that is closed under the basic operations of $\mathbf{A}$. We denote the collection of all subuniverses of $\mathbf{A}$ by $\mathrm{Sub}\,\mathbf{A}$. This collection is a closed set system, by Example 2f(xviii). The associated closure operator will be written $\mathrm{Sg}^{\mathbf{A}}$.

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[1] P.F. Strawson, On Referring, *Mind* 59 (1950): 320-344.

Of course $A \in \mathrm{Sub}\,\mathbf{A}$. A subuniverse $B \neq A$ is called a *proper* subuniverse of $\mathbf{A}$. An algebra is *minimal* if it has no proper subuniverses.

A set $X \subseteq A$ is *dense* in $\mathbf{A}$, or *generates* $\mathbf{A}$, if $\mathrm{Sg}^{\mathbf{A}}(X) = A$. In particular, $\mathbf{A}$ is minimal if and only if $\emptyset$ is dense in $\mathbf{A}$.

Examples

i. Let t and f be two objects that are not natural numbers, and $A = \mathbb{N} \cup \{\mathsf{t}, \mathsf{f}\}$. Define an interpretation I in A of three nullary operation symbols $\mathbf{t}$, $\mathbf{f}$ and 0, three unary operation symbols N, B and S, and one binary operation symbol Eq, by

$$I(N) = \Delta_{\mathbb{N}}, I(B) = \Delta_{\{\mathsf{t},\mathsf{f}\}};$$
$$I(\mathbf{t}) = \mathsf{t}, I(\mathbf{f}) = \mathsf{f}, I(0) = 0;$$
$$I(S) = \{\langle n+1, n\rangle \mid n \in \mathbb{N}\};$$
$$I(Eq) = (\{\mathsf{t}\} \times \Delta_{\mathbb{N}}) \cup (\{\mathsf{f}\} \times (\mathbb{N}^2 - \Delta_{\mathbb{N}}).$$

The subuniverse A is generated by the void set, in symbols: $\mathrm{Sg}^{\mathbf{A}}(\emptyset) = A$. Indeed, A is the *only* subuniverse of $\mathbf{A}$. For suppose X is a subuniverse of $\mathbf{A}$. Then t and f must belong to X. Likewise $0 \in X$. Moreover, if $n \in X \cap \mathbb{N}$, $n + 1 = S^{\mathbf{A}}(n) \in X$ as well. By mathematical (incomplete) induction, $\mathbb{N} \subseteq X$. Thus $\mathbf{A}$ has no proper subuniverses: it is a minimal algebra.

ii. Recall our convention that a predicate is an operation that evaluates, if at all, to its first argument. It follows that every subset of the universe is closed under predicates, and hence that in algebras obtained from relational structures every subset is a subuniverse.

b1 Structural induction. One can prove that all the elements of a given subuniverse have a property P by showing
1° that all the elements of some generating set have P, and
2° if $x_1,\ldots, x_n$ have P, and Q is any operation of the algebra, then $Q(x_1\ldots x_n)$ has P.

b2 Subuniverse Generation Theorem. Let $\mathbf{A}$ be an algebra, and $X \subseteq A$. Put $X_0 = X$, and for $k \in \mathbb{N}$,

$$X_{k+1} = X_k \cup \bigcup_{Q \in \mathrm{Nom}\,\mathbf{A}} Q^{\mathbf{A}}[X_k^{<\omega}].$$

Then

$$\mathrm{Sg}^{\mathbf{A}} X = \bigcup_{k=0}^{\infty} X_k.$$

Proof. For each $Q \in \mathrm{Nom}\,\mathbf{A}$, $\mathcal{U}_Q := \{U \subseteq A \mid Q^{\mathbf{A}}[U^{<\omega}] \subseteq U\}$ is an algebraic closure system, and $\mathrm{Sub}\,\mathbf{A} = \bigcap_Q \mathcal{U}_Q$. Let $\mathbf{C}_Q$ be the closure operator determined by $\mathcal{U}_Q$, and define F_Q: $\mathcal{P}A \longrightarrow \mathcal{P}A$ by $F_Q(U) = U \cup Q^{\mathbf{A}}[U^{<\omega}]$. Then

$$\mathbf{C}_Q(U) = \bigcup_{k=0}^{\infty} F_Q^k(U),$$

and $X_{k+1} = \bigcup_Q F_Q(X_k)$. Hence by Corollary 2f3.7, $\mathrm{Sg}^{\mathbf{A}} X = \bigcup_k X_k$. ⊠

Define the *complexity* of an element a of $\mathrm{Sg}^{\mathbf{A}} X$ to be the least k such that $a \in X_k$. Then this theorem warrants another form of induction for $\mathrm{Sg}^{\mathbf{A}} X$, induction *on the complexity* of a. If one can prove that a has property P from the assumption that elements of lower complexity have P, then every element of $\mathrm{Sg}^{\mathbf{A}}X$ has property P.

b3 Corollary. The closure operator $\mathrm{Sg}^{\mathbf{A}}$ is algebraic.

Proof. Immediate by Corollary 2f3.7. ☒

So the subuniverses of an algebra from an algebraic closure system, and a fortiori an algebraic lattice. The converse holds as well.

b4 Theorem (Birkhoff & Frink). Every algebraic lattice is isomorphic to the subuniverse lattice of some algebra.

Proof. Let $\mathbf{L}$ be an algebraic lattice. Then by Theorem 2f.3.9, there exists an algebraic closure operator $\mathbf{C}$ such that $\mathbf{L} \cong \langle \mathrm{Ran}\,\mathbf{C}, \subseteq \rangle$. Let $A = \bigcup \mathrm{Ran}\,\mathbf{C}$. Take $\mathcal{T} = A \times \mathbb{N}$. For $t = \langle a, n \rangle \in \mathcal{T}$, let $I(t)$ be the n-ary operation on A defined by

$$I(t)(a_1,\ldots, a_n) = a \text{ if } a \in \mathbf{C}\{a_1,\ldots, a_n\};$$
$$\uparrow \text{ otherwise.}$$

Let $\mathbf{A} = \langle A, I \rangle$; then $\mathrm{Sg}^{\mathbf{A}} = \mathbf{C}$. For, suppose $X \subseteq A$.

• $\mathrm{Sg}^{\mathbf{A}}X \subseteq \mathbf{C}(X)$: by the Subuniverse Generation Theorem, $\mathrm{Sg}^{\mathbf{A}}X = \bigcup_k X_k$, where $X_0 = X$, and

$$X_{k+1} = X_k \cup \bigcup_{t \in \mathcal{T}} I(t)[X_k^{<\omega}].$$

Now $X_0 \subseteq \mathbf{C}(X)$; and if $X_k \subseteq \mathbf{C}(X)$, and $I(t)(a_1,\ldots, a_n){\downarrow}$ for certain $a_1,\ldots, a_n \in X_k$, then $I(t)(a_1,\ldots, a_n) \in \mathbf{C}\{a_1,\ldots, a_n\} \subseteq \mathbf{C}(X_k) \subseteq \mathbf{C}\mathbf{C}(X) = \mathbf{C}(X)$: so $X_{k+1} \subseteq \mathbf{C}(X)$.

• $\mathbf{C}(X) \subseteq \mathrm{Sg}^{\mathbf{A}}X$: if $a \in \mathbf{C}(X)$, then there is a finite set $\{x_1,\ldots, x_n\} \subseteq X$ such that $a \in \mathbf{C}\{x_1,\ldots, x_n\}$. Take $t = \langle a, n \rangle$; then $a = I(t)(x_1,\ldots, x_n) \in \mathrm{Sg}^{\mathbf{A}}X$. ☒

Now let $\mathbf{A} = \langle A, I \rangle$ and $\mathbf{B} = \langle B, J \rangle$ be similar algebras. We say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$, or that $\mathbf{A}$ is an *extension* of $\mathbf{B}$, and write $\mathbf{B} \le \mathbf{A}$, if $B \in \mathrm{Sub}\,\mathbf{A}$, and for every symbol $S \in \mathrm{Nom}\,\mathbf{A}$, $J(S) = I(S)_B$. If $\mathbf{B} \le \mathbf{A}$ and $\mathbf{B} \ne \mathbf{A}$, then $\mathbf{B}$ is a *proper subalgebra* of $\mathbf{A}$, and $\mathbf{A}$ a *proper extension* of $\mathbf{B}$; notation $\mathbf{B} < \mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are structures of some particular type, say orders, categories, or groups, we speak of sub-whatever: suborders, subcategories, subgroups, and so on.

