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Limit theorems for reflected Ornstein–Uhlenbeck processes

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This paper studies one-dimensional Ornstein–Uhlenbeck (OU) processes, with the distinguishing feature that they are reflected on a single boundary (put at level 0) or two boundaries (put at levels 0 and $d > 0$). In the literature, they are referred to as reflected OU (ROU) and doubly reflected OU (DROU), respectively. For both cases, we explicitly determine the decay rates of the (transient) probability to reach a given extreme level. The methodology relies on sample-path large deviations, so that we also identify the associated most likely paths. For DROU, we also consider the 'idleness process' L_t and the 'loss process' U_t, which are the minimal non-decreasing processes, which make the OU process remain ≥ 0 and $\leq d$, respectively. We derive central limit theorems (CLTs) for U_t and L_t, using techniques from stochastic integration and the martingale CLT.

Keywords and Phrases: Ornstein–Uhlenbeck processes, reflection, large deviations, central limit theorems.

1 Introduction

Ornstein–Uhlenbeck (OU) processes are Markovian, mean-reverting Gaussian processes. They well describe various real-life phenomena and allow a relatively high degree of analytical tractability. As a result, they have found widespread use in a broad range of application domains, such as finance, life sciences, and operations research. In many situations, though, the stochastic process involved is not allowed to cross a certain boundary, or is even supposed to remain within two boundaries.

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The resulting reflected (denoted in the sequel by ROU) and doubly reflected OU (DROU) processes have hardly been studied, though, a notable exception being the works by Ward and Glynn (2003, 2003, 2005), where ROU processes are used to approximate the number-in-system processes in M/M/1 and GI/GI/1 queues with reneging under a specific, reasonable scaling; the DROU process can be seen as an approximation of the associated finite-buffer queue. Srikant and Whitt (1996) also showed that the number-in-system process in a GI/M/n loss model can be approximated by ROU. For other applications, we refer to, for example, the introduction of Giorno *et al*. (2012) and references therein.

Throughout this paper, a probability space $(\Omega,\mathscr{F},\mathbb{P})$ equipped with a filtration $\mathbf{F} = \{\mathscr{F}_t\}_{t\in\mathbb{R}_+}$ is fixed. As known, the OU process is defined as the unique strong solution to the stochastic differential equation (SDE):

$$\mathrm{d}X_t = (\alpha - \gamma X_t)\mathrm{d}t + \sigma \mathrm{d}B_t, \qquad X_0 = x\in\mathbb{R},$$

where $\alpha \in \mathbb{R}$, $\gamma, \sigma > 0$, and B_t is a standard Brownian motion. The choice $\sigma > 0$ is only made for definiteness; from a distributional point of view, nothing changes if it is replaced with $-\sigma$. The OU process is *mean reverting* towards the value α/γ. To incorporate reflection at a lower boundary 0, thus constructing ROU, the following SDE is used, where we, throughout the paper, additionally assume $\alpha > 0$,

$$\mathrm{d}Y_t = (\alpha - \gamma Y_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}L_t, \qquad Y_0 = x \geqslant 0,$$

where L_t could be interpreted as an 'idleness process'. More precisely, L_t is defined as the minimal non-decreasing process such that $Y_t \geqslant 0$ for $t \geqslant 0$; it holds that $\int_{[0,T]} 1_{\{Y_t>0\}}\mathrm{d}L_t = 0$ for any $T > 0$.

Likewise, reflection at two boundaries can be constructed. DROU is defined through the SDE

$$\mathrm{d}Z_t = (\alpha - \gamma Z_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}L_t - \mathrm{d}U_t, \qquad Z_0 = x\in[0, d],$$

where U_t is the 'loss process' at the boundary d, that is, we have $\int_{[0,T]} 1_{\{Z_t>0\}}\mathrm{d}L_t = 0$ as well as $\int_{[0,T]} 1_{\{Z_t<d\}}\mathrm{d}U_t = 0$ for any $T > 0$. In the case of DROU, we assume that the upper boundary d is larger than α/γ throughout this paper, to guarantee that hitting d does not happen too frequently (which is a reasonable assumption for most of applications). For the existence of a unique solution to the aforementioned SDEs with reflecting boundaries, we refer to, for example, Tanaka (1979). In the context of queues with finite capacity, U_t is the continuous analog to the cumulative amount of loss over $[0,t]$, and that explains why we refer to it as the loss process.

The first objective of this paper is to obtain insight into transient rare-event probabilities. We do so for an ROU process with 'small perturbations', that is, a process given through the SDE

$$\mathrm{d}Y_t^\epsilon = \left(\alpha - \gamma Y_t^\epsilon\right)\mathrm{d}t + \sqrt{\epsilon}\sigma \mathrm{d}B_t + \mathrm{d}L_t^\epsilon,$$

with $\epsilon > 0$ is typically small. The transient distribution (at time $T \geq 0$, for any initial value $x \geq 0$) of the OU process being explicitly known (it actually has a Normal distribution), we lack such results for the ROU process. (As an aside, we note that the *stationary* distribution of ROU *is* known (Ward and Glynn, 2003); it is a truncated Normal distribution.) This motivates the interest in large-deviations asymptotics of the type

$$\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(Y_T^{\epsilon}\geqslant b|Y_0^{\epsilon}=x\right), \tag{1}$$

for $x\geq 0$, $T\geq 0$, and $b>\mathbb{E}\left(Y_T^{\epsilon}|Y_0^{\epsilon}=x\right)$ (so that the event under consideration is rare). We follow the method used for computing blocking probabilities of the Erlang queue in Shwartz and Weiss (1995), that is, relying on sample-path large deviations. In our strategy, a first step is to study the aforementioned decay rate for the 'normal' (that is, non-reflected) OU process. This decay rate is computed as the solution of a certain variational problem, relying on standard calculus-of-variations: it minimizes an 'action functional' over all paths f such that $f(0)=x$ and $f(T)\geq b$. The optimizing path $f^{\star}$ has the informal interpretation of 'most likely path' (or 'minimal cost path'): given the rare event under study happens, with overwhelming probability, it does so through a path 'close to' $f^{\star}$. In fact, $f^{\star}$ does not hit level 0 between 0 and T. For ROU, one can compute the cost of staying at the boundary 0, and performing all calculations, it turns out that the most likely path for ROU stays away from 0 and coincides with the most likely path for OU.

The computations for OU are presented in Section 2. The results are in line with what could be computed from the explicitly known distribution of X_T^{ϵ} conditional on $X_0^{\epsilon}=x$ but provide us, additionally, with the most likely path. Section 3 then focuses on the computation of the decay rate for ROU. Earlier, we described the intuitively appealing approach we followed, but it should be emphasized that at the technical level, there are some non-trivial steps to be taken. The primary complication is that the local large-deviations rate function at the reflecting boundary is different from this function in the interior (Doss and Priouret, 1983). Inspired by Robert (1976), we derive explicit expressions of the large-deviations rate function for ROU by properties of the reflection map in the deterministic Skorokhod problem. Unfortunately, calculus-of-variation techniques cannot be used immediately to identify the most likely path; this is due to the fact that we need to minimize over all non-negative continuous paths. However, the non-negativity of the optimizing path for the OU process facilitates the computation of the decay rates for ROU. In Section 4, we compute the decay rate for DROU by the same strategy as the one for ROU.