Examples.

iii. Under the assumption that $\mathbb{N} \subseteq \mathbb{Q}$, $\langle \mathbb{N}, <, 0, 1, +, \cdot \rangle \le \langle \mathbb{Q}, <, 0, 1, +, \cdot \rangle$. Observe that *every* subset of $\mathbb{Q}$ is closed under the operation representing $<$, as a general consequence of our construction of such operations.

iv. A subalgebra of a group $\langle G, \cdot, e \rangle$ is a monoid, but not necessarily a group. However, if we include inversion in the nominator, subalgebras of groups will always be groups.

v. A *set lattice* is a subalgebra of a lattice $\langle \mathcal{P}X, \cup, \cap \rangle$.

vi. A *field of sets* is a subalgebra of a powerset algebra $\mathcal{P}X$.

vii. By §2f4, we have complete lattices $\mathbf{Tr}A$, of all the transitive relations contained in A^2, and $\mathbf{Qo}\,A$, of quasi-orderings of A. Any nonempty set of quasi-orderings of A has the same infimum and supremum in $\mathbf{Tr}A$ as in $\mathbf{Qo}\,A$. So in particular, $\mathbf{Tr}A$ is a sublattice of $\mathbf{Qo}\,A$. Yet $\bigvee^{\mathbf{Qo}\,A}\emptyset = \Delta_A$, whereas $\bigvee^{\mathbf{Tr}A}\emptyset = \emptyset$: $\mathbf{Tr}A$ is not a *complete sublattice* of $\mathbf{Qo}\,A$, nor a *bounded sublattice*.

viii. For any categories $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \le \mathbf{B}$ implies $\mathbf{A}^\partial \le \mathbf{B}^\partial$.

ix. Let **C** be a category. Define, for any $x, z \in C$,

$$\{x{\leftarrow}z\}^{\mathbf{C}},$$

the *homclass* of x from z, as the class of all $y \in C$ such that $x \circ y \circ z$ exists. We write simply $x{\leftarrow}z$ if the category **C** is understood. If **C** has a clear notion of *object*, x and z may be objects instead of arrows; then

$$\{x{\leftarrow}z\}^{\mathbf{C}} = \{1_x{\leftarrow}1_z\}^{\mathbf{C}}.$$

The category **C** is said to be *locally small* if for all $x, z \in C$, $x{\leftarrow}z$ is a set. (We may then speak of the *homset* of x from z.) All the large categories that we have come across — **Rel**, **Set**, **Alg** — are locally small. Observe that

$$\{x{\leftarrow}z\}^{\mathbf{C}^{\partial}} = \{z{\leftarrow}x\}^{\mathbf{C}}.$$

Now let $\mathbf{A} \leq \mathbf{B}$ be categories. Then clearly for all $a, b \in A$,

$$\{a{\leftarrow}b\}^{\mathbf{A}} \subseteq \{a{\leftarrow}b\}^{\mathbf{B}}.$$

We say that **A** is a *full* subcategory of **B** if the reverse inclusion holds as well.

Observe that **Set** (or, to be precise, the variant of **Set** that has arrows

$$\langle Y, f, \mathrm{Dom} f\rangle$$

instead of $\langle Y, f\rangle$) is *not* a full subcategory of **Rel**. If **A** is a full subcategory of **B**, then $\mathbf{A}^{\partial}$ is a full subcategory of $\mathbf{B}^{\partial}$.

x. A *module* consists of a ring **R** and an abelian group **A**, with disjoint universes R and A (hence two distinct zero elements, say 0 and **0**) and an extended product operation. If the domain of $\cdot$ in the module **M** includes both $R \times A$ and $A \times R$, we call **M** a *bimodule*; if it includes only $R \times A$, **M** is a *left module*; and if it includes only $A \times R$, **M** is a *right module*. In practice, the ring component of a module is relatively fixed, and we call a module with ring component **R** an **R**-module. Thus we have **R**-bimodules and left and right **R**-modules. A left **R**-module satisfies, for all r, $s \in R$ and all a, $b \in A$, the equations

(lm1) $r \cdot (a + b) = r \cdot a + r \cdot b$,

(lm2) $(r + s) \cdot a = r \cdot a + s \cdot a$,

(lm3) $(r \cdot s) \cdot a = r \cdot (s \cdot a)$.

Symmetrically, a right **R**-module satisfies finitely

(rm1) $(a + b) \cdot r = a \cdot r + b \cdot r$,

(rm2) $a \cdot (r + s) = a \cdot r + a \cdot s$,

(rm3) $a \cdot (r \cdot s) = (a \cdot r) \cdot s$.

An **R**-bimodule satisfies all six equation schemes.

A *sub*-**R**-*module* of an **R**-module **M** is a subalgebra of **M** that includes the entire ring component R.

Let **A** be an algebra, and $X \subseteq A$. The subalgebra of **A** with universe $\mathrm{Sg}^{\mathbf{A}}X$ is called the subalgebra of **A** *generated by* X, and denoted by $\mathbf{Sg}^{\mathbf{A}}X$. If $\mathbf{A} = \mathbf{Sg}^{\mathbf{A}}X$, we say X is a *generating set* of **A**, and **A** is *X-generated*. An algebra **A** is *finitely generated* if it has a finite generating set.

b5 Proposition. The subalgebra relation is an ordering of the class of all algebras.

§c Homomorphisms

Homomorphisms may be said to constitute the most important kind of alikeness between algebras.

c1 Definition. Let **A** and **B** be algebras. A mapping $f\colon A \longrightarrow B$ is a *homomorphism* from **A** to **B** if for each operation symbol Q, whenever $\langle a_0,\dots, a_{n-1}\rangle \in \mathrm{Dom}\, Q^{\mathbf{A}}$,

$$f(Q^{\mathbf{A}}(a_0,\dots, a_{n-1})) = Q^{\mathbf{B}}(f(a_0),\dots,f(a_{n-1})).$$

(Recall our convention that '… = N' can only hold if the expression 'N' makes sense. So this definition implies in particular

$$\text{if } \langle a_0,\dots, a_{n-1}\rangle \in \mathrm{Dom}\, Q^{\mathbf{A}}, \text{ then } \langle f(a_0),\dots,f(a_{n-1})\rangle \in \mathrm{Dom}\, Q^{\mathbf{B}}.)$$

We write $f\colon \mathbf{A} \longrightarrow \mathbf{B}$ to express that f is a homomorphism from **A** to **B**. If **A** = **B**, f is called a*n endomorphism* of **A**. An injective homomorphism is also called an *embedding*. For embeddings we use the notation $f\colon\ \mathbf{A} \hookrightarrow \mathbf{B}$. Likewise we use $\twoheadrightarrow$ and $\hookrightarrow\!\!\!\!\rightarrow$ to express, respectively, surjectivity and bijectivity.

Examples.
i. A homomorphism from a category to a category is commonly called a *functor*. A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ is *faithful* if for all $x, z \in Id^{\mathbf{C}}$, the restriction of F to $(z{\leftarrow}x)^{\mathbf{C}}$ is an injection. Thus a faithful functor is a kind of local embedding. We also have a local notion of surjection: a functor is *full* if for all $x, z \in Id^{\mathbf{C}}$, the restriction of F to $(z{\leftarrow}x)^{\mathbf{C}}$ is a surjection onto $(Fz{\leftarrow}Fx)^{\mathbf{D}}$.
ii. Suppose $\mathbf{A} = \langle A, \le\rangle$ and $\mathbf{B} = \langle B, \prec\rangle$ are quasi-orders, construed as algebras, and $f\colon \mathbf{A} \longrightarrow \mathbf{B}$. Then if $x \le y$ exists, $f(x \le y) = f(x) \prec f(y)$. In relational terms:

$$\text{if } x \le y, \text{ then } f(x) \prec f(y).$$

A function with this property is called *isotone*. A function $g\colon A \longrightarrow A$ such that $x \le g(x)$ for all $x \in A$ is called *increasing*.

c2 Proposition. Let **A**, **B** and **C** be algebras.
(i) 1_A is an endomorphism of **A**.
(ii) If $f\colon \mathbf{A} \longrightarrow \mathbf{B}$ and $g\colon \mathbf{B} \longrightarrow \mathbf{C}$, then $g \circ f$ is a homomorphism from **A** to **C**.

Proof. (i) Trivial.
(ii) Clearly, $g \circ f$ is a mapping of A into C.
If $\langle a_0,\dots, a_{n-1}\rangle \in \mathrm{Dom}\, Q^{\mathbf{A}}$, then

$$f(Q^{\mathbf{A}}(a_0,\dots, a_{n-1})) = Q^{\mathbf{B}}(f(a_0),\dots,f(a_{n-1}))$$

since f is a homomorphism; and

$$g(Q^{\mathbf{B}}(f(a_0),\dots,f(a_{n-1}))) = Q^{\mathbf{C}}(g(f(a_0),\dots, g(f(a_{n-1})))$$

since g is a homomorphism. ⊠

This proposition implies that we have a category **Alg** of homomorphisms. To be precise, the arrows of **Alg** are the triples $\langle \mathbf{B}, f, \mathbf{A}\rangle$ where $f\colon \mathbf{B} \longleftarrow \mathbf{A}$ is a homomorphism. Arrows are composable if adjacent elements match, that is,

$$\langle \mathbf{D}, g, \mathbf{C}\rangle \circ \langle \mathbf{B}, f, \mathbf{A}\rangle \text{ exists} \Leftrightarrow \mathbf{C} = \mathbf{D},$$

and then the composite is $\langle \mathbf{D}, g \circ f, \mathbf{A}\rangle$. Identity arrows are triples of the form $\langle \mathbf{A}, 1_A, \mathbf{A}\rangle$. — In practice, of course, we prefer to keep the outer elements of these triples implicit.

In general, a *concrete category* is a category $\mathbf{C}$ with a faithful functor

$$U\colon \mathbf{C} \longrightarrow \mathbf{Set}.$$

A typical concrete category has objects consisting in a set with additional structure; since U obliterates this structure, it is called the *forgetful functor* of the concrete category. For example, the category **Alg** has forgetful functor

$$\langle \mathbf{B}, f, \mathbf{A}\rangle \longmapsto \langle B, f\rangle.$$

In general, if $U\colon \mathbf{C} \longrightarrow \mathbf{D}$ is a forgetful functor, properties of $\mathbf{D}$-arrows may be attributed to elements of C, since these are, in a clear sense specified by U, also elements of D. This is how we can call a homomorphism 'injective' or 'surjective'.

The construction of **Alg** as a class of triples represents a general pattern. Suppose we have objects for which some underlying set is specified, say A underlies the object $\mathfrak{A}$, B underlies $\mathfrak{B}$, and so on; and for every pair $\langle \mathfrak{B}, \mathfrak{A}\rangle$ a set $(\mathfrak{B} \leftarrow \mathfrak{A})$ of functions from A to B has been defined. If

for every $\mathfrak{A}$, $\qquad 1_A \in (\mathfrak{A} \leftarrow \mathfrak{A})$, $\qquad$ and

for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, $\qquad f \in (\mathfrak{C} \leftarrow \mathfrak{B})\ \&\ g \in (\mathfrak{B} \leftarrow \mathfrak{A}) \Rightarrow f \circ g \in (\mathfrak{C} \leftarrow \mathfrak{A})$,

then the triples $\langle \mathfrak{B}, f, \mathfrak{A}\rangle$, with $f \in (\mathfrak{B} \leftarrow \mathfrak{A})$ — or the sets $(\mathfrak{B} \leftarrow \mathfrak{A})$ — will be said to *form a concrete category*, with composition and identity defined in analogy to **Alg**.

c3 Proposition. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. If f and g are homomorphisms from $\mathbf{A}$ into $\mathbf{B}$, then $\{a \in A \mid f(a) = g(a)\}$ is a subuniverse of $\mathbf{A}$.

Proof. Put

$$X := \{a \in A \mid f(a) = g(a)\},$$

and suppose $\langle a_0, \ldots, a_{n-1}\rangle \in \operatorname{Dom}(Q^{\mathbf{A}})_X$. Then

$$\begin{aligned} f(Q^{\mathbf{A}}(a_0, \ldots, a_{n-1})) &= Q^{\mathbf{B}}(f(a_0), \ldots, f(a_{n-1})) = Q^{\mathbf{B}}(g(a_0), \ldots, g(a_{n-1})) \\ &= g(Q^{\mathbf{A}}(a_0, \ldots, a_{n-1})), \end{aligned}$$

so X is closed under $Q^{\mathbf{A}}$, and since Q was arbitrary, X is a subuniverse of $\mathbf{A}$. ☒

Corollary. If f and g are homomorphisms of $\mathbf{A}$ that coincide on a dense set of $\mathbf{A}$, then $f = g$.

c4 Proposition. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras and f a homomorphism from $\mathbf{A}$ into $\mathbf{B}$. Then for any $Y \in \operatorname{Sub}\mathbf{B}$, $f^{-1}[Y] \in \operatorname{Sub}\mathbf{A}$.

Proof. Put $X = f^{-1}[Y]$, and suppose $\langle a_0, \ldots, a_{n-1}\rangle \in \operatorname{Dom}(Q^{\mathbf{A}})_X$. Then

$$f(Q^{\mathbf{A}}(a_0, \ldots, a_{n-1})) = Q^{\mathbf{B}}(f(a_0), \ldots, f(a_{n-1})) \in Y$$

since Y is a subuniverse, so X is closed under $Q^{\mathbf{A}}$, and since Q was arbitrary, X is a subuniverse of $\mathbf{A}$. ☒

c5 Theorem. A homomorphism is a monomorphism (in **Alg**) if and only if it is injective.

Proof. (⇒) Suppose f: $\mathbf{A} \longrightarrow \mathbf{B}$ is not injective; let a_0, a_1 be distinct elements of A such that $f(a_0) = f(a_1)$. Let $g = \{a_0 \hookleftarrow 0\}$, $h = \{a_1 \hookleftarrow 0\}$. Then

$$g, h\colon \{0\} \longrightarrow \mathbf{A},$$

and $f \circ g = f \circ h$ although g and h are distinct.
(⇐) Injectivity is a property of homomorphisms in their capacity of mappings between sets. So if f: $\mathbf{A} \longrightarrow \mathbf{B}$ is injective, and $f \circ g = f \circ h$, for homomorphisms g, h: $\mathbf{X} \longrightarrow \mathbf{A}$, then $f \circ g = f \circ h$ in **Set**, and since injective mappings are monomorphisms in **Set** (example 2d.vi), $g = h$. ⊠

c6 Theorem. A homomorphism f: $\mathbf{A} \longrightarrow \mathbf{B}$ is an epimorphism in **Alg** if and only if $f[A]$ is dense in $\mathbf{B}$.