The second part of the paper focuses on DROU, with emphasis on properties of the loss process U_t (and also the idleness process L_t), for t large. Zhang and Glynn's martingale approach, as developed in Zhang and Glynn (2011), is employed to tackle a problem of this type. With $h(\cdot)$ being a twice continuously differentiable real function, we apply Itô's formula on $h(Z_t)$ and require $h(\cdot)$ to satisfy certain ordinary differential equations (ODEs) and specific initial and boundary conditions in order to construct martingales related to U_t and L_t. The presence of Z_t in the drift term leads to ODEs with non-constant coefficients, which seriously complicates the derivation of exact solutions. In Section 5, we use this approach to identify a central limit theorem (CLT) for U_t: we find explicit expressions for q_U and η_U such that $(U_t - q_U t)/\sqrt{t}$ converges to a Normal random variable with mean 0 and variance η_U^2; a similar result is established for L_t.

2 Transient asymptotics for Ornstein–Uhlenbeck

The primary goal of this section is to compute the decay rate (1) with Y^ϵ replaced by X^ϵ; in other words, we now consider the OU case (that is, no reflection). Before we attack this problem, we first identify the OU process' average behavior. To this end, we first describe the so-called zeroth-order approximation of one-dimensional diffusion processes. The SDE (more general than the one defining OU) we here consider is

$$\mathrm{d}J_t^\epsilon = b\left(J_t^\epsilon\right)\mathrm{d}t + \sqrt{\epsilon}\sigma\left(J_t^\epsilon\right)\mathrm{d}B_t, \qquad J_0^\epsilon = x,$$

and the corresponding ODE is

$$\mathrm{d}x(t) = b(x(t))\mathrm{d}t, \quad x(0) = x.$$

Theorem 1. *(Freidlin and Wentzell, 1984, Thm. 2.1.2)*
Suppose that $b(\cdot)$ and $\sigma(\cdot)$ are Lipschitz continuous and increase no faster than linearly, that is,

$$[b(x) - b(y)]^2 + [\sigma(x) - \sigma(y)]^2 \leqslant K^2|x - y|^2,$$
$$b^2(x) + \sigma^2(x) \leqslant K^2(1+|x|^2),$$

where K is a constant. Then for all $t > 0$ and $\epsilon > 0$, we have

$$\mathbb{E}|J_t^\epsilon - x(t)|^2 \leqslant \epsilon a(t),$$

where $a(t)$ is a monotone increasing function, which is expressed in terms of $|x|$ and K. Moreover, for all $t > 0$ and $\delta > 0$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} \left|J_s^\epsilon - x(s)\right| > \delta\right) = 0.$$

In the specific case of OU processes, the corresponding small perturbation process X_t^ϵ (on a finite time interval) satisfies

$$\mathrm{d}X_t^\epsilon = \left(\alpha - \gamma X_t^\epsilon\right)\mathrm{d}t + \sqrt{\epsilon}\sigma \mathrm{d}B_t, \qquad X_0 = x \geqslant 0. \tag{2}$$

It is readily checked that the limiting process $x(t)$ is given by

$$\dot{x}(t) = \alpha - \gamma x(t), \quad x(0) = x,$$

which has the solution

$$x(t) = \frac{\alpha}{\gamma} + \left(x - \frac{\alpha}{\gamma}\right)e^{-\gamma t}.$$

Note that $x(t) = \mathbb{E}X_t^\epsilon$. Popularly, as $\epsilon \downarrow 0$, with high probability, X_t^ϵ is contained in any δ-neighborhood of $x(t)$ on the interval $[0,T]$. Assuming that $b > x(T)$, it is now seen that the probability of our interest, of which we wish to identify the decay rate, relates to a rare event.