Proof. Let f: $\mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism; put $D := \mathrm{Sg}^{\mathbf{B}} f[A]$.
(⇒) Suppose $D \neq B$. Let C be a set disjoint with B, and in one-one correspondence with $B - D$; let ϕ: $B - D \longrightarrow C$ be a bijection. Put $\psi = 1_D \cup \phi$. Take $M = B \cup C$; define an extension $\mathbf{M} \geq \mathbf{B}$ by putting, for each $Q \in \mathrm{Nom}\mathbf{B}$, and $x_0,\ldots, x_{n-1} \in D \cup C$,

$$Q^{\mathbf{M}}(x_0,\ldots, x_{n-1}) \simeq \psi(Q^{\mathbf{B}}(\psi^{-1}(x_0),\ldots, \psi^{-1}(x_{n-1}))).$$

Then 1_B and ψ are distinct homomorphisms from $\mathbf{B}$ to $\mathbf{M}$, but $1_B \circ f = \psi \circ f$; so f is not an epimorphism.
(⇐) Suppose $f[A]$ is dense. Let g, h: $\mathbf{B} \longrightarrow \mathbf{C}$, be homomorphisms; assume that $g \circ f = h \circ f$. Put $X := \{b \in B \,|\, h(b) = g(b)\}$. Then $X \supseteq f[A]$; since by Proposition 3, $X \in \mathrm{Sub}\,\mathbf{B}$, X must be B, and $g = h$. ⊠

§d Natural transformations

Let S, T: $\mathbf{B} \longleftarrow \mathbf{C}$ be functors. A *natural transformation* from S to T is a mapping τ: $B \longleftarrow C$ such that for all $x, y \in C$,

(1) $$xy{\downarrow} \Rightarrow \tau_x \circ Sy = \tau_{xy} = Tx \circ \tau_y .$$

We write τ: $T \leftarrow\!\!\!\cdot\; S$ or τ: $S \;\cdot\!\!\!\rightarrow T$.

Proposition. Let S, T: $\mathbf{B} \longleftarrow \mathbf{C}$ be functors. Every mapping σ: $B \longleftarrow Id^{\mathbf{C}}$ that satisfies for all $c \in C$

(2) $$\sigma_{\mathsf{b}(c)} \circ Sc = Tc \circ \sigma_{\mathsf{d}(c)}$$

can be extended in exactly one way to a natural transformation from S to T. Conversely, if τ: $S \;\cdot\!\!\!\rightarrow T$ is a natural transformation, then $\sigma = \tau{\restriction} Id^{\mathbf{C}}$ satisfies (2).

Proof. Define τ_c to be $\sigma_{\mathsf{b}(c)} \circ Sc$, or equivalently, $Tc \circ \sigma_{\mathsf{d}(c)}$. This definition is forced upon us, for $c = \mathsf{b}(c) \circ c$, hence by (1) $\tau_{\mathsf{b}(c)} \circ Sc = \tau_c$, and $c = c \circ \mathsf{d}(c)$, so again by (1) $\tau_c = Tc \circ \tau_{\mathsf{d}(c)}$. And it works: if $xy{\downarrow}$, then

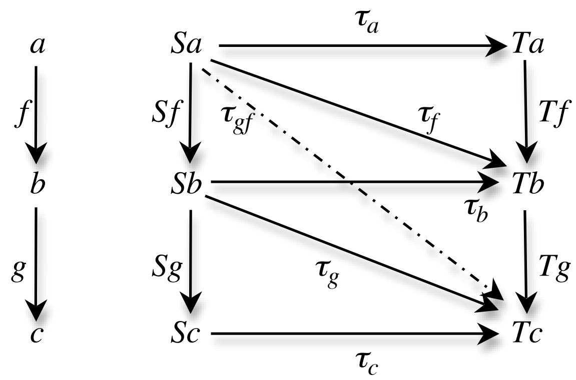
$$\tau_x \circ Sy = \tau_{\mathsf{b}(x)} \circ Sx \circ Sy = \tau_{\mathsf{b}(x)} \circ S(xy) = \tau_{xy},$$

and analogously $\tau_{xy} = Tx \circ \tau_y$. ⊠

By extension, we also call mappings as in (2) ‘natural transformations’; and moreover, since a natural transformation can be given as a mapping of identity arrows, it can also be given as a mapping of objects.

Suppose a, b and c are objects of $\mathbf{C}$, and f: $a \longrightarrow b$ and g: $b \longrightarrow c$ arrows. Then the preceding is represented by the following diagram of arrows:

(3)



The formulas (1) and (2) express that this diagram *commutes*, that is, any two ways of getting from one object in the diagram to another by following arrows are equivalent.

Examples. Functors S, T: $\mathbf{B} \longleftarrow \mathbf{C}$ may be viewed as constructions of objects in $\mathbf{B}$ from objects in the base category; being functors, these constructions have the property that they lift relations (arrows) between objects in $\mathbf{C}$ to relations between the corresponding constructs in $\mathbf{B}$. A transformation of constructs Sc into constructs Tc, is called 'natural' if the arrows embodying the transformation commute with relations given in $\mathbf{C}$ and their derivatives in $\mathbf{B}$.

i. Let $\mathcal{N}$ be a nominator consisting entirely of unary operation symbols, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ a *quomorphism* of $\mathcal{N}$-algebras: that is, ϕ is a partial function from A to B, and whenever $a \in \operatorname{Dom} Q^{\mathbf{A}} \cap \operatorname{Dom} \phi$,

$$\phi(Q^{\mathbf{A}}(a)) = Q^{\mathbf{B}}(\phi(a)).$$

Let $\mathbf{N}$ be the monoid of finite sequences of elements of $\mathcal{N}$, and $\mathbf{Pfn}$ the category of partial functions, with arrows $\langle Y, \psi, X\rangle$, where X and Y are sets, and $\psi \subseteq Y \times X$ is a function (cf. §2b:1). Define functors S, T: $\mathbf{N} \longrightarrow \mathbf{Pfn}$:

$$S(Q_0 \dots Q_{n-1}) = Q_0^{\mathbf{A}} \circ \dots \circ Q_{n-1}^{\mathbf{A}},$$
$$T(Q_0 \dots Q_{n-1}) = Q_0^{\mathbf{B}} \circ \dots \circ Q_{n-1}^{\mathbf{B}}$$

— in particular, $S(\varepsilon) = 1_A$ and $T(\varepsilon) = 1_B$. Define τ: $N \longrightarrow Pfn$ by

$$\tau(Q_0 \dots Q_{n-1}) = \phi \circ Q_0^{\mathbf{A}} \circ \dots \circ Q_{n-1}^{\mathbf{A}}.$$

Then τ is a natural transformation from S to T. Observe that by the Proposition above we fix τ by stipulating $\tau_\varepsilon = \phi$.

ii. For any set X, let SX be X^2, and for a function f: $X \longrightarrow Y$, $Sf = f^2$, mapping $\langle x_0, x_1\rangle \in X^2$ to $\langle f(x_0), f(x_1)\rangle \in Y^2$; and

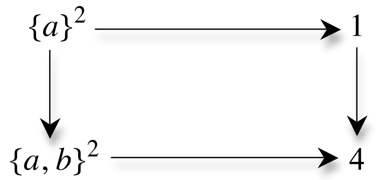
$$TX = X^{(2)} = \{\{x_0, x_1\} \mid x_0, x_1 \in X\},$$

with Tf mapping $\{x_0, x_1\}$ to $\{f(x_0), f(x_1)\}$. For any set Z, define τ_Z: $Z^2 \longrightarrow Z^{(2)}$ by

$$\tau_Z(z_0, z_1) = \{z_0, z_1\}.$$

Then t is a natural transformation from S to Z.

iii. Let **Inj** ≤ **Set** be the category of injective mappings. Let S be as in the previous example; and let τ_X be some bijection of X^2 onto $|X^2|$, the least ordinal equipollent with X^2. We turn the assignment $X \longmapsto |X^2|$ into a functor from **Inj** to **Inj** by stipulating that Tf is the canonical embedding of $|X^2|$ into $|Y^2|$. Under these conditions, τ cannot be natural. For, consider the square



.

Let f: $\{a\} \longrightarrow \{a, b\}$ be a mapping; suppose $f(a) = a$. Then since $Tf(\tau_{\{a\}}(a, a)) = 0$, if τ is natural, $\tau_{\{a,b\}}(a, a) = 0$. But the same argument for g: $\{a\} \longrightarrow \{a, b\}$ with $g(a) = b$ would show that $\tau_{\{a,b\}}(b, b) = 0$. Therefore the proposed transformation is not natural.

Natural transformations may be regarded as arrows in two kinds of categories.
a) Let F, G and H be functors from some category **C** to a category **D**;
α: $F \dashrightarrow G$ and β: $G \dashrightarrow H$ natural transformations. We define, for any composite $xy \in C$,

(4) $$(\beta \bullet \alpha)_{xy} = \beta_x \circ \alpha_y.$$

This is unambiguous, since if $uv = xy$, then by (1)

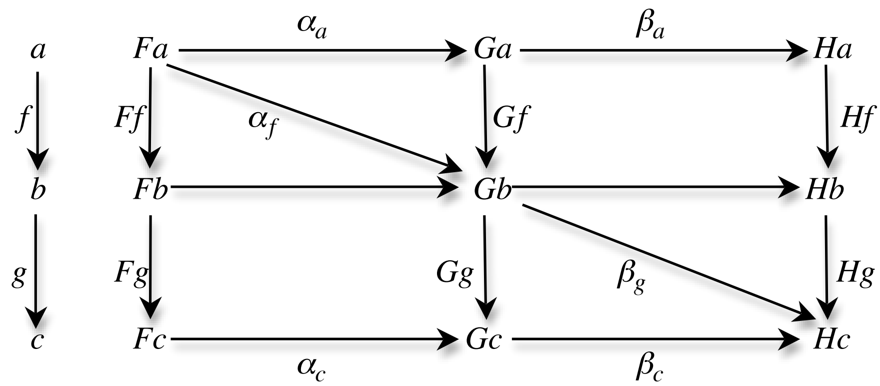
$$\beta_u \circ \alpha_v = \beta_u \circ Gv \circ \alpha_{\mathsf{d}(x)} = \beta_{uv} \circ \alpha_{\mathsf{d}(x)} = \beta_{xy} \circ \alpha_{\mathsf{d}(y)} = \beta_x \circ \alpha_y,$$

and it covers all of C since $x = x \circ \mathsf{d}(x)$. The operation $\bullet$ is seen to be associative from the Kleene-equalities

$$(\gamma \bullet (\beta \bullet \alpha))_{xyz} \simeq \gamma_x \circ \beta_y \circ \alpha_z \simeq ((\gamma \bullet \beta) \bullet \alpha)_{xyz}.$$

It is easily checked that $\beta \bullet \alpha$ is a natural transformation from F to H — and it shows clearly in the diagram below.

(5)



It is clear from (1) that a functor F from **C** to **D** is a natural transformation from F to F; and by comparing (1) with (4), we find for α: $F \dashrightarrow G$ that

$$\alpha \bullet F = \alpha = G \bullet \alpha.$$

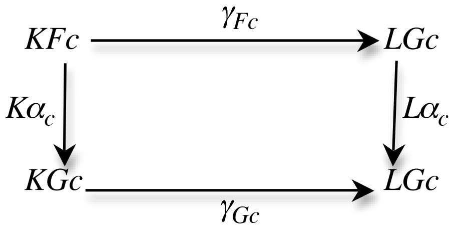
So the natural transformations between functors from **C** to **D** form a category, with composition $\bullet$, and for α: $F \dashrightarrow G$,

$$\mathsf{b}(\alpha) = G \text{ and } \mathsf{d}(\alpha) = F.$$

We denote this category by $\mathbf{D}^{\mathbf{C}}$.

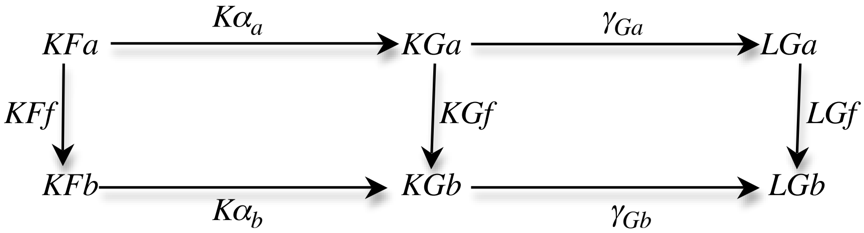
b) Let F and G be functors from $\mathbf{C}$ to $\mathbf{D}$, $\alpha\colon F \dot{\rightarrow} G$; and K and L functors from $\mathbf{D}$ to $\mathbf{E}$, with a natural transformation $\gamma\colon K \dot{\rightarrow} L$. Then for any $c \in \mathrm{Ob}\,\mathbf{C}$, $\gamma_{Gc} \circ K(\alpha_c) = L(\alpha_c) \circ \gamma_{Fc}$, as appears from the diagram below.

(4)



The assignment $c \longmapsto \gamma_{Gc} \circ K(\alpha_c)$ is a natural transformation from KF to LG: in the diagram below, the lefthand square commutes because α is natural, and the righthand square because γ is natural.

(5)



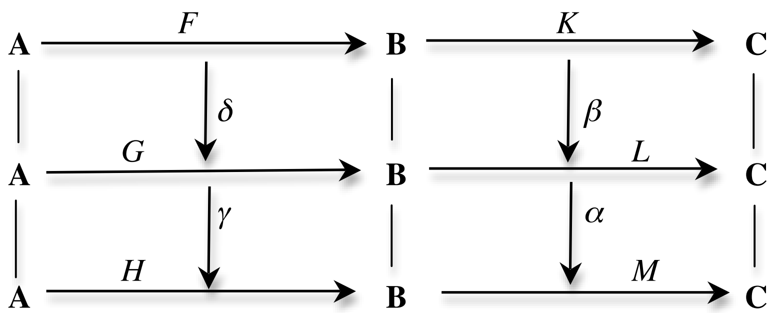
We view this assignment as the composite $\gamma \circ \alpha$ of arrows $\gamma\colon \mathbf{D} \longrightarrow \mathbf{E}$ and $\alpha\colon \mathbf{C} \longrightarrow \mathbf{D}$. This composition is transitive: if α and γ are as above, and $\delta\colon \mathbf{E} \longrightarrow \mathbf{B}$ is a natural transformation from M to N, then for any $c \in \mathrm{Ob}\,\mathbf{C}$,

$$
\begin{aligned}
(\delta \circ (\gamma \circ \alpha))(c) &= \delta_{LGc} \circ M((\gamma \circ \alpha)(c)) \text{ since } \gamma \circ \alpha\colon KF \dot{\rightarrow} LG \text{ and } \delta\colon M \dot{\rightarrow} N \\
&= \delta_{LGc} \circ M(\gamma_{Gc} \circ K(\alpha_c)) = \delta_{LGc} \circ M\gamma_{Gc} \circ MK(\alpha_c) \\
&= (\delta \circ \gamma)(Gc) \circ MK(\alpha_c) \text{ since } \delta\colon M \dot{\rightarrow} N \text{ and } \gamma\colon K \dot{\rightarrow} L \\
&= ((\delta \circ \gamma) \circ \alpha)(c) \text{ since } \alpha\colon F \dot{\rightarrow} G\text{; and } \delta \circ \gamma\colon MK \dot{\rightarrow} NL.
\end{aligned}
$$

The identity arrows for this composition are the identical transformations of the identity functors; thus $\mathrm{id}_{\mathbf{C}}$ assigns 1_c to every object c of $\mathbf{C}$, and this assignment is a natural transformation $1_{\mathbf{C}} \dot{\rightarrow} 1_{\mathbf{C}}$.

After the layout of the diagram

(6)



the operation $\bullet$ is called *vertical composition* and $\circ$ *horizontal composition*. The two are related by the *exchange law*:

(7) $$(\alpha \bullet \beta) \circ (\gamma \bullet \delta) \simeq (\alpha \circ \gamma) \bullet (\beta \circ \delta).$$

The truth of (7), assuming diagram (6) and $a \in \mathrm{Ob}\,\mathbf{A}$, may be demonstrated by calculation as follows:

$$((\alpha \bullet \beta) \circ (\gamma \bullet \delta))(a) = (\alpha \bullet \beta)(Ha) \circ K((\gamma \bullet \delta)(a)) \text{ since } \gamma \bullet \delta\colon F \dot{\rightarrow} H \text{ and } \alpha \bullet \beta\colon K \dot{\rightarrow} M$$

$= \alpha_{Ha} \circ \beta_{Ha} \circ K(\gamma_a) \circ K(\delta_a)$
$= \alpha_{Ha} \circ L(\gamma_a) \circ \beta_{Ga} \circ K(\delta_a)$ since $\beta\colon K \dashrightarrow L$ and $\gamma_a\colon Ga \longrightarrow Ha$
$= (\alpha \circ \gamma)(a) \circ (\beta \circ \delta)(a)$ since $\alpha\colon L \dashrightarrow M$ and $\gamma\colon G \dashrightarrow H$, and
$\beta\colon K \dashrightarrow L$ and $\delta\colon F \dashrightarrow G$
$= ((\alpha \circ \gamma) \bullet (\beta \circ \delta))(a)$.