We now recall the Freidlin–Wentzell theorem (Dembo and Zeitouni, 1998, Thm. 5.6.7), which is the cornerstone behind the results of this section. To this end, we first define $C_{[0,T]}(\mathbb{R})$ as the space of continuous functions from $[0,T]$ to $\mathbb{R}$, with the uniform norm $\|f\|_\infty := \sup_{t\in[0,T]}|f(t)|$ and the metric $d(f,g) := \|f-g\|_\infty$. The Freidlin–Wentzell result now states that X^ϵ satisfies the sample-path large deviations principle (LDP) with the good rate function

$$I_x(f) := \begin{cases} (2\sigma^2)^{-1}\int_0^T \left(f'(t) - \alpha + \gamma f(t)\right)^2 \mathrm{d}t & \text{if } f\in H_x, \\ \infty & \text{if } f\notin H_x, \end{cases}$$

where $H_x := \left\{f : f(t) = x + \int_0^t \phi(s)\mathrm{d}s,\ \phi\in L_2([0,T])\right\}$. The LDP states that for any closed set F and open set G in $(C_{[0,T]}(\mathbb{R}), \|\cdot\|_\infty)$,

$$\limsup_{\epsilon\to 0} \epsilon\log\mathbb{P}\left(X_t^\epsilon\in F\right) \leqslant -\inf_{f\in F} I_x(f),$$
$$\liminf_{\epsilon\to 0} \epsilon\log\mathbb{P}\left(X_t^\epsilon\in G\right) \geqslant -\inf_{f\in G} I_x(f).$$

These upper and lower bounds obviously match for I_x-continuity sets S, that is, sets S such that $\inf_{f\in \mathrm{Cl}\, S} I_x(f) = \inf_{f\in \mathrm{int}\, S} I_x(f)$.

We now return to the decay rate under consideration. Let us first introduce some notation, following standard conventions in Markov process theory. We write $\mathbb{P}_x(E)$ for the probability of an event E in terms of the process X^ϵ if this process starts in x. We will mainly work with a fixed time horizon $T > 0$ and write $X_\cdot$ for $\{X_t,\ t\in[0,T]\}$. Our first step is to express the probability under study in terms of probabilities featuring in the sample-path LDP. Observe that we can write $\mathbb{P}_x\left(X_T^\epsilon \geq b\right) = \mathbb{P}_x\left(X_\cdot^\epsilon\in S\right)$, with

$$S := \bigcup_{a\geq b} S_a, \quad S_a := \left\{f\in C_{[0,T]}(\mathbb{R}) : f(0) = x, f(T) = a\right\}.$$

Later, we first solve a calculus-of-variation problem to find $\inf_{f\in S_a} I_x(f)$ explicitly. Second, we prove that S is an I_x-continuity set. A combination of these findings gives us an expression for the decay rate.

Proposition 1. *Let $a \geq b > x(T)$. Then*

$$\inf_{f\in S_a} I_x(f) = \frac{[a - x(T)]^2}{[1 - e^{-2\gamma T}](\sigma^2/\gamma)} = \frac{\left[a - \left(\frac{\alpha}{\gamma} + \left(x - \frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{[1 - e^{-2\gamma T}](\sigma^2/\gamma)}.$$

The optimizing path is given by

$$f^\star(t) = \big(C - \frac{\alpha}{\gamma}\big)e^{\gamma t} + (x - C)e^{-\gamma t} + \frac{\alpha}{\gamma}, \quad \text{where} \quad C := \frac{a - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma}e^{\gamma T} - xe^{-\gamma T}}{e^{\gamma T} - e^{-\gamma T}}.$$

Moreover, $f^\star(t) \geq 0$ on $t \in [0, \infty)$ when the starting point $x \geq 0$; $f^\star(t) \in [0,d]$ on $t \in [0,T]$ when the starting point $x \in [0,d]$, $a \in [0,d]$ and $\alpha/\gamma < d$.