The exchange law implies among other things that

$$(\alpha \bullet \beta) \circ F \simeq (\alpha \bullet \beta) \circ (F \bullet F) \simeq (\alpha \circ F) \bullet (\beta \circ F).$$

We sometimes omit the composition symbol $\circ$, thus writing $\alpha F \bullet \beta F$; $\bullet$ is never omitted. The category of natural transformations with vertical composition will be will be indicated by **Nat**.

(homomorphisms as nat transfos) (homomorphisms and term operations: hoofstuk 7. Clones as categories.)

§E Direct Products

Roughly speaking, the direct product of a family of structures is a structure that combines the information residing in the elements of the family, provided every element contains information.

1 Definition. Let $A = \langle \mathbf{A}_i \mid i \in I\rangle$ be a family of algebras; say $\mathbf{A}_i = \langle A_i, J_i\rangle$. The *direct product* of A is the algebra $\mathbf{B} = \langle B, K\rangle$ defined by

$$B = \prod_{i \in I} A_i\,;$$
$$\mathrm{Dom}(K) = \bigcap_{i \in I} \mathrm{Dom}(J_i);$$

and for every $n \in \mathbb{N}$, for every n-ary operation symbol $Q \in \mathrm{Dom}(K)$,

$$K(Q) = \{\langle b_0, \langle b_1,\ldots, b_n\rangle\rangle \mid \text{for all } i \in I, b_0(i) = J_i(Q)(b_1(i),\ldots, b_n(i))\}.$$

The notation for the direct product of $\langle \mathbf{A}_i \mid i \in I\rangle$ is

$$\prod_{i \in I} \mathbf{A}_i.$$

Single-line notations are $\prod_i \mathbf{A}_i$, $\prod(\mathbf{A}_i \mid i \in I)$, and $\prod A$.

If for every index $i \in I$, $\mathbf{A}_i$ is the same algebra $\mathbf{A}$, we also write $\mathbf{A}^I$ instead of $\prod_i \mathbf{A}_i$; we say that $\mathbf{A}^I$ is a *direct power* of $\mathbf{A}$. If $|I| = 2$, say $I = \{0, 1\}$, we write $\mathbf{A}_0 \times \mathbf{A}_1$, or, if $\mathbf{A}_0 = \mathbf{C}$ and $\mathbf{A}_1 = \mathbf{D}$, $\mathbf{C} \times \mathbf{D}$; more in general, a product of n algebras $\mathbf{A}_0, \ldots, \mathbf{A}_{n-1}$ is denoted by $\mathbf{A}_0 \times \ldots \times \mathbf{A}_{n-1}$.

The algebras $\mathbf{A}_i$ are called *direct factors* of $\prod_i \mathbf{A}_i$.

2 Proposition. The projections $e_j\colon \langle a_i \mid i \in I\rangle \mapsto a_j$, for $j \in I$, are homomorphisms from $\prod(\mathbf{A}_i \mid i \in I)$ to $\mathbf{A}_j$.

Proof. Let $\mathbf{B} = \prod_i \mathbf{A}_i$.

If Q is an n-ary operation symbol, and $b_0 = Q^{\mathbf{B}}(b_1,\ldots, b_n)$, then $b_0(j) = Q^{\mathbf{A}_j}(b_1(j),\ldots, b_n(j))$, written differently: $e_j(b_0) = Q^{\mathbf{A}_j}(e_j(b_1),\ldots, e_j(b_n))$. ⊠

The conditions under which projections are surjective were stated in §1H3. It is easily seen that if Q is an operation symbol and $\mathbf{B} = \prod(\mathbf{A}_i \mid i \in I)$, $Q^{\mathbf{B}}$ is total if and only if either some A_i is void, or *every* $Q^{\mathbf{A}_i}$ is total.

Examples
i. Let $\mathbf{N} = \langle \mathbb{N}, <, 0, S, + \rangle$. Then $0^{\mathbf{N}\times\mathbf{N}} = \langle 0, 0 \rangle$, $\langle 2, 1 \rangle <^{\mathbf{N}\times\mathbf{N}} \langle 6, 2 \rangle$, $\langle 2, 1 \rangle +^{\mathbf{N}\times\mathbf{N}} \langle 3, 2 \rangle = \langle 5, 3 \rangle$, etc.
ii. Let $\mathbf{R} = \langle \mathbb{R}, 0, + \rangle$. Then $\mathbf{R} \times \mathbf{R}$ is the real plane with vector addition.
iii. Let $\mathbf{N}_0 = \langle \mathbb{N}, 0, +, - \rangle$, with subtraction defined by

$$x - y = z \text{ if and only if } x = z + y,$$

and $\mathbf{B}$ the expansion of $\mathbf{N}_0 \times \mathbf{N}_0$ by a product operation $\cdot$ defined by

$$\langle x, y \rangle \cdot \langle u, v \rangle = \langle xu + yv, xv + yu \rangle.$$

(On the righthand side we use the ordinary multiplication of natural numbers.) Define f: $B \longrightarrow \mathbb{Z}$ by $f(x, y) = x - y$. Then f is a homomorphism from $\mathbf{B}$ onto the ring $\mathbf{Z} = \langle \mathbb{Z}, 0, +, -, \cdot \rangle$ of integers.

E3 Lemma. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras, $C \subseteq B$, f: $\mathbf{B} \longrightarrow \mathbf{A}$, $g = f \restriction C$, and $\mathbf{C} = C_{\mathbf{B}}$. Then
(i) g: $\mathbf{C} \longrightarrow \mathbf{A}$;
(ii) $g \in \operatorname{Sub}(\mathbf{A} \times \mathbf{B}) \Leftrightarrow C \in \operatorname{Sub}\mathbf{B}$.

Proof of (ii):
($\Rightarrow$) Suppose $\langle c_0, \ldots, c_{n-1} \rangle \in C^n \cap \operatorname{Dom}(Q^{\mathbf{B}})$. Then

$$\langle g(c_0), \ldots, g(c_{n-1}) \rangle = \langle f(c_0), \ldots, f(c_{n-1}) \rangle \in \operatorname{Dom}(Q^{\mathbf{A}}),$$

so $\langle \langle g(c_0), c_0 \rangle \ldots, \langle g(c_{n-1}), c_{n-1} \rangle \rangle \in \operatorname{Dom}(Q^{\mathbf{A}\times\mathbf{B}})$. Since

$$\langle g(c_0), c_0 \rangle \ldots, \langle g(c_{n-1}), c_{n-1} \rangle \in g \in \operatorname{Sub}(\mathbf{A} \times \mathbf{B}),$$

it follows that g contains $Q^{\mathbf{A}\times\mathbf{B}}(\langle g(c_0), c_0 \rangle \ldots, \langle g(c_{n-1}), c_{n-1} \rangle)$. But this is

$$\langle Q^{\mathbf{A}}(g(c_0), \ldots, g(c_{n-1})), Q^{\mathbf{B}}(c_0, \ldots, c_{n-1}) \rangle,$$

so we find $Q^{\mathbf{B}}(c_0, \ldots, c_{n-1}) \in C$. Since Q and $c_0, \ldots, c_{n-1}$ were arbitrary, we may conclude that $C \in \operatorname{Sub}\mathbf{B}$.
($\Leftarrow$) Suppose $c_0, \ldots, c_{n-1} \in C$, and

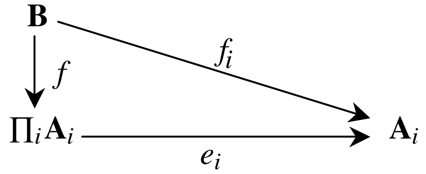
$$\langle \langle g(c_0), c_0 \rangle \ldots, \langle g(c_{n-1}), c_{n-1} \rangle \rangle \in \operatorname{Dom}(Q^{\mathbf{A}\times\mathbf{B}}).$$

Then certainly $\langle c_0, \ldots, c_{n-1} \rangle \in \operatorname{Dom}(Q^{\mathbf{B}})$, so $Q^{\mathbf{B}}(c_0, \ldots, c_{n-1}) \in C$, hence

$$Q^{\mathbf{A}\times\mathbf{B}}(\langle g(c_0), c_0 \rangle \ldots, \langle g(c_{n-1}), c_{n-1} \rangle) = \langle Q^{\mathbf{A}}(g(c_0), \ldots, g(c_{n-1})), Q^{\mathbf{B}}(c_0, \ldots, c_{n-1}) \rangle \in g. \quad \boxtimes$$

4 Corollary. If f: $\mathbf{B} \longrightarrow \mathbf{A}$, then $f \in \operatorname{Sub}(\mathbf{A} \times \mathbf{B})$.

5 Theorem. Let f_i: $\mathbf{B} \longrightarrow \mathbf{A}_i$ be homomorphisms, for all i in a set I. Then there exists precisely one homomorphism f: $\mathbf{B} \longrightarrow \prod_i \mathbf{A}_i$ that has the property that $e_i \circ f = f_i$ for all $i \in I$. If f_i is injective for at least one $i \in I$, then f is injective.



Proof. Put $\mathbf{A} := \prod_i \mathbf{A}_i$. Define $f(b)$, for $b \in B$, as $\langle f_i(b) \mid i \in I \rangle$; it is evident beforehand that this is the only way we can bring about that $e_i \circ f = f_i$ for all $i \in I$.

So we need only show that this f is a homomorphism. Let Q be an n-ary operation symbol such that $b_0 = Q^{\mathbf{B}}(b_1, \ldots, b_n)$. Then

$$\begin{aligned} f(b_0) &= \langle f_i(b_0) \mid i \in I \rangle && \text{by definition} \\ &= \langle Q^{\mathbf{A}_i}(f_i(b_1), \ldots, f_i(b_n)) \mid i \in I \rangle && \text{since the } f_i \text{ are homomorphisms} \\ &= Q^{\mathbf{A}}(\langle f_i(b_1) \mid i \in I \rangle, \ldots, \langle f_i(b_n) \mid i \in I \rangle) && \text{by definition} \\ &= Q^{\mathbf{A}}(f(b_1), \ldots, f(b_n)). \end{aligned}$$

Suppose f_j is injective, and $f(b_0) = f(b_1)$. Then

$$f_j(b_0) = e_j(f(b_0)) = e_j(f(b_1)) = f_j(b_1),$$

so $b_0 = b_1$. ⊠

The homomorphism f defined in the proof above may be denoted by

$$(f_i \mid i \in I),$$

or $(f_i)_{i \in I}$. For finite I we have more suggestive notations: $(f_0, \ldots, f_{n-1})$ if $I = n$; (g, h) if $I = 2$ and $f_0 = g$ and $f_1 = h$, and so on.

Corollary. Let $\boldsymbol{A} = \langle \mathbf{A}_i \mid i \in I \rangle$ and $\boldsymbol{B} = \langle \mathbf{B}_i \mid i \in I \rangle$ be families of algebras with the same index set I. Let for every $i \in I$ a homomorphism g_i: $\mathbf{B}_i \longrightarrow \mathbf{A}_i$ be given; let π_i be the projection from $\prod \boldsymbol{A}$ to $\mathbf{A}_i$, and ρ_i the projection from $\prod \boldsymbol{B}$ to $\mathbf{B}_i$. Then there exists precisely one homomorphism g: $\prod \boldsymbol{B} \longrightarrow \prod \boldsymbol{A}$ that has the property that $\pi_i \circ g = g_i \circ \rho_i$ for all $i \in I$. If g_i is injective for all $i \in I$, then g is injective.

Proof. Apply the theorem with $\mathbf{B} = \prod \boldsymbol{B}$ and $f_i = g_i \circ \rho_i$.

Suppose all the g_i are injective. If $g(x) = g(y)$, then for every i,

$$g_i(\rho_i(x)) = \pi_i(g(x)) = \pi_i(g(y)) = g_i(\rho_i(y)),$$

hence $\rho_i(x) = \rho_i(y)$. So $x = \langle \rho_i(x) \mid i \in I \rangle = \langle \rho_i(y) \mid i \in I \rangle = y$. ⊠

We use product notation for the g defined in the proof of the Corollary: $g = \prod(g_i \mid i \in I)$, $g = h \times k$ and so on. *In principle* this is ambiguous; but we are seldom interested in products of functions qua sets.

ii. Let $\mathbf{A}$ be a small category. Define for $f, g, x \in A$:

$$(f \leftarrow g)^{\mathbf{A}}(x) \simeq f \circ x \circ g \qquad (1).$$

This formula defines a functor $(\leftarrow)^{\mathbf{A}}$: $\mathbf{A} \times \mathbf{A}^{\partial} \longrightarrow \mathbf{Set}$.

1° For identity elements a, b of $\mathbf{A}$, define

$$\{a \leftarrow b\}^{\mathbf{A}} = \{x \mid (a \circ g \circ b)\downarrow\} \qquad (2).$$

Then $(a \leftarrow b)^{\mathbf{A}}$ is the identical function on $\{a \leftarrow b\}^{\mathbf{A}}$, and clearly $(f \leftarrow g)^{\mathbf{A}}$ maps $\{\mathsf{d}f \leftarrow \mathsf{b}g\}^{\mathbf{A}}$ to $\{\mathsf{b}f \leftarrow \mathsf{d}g\}^{\mathbf{A}}$.

2° Let $\langle f_1, g_1 \rangle$ and $\langle f_2, g_2 \rangle$ be composable arrows of $\mathbf{A} \times \mathbf{A}^{\partial}$. The composite is

$$\langle f_1 \circ f_2, g_2 \circ g_1 \rangle,$$

and indeed for $x \in A$,

$$(f_1 \circ f_2 \leftarrow g_2 \circ g_1)^{\mathbf{A}}(x) \simeq f_1 \circ f_2 \circ x \circ g_2 \circ g_1 \simeq (f_1 \leftarrow g_1)^{\mathbf{A}}((f_2 \leftarrow g_2)^{\mathbf{A}}(x)).$$

Now let a be an identity element (or, equivalently, an object) in $\mathbf{A}$. The *covariant homfunctor* $(\leftarrow a)^{\mathbf{A}}$ maps every object b of $\mathbf{A}$ to the set $\{b\leftarrow a\}^{\mathbf{A}}$, and every $f\colon c \longleftarrow b$ to the map

$$(f\leftarrow a)^{\mathbf{A}}\colon g \longmapsto f \circ g$$

from $\{b\leftarrow a\}^{\mathbf{A}}$ to $\{c\leftarrow a\}^{\mathbf{A}}$. The *contravariant homfunctor* $(a\leftarrow)^{\mathbf{A}}$ maps every object b of $\mathbf{A}$ to the set $\{a\leftarrow b\}^{\mathbf{A}}$, and every $f\colon b \longleftarrow c$ to the map

$$(a\leftarrow f)^{\mathbf{A}}\colon g \longmapsto g \circ f.$$

§f Infinitary operations

A finitary operation Q on a set A maps families $\langle a_i \mid i < n_Q\rangle$ into A. There are advantages, of practicality and cardinal simplicity, to natural numbers as index sets; but there is no reason in principle why we should not be more liberal. With maximal generality, for the moment, an I-ary operation on A, where I is any set, is a mapping of I-indexed families $\langle a_i \mid i \in I\rangle$, of elements of A, into A.

Example 1

Let $\langle A, O\rangle$ be a 1° countable Hausdorff space, and + a binary operation on A. Suppose $\langle a_i \mid i < \omega\rangle$ is a sequence of elements of A such that the secondary sequence

$$s_0 = a_0,$$

$$s_{n+1} = s_n + a_{n+1}$$

converges. Then we define

$$\sum_{i=0}^{\infty} a_i := \lim_{n\longrightarrow\infty} s_n.$$

This infinitary summation is an ω-ary operation. In particular, the infinite series of analysis are of this type.

Example 2 (Lehmann-Pásztor)

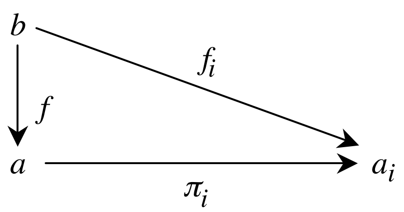
An ω-complete order is an ordered set $\langle X, \leq\rangle$ in which every chain

$$x_0 \leq x_1 \leq x_2 \leq \ldots$$

has a least upper bound. In this case, *least upper bound* is an ω-ary operation.

Example 3

Let $\mathbf{C}$ be a category, and $\boldsymbol{a} = \langle a_i \mid i \in I\rangle$ a family of objects of $\mathbf{C}$. A *product* of $\boldsymbol{a}$ is an object a of $\mathbf{C}$ with a family of arrows $\pi_i\colon a \longrightarrow a_i$ that every family $\langle f_i\colon b \longrightarrow a_i \mid i \in I\rangle$ of arrows uniquely factors through: there is a unique $f\colon b \longrightarrow a$ such that for all $i \in I$, $\pi_i \circ f = f_i$.



In various categories (in **Set**, for example, and in **Alg** — see §D) there exists a uniform construction of products, that can be thought of as a class of infinitary operations on objects. And even if there is no question of a uniform construction, we may still *assume* there is an operation that chooses a product for each family that has products. The product of $\boldsymbol{a}$ is denoted by

$$\prod \boldsymbol{a},\ \prod(a_i \mid i \in I) \text{ or } \prod_{i \in I} a_i,$$

or abbreviations such as $\prod_i a_i$ or $\prod a_i$. For the unique arrow f we use the notation $(f_i \mid i \in I)$. The projection may also be viewed as the result of an I-ary operation: $\pi_i^{\boldsymbol{a}}\colon \prod \boldsymbol{a} \longrightarrow a_i$.

The product notation may be generalized to arbitrary arrows: if

$$\boldsymbol{f} = \langle f_i\colon a_i \longrightarrow b_i \mid i \in I \rangle$$

is a family of arrows, then $\prod \boldsymbol{f} = (f_i \circ \pi_i^{\boldsymbol{a}} \mid i \in I)$.

§ Historical notes

The trick of passing relations for partial operations has occurred to several people at different times, probably independently. The earliest inventors that we are aware of are Lehmann and Pasztor [1982].

Exercises

§b

1. Draw Hasse-diagrams for the Boolean algebras $\mathcal{P}(0)$, $\mathcal{P}(1)$, $\mathcal{P}(2)$ and $\mathcal{P}(3)$. (Taking $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = 2 \cup \{2\}$.) Sketch the subuniverse lattices, assuming the nominator is $\{0, 1, \neg, \wedge, \vee\}$.

2. Let $\mathbf{A}$ be an algebra. By Corollary 3 and Theorem 14.3.8, $\mathbf{Sub}(\mathbf{A}) := \langle \mathrm{Sub}(\mathbf{A}), \subseteq \rangle$ is an algebraic lattice. Are there algebras $\mathbf{A}$ such that $\mathbf{Sub}(\mathbf{A})$ is not distributive?

3. Verify that a set lattice is indeed a lattice, and that a field of sets is a Boolean algebra.

4. Show that a subalgebra of a category $\mathbf{C} = \langle C, \circ, \mathsf{d}, \mathsf{b} \rangle$ is a category.

5. Prove Proposition 4.

6. Let $\mathbf{N}$ be the algebra with universe $\mathbb{N}$ and for each $m \in \mathbb{N}$ a single m-ary basic operation Q_m, defined by: $Q_m(n_0,\dots, n_{m-1}) = m$ if $n_0,\dots, n_{m-1}$ are all distinct,
0 otherwise.
Prove that $\mathbf{N}$ is a minimal algebra.

7. A category $\mathbf{C} = \langle C, \circ, \mathsf{d}, \mathsf{b} \rangle$ is a *quasi-order* if $\forall c, d \in \mathrm{Ob}\mathbf{C}\ |\mathbf{C}(c, d)| \leq 1$. A quasi-order is an *order* if it satisfies $Iso = Id$. Show:
(a) A subcategory of a quasi-order is a quasi-order.
(b) A subcategory of an order is an order.
(c) Every small quasi-order has a subcategory with the same objects (identity arrows) that is an order.

8. Show that left modules satisfy $0 \cdot a = \mathbf{0}$.

9. Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice. Show that $D \subseteq L$ is an ideal if and only if it is a downwards closed subuniverse.

§c

1. Prove: if $\langle f_i \mid i \in I \rangle$ is a family of homomorphisms from $\mathbf{A}$ into $\mathbf{B}$, then

$$\{a \in A \mid \text{for all } i, j \in I, f_i(a) = f_j(a)\}$$

is a subuniverse of $\mathbf{A}$.

2. A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is *constant* if for all $x, y \in A$, $f(x) = f(y)$. Show that the value of a constant functor is an identity element.

3. Let $\mathbf{A}$ and $\mathbf{B}$ be lattice orders. Show by example that a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ (that is, an isotone mapping) is not necessarily a homomorphism from $\mathbf{A}^{@}$ to $\mathbf{B}^{@}$.

§d

1. Let **A**, **B** be algebras, $f: B \longrightarrow A$ a mapping. Suppose $f \in \mathrm{Sub}(\mathbf{A} \times \mathbf{B})$.
(a) Show that it is not necessarily true that $f: \mathbf{B} \longrightarrow \mathbf{A}$.
(b) Show that $f: \mathbf{B} \longrightarrow \mathbf{A}$ if **A** is total.

2 (Mac Lane). Laat **B**, **C** en **D** kategorieën zijn. Laat voor alle $b \in Id^{\mathbf{B}}$ en $c \in Id^{\mathbf{C}}$ functoren

$$L_c: \mathbf{B} \longrightarrow \mathbf{D}, \quad M_b: \mathbf{C} \longrightarrow \mathbf{D}$$

gegeven zijn zo dat

$$\forall b, c: L_c(b) = M_b(c).$$

Dan bestaat er een functor $S: \mathbf{B} \times \mathbf{C} \longrightarrow \mathbf{D}$ zo dat

$$\forall b \in Id^{\mathbf{B}}\ \forall g \in C\ \ S(b, g) = M_b(g) \text{ en}$$
$$\forall c \in Id^{\mathbf{C}}\ \forall f \in B\ \ S(f, c) = L_c(f)$$

dan en slechts dan als voor iedere $f \in B$ en $g \in C$

$$M_{\mathsf{b}f}(g) \circ L_{\mathsf{d}g}(f) = L_{\mathsf{b}g}(f) \circ M_{\mathsf{d}f}(g).$$

Merk op dat uit het bewijs blijkt dat S uniek bepaald is.

§e

(Mac Lane). Laat $S, S': \mathbf{B} \times \mathbf{C} \longrightarrow \mathbf{D}$ functoren zijn. Zij $\alpha: Id^{\mathbf{B}} \times Id^{\mathbf{C}} \longrightarrow D$ een functie zo dat voor alle b, c

$$\alpha(b, c) \in \big(S'(b, c) \leftarrow S(b, c)\big)^{\mathbf{D}}.$$

Dan is α een natuurlijke transformatie van S in S' dan en slechts dan als voor alle b, c

(1) $\qquad \forall f \in B\ \ S'(f, c) \circ \alpha(\mathsf{d}f, c) = \alpha(\mathsf{b}f, c) \circ S'(f, c)$ en

(2) $\qquad \forall g \in C\ \ S'(b, g) \circ \alpha(b, \mathsf{d}g) = \alpha(b, \mathsf{b}g) \circ S'(b, g).$

Conditie (1) wordt informeel onder woorden gebracht als *α is natuurlijk in b*, conditie (2) als *α is natuurlijk in c*.

§E

1. Let I be a set, and **C** a category in which all I-indexed families have a product. Let $\prod$ be the I-ary product operation in **C**. Show that $\prod$ is a functor from $\mathbf{C}^I$ to **C**.