Proof Obviously,

$$\inf_{f \in S_a} I_x(f) = \inf\left\{\frac{1}{2\sigma^2}\int_0^T \big(f'(t) + \gamma f(t) - \alpha\big)^2 \mathrm{d}t,\ f \in H_x \cap S_a\right\}.$$

According to Euler's necessary condition (Shwartz and Weiss, 1995, Thm. C.13), the initial condition, and the boundary condition, we have that the optimizing path satisfies

$$f''(t) - \gamma^2 f(t) + \alpha\gamma = 0, \quad f(0) = x, \quad f(T) = a.$$

The general solution of the ODE (unique up to the choice of the two constants) reads

$$f(t) = C_1 e^{\gamma t} + C_2 e^{-\gamma t} + \frac{\alpha}{\gamma}.$$

It is now readily checked that the stated expression follows, by imposing the initial condition and the boundary condition. Hence,

$$\inf_{f \in S_a} I_x(f) = \frac{(2C_1\gamma)^2}{2\sigma^2}\int_0^T e^{2\gamma t}\mathrm{d}t = \frac{\big[a - \big(\frac{\alpha}{\gamma} + \big(x - \frac{\alpha}{\gamma}\big)e^{-\gamma T}\big)\big]^2}{[1 - e^{-2\gamma T}](\sigma^2/\gamma)}.$$

We proceed with proving that $f^\star(t) \geq 0$ and $f^\star(t) \in [0,d]$ on $t \in [0, \infty)$ under the two stipulated assumptions. First, we note that $x(t) = xe^{-\gamma t} + (1 - e^{-\gamma t})\,\alpha/\gamma$, a convex combination of x and α/γ. As both of these are non-negative by assumption, so is $x(t)$. For $f^\star(t)$, we have the following alternative expressions with $q(t) := \sinh(\gamma t)/\sinh(\gamma T)$, as a direct computation shows:

$$\begin{aligned} f^\star(t) &= x(t) + (a - x(T))q(t) \\ &= q(t)a + (e^{-\gamma t} - q(t)e^{-\gamma T})x + (1 - e^{-\gamma t} - q(t)(1 - e^{-\gamma T}))\frac{\alpha}{\gamma}. \end{aligned}$$

It follows from the first equality that $f^\star(t) \geq x(t)$, because $a \geq x(T)$, and hence $f^\star(t)$ is non-negative. Moreover, the second equality shows that $f^\star(t)$ is a convex combination of a, x, and α/γ; see succeeding discussion. As all three of these are assumed to be less than d, the same holds true for $f^\star(t)$.

Finally, we show that we indeed have the claimed convex combination, by showing that all coefficients are non-negative and sum to one. The latter is obvious, as well as $q(t)\in[0,1]$. Furthermore $e^{-\gamma t}-q(t)e^{-\gamma T}\geq(1-q(t))e^{-\gamma T}\geq 0$. To prove that the third coefficient is non-negative, we use the basic equality

$$\sinh(x)=\frac{(1+e^{x})(1-e^{-x})}{2}.$$

Then observe that

$$\begin{aligned}1-e^{-\gamma t}-q(t)(1-e^{-\gamma T}) &= 1-e^{-\gamma t}-\frac{(1+e^{\gamma t})(1-e^{-\gamma t})}{(1+e^{\gamma T})(1-e^{-\gamma T})}(1-e^{-\gamma T})\\ &=(1-e^{-\gamma t})\left(1-\frac{1+e^{\gamma t}}{1+e^{\gamma T}}\right)\geq 0.\end{aligned}$$

This completes the proof. □

Proposition 2. *S is an I_x-continuity set.*

Proof Consider the topological space $(C_{[0,T]}(\mathbb{R}),\tau)$, where the topology τ is induced by the metric $d(f,g)$. We next consider $\bar{S}_x=\{f\in C_{[0,T]}(\mathbb{R}):f(0)=x\}$ with the subspace topology $\tau_{\bar{S}_x}=\{U\cap\bar{S}_x:U\in\tau\}$. The set S is a closed subset in $\bar{S}_x$ because the coordinate mapping $f\mapsto f(T)$ is τ-continuous. By the same property and the fact that the coordinate mapping is τ-open, the $\tau_{\bar{S}_x}$-interior of S is int $S=\{f\in C_{[0,T]}(\mathbb{R}):f(0)=x,f(T)>b\}$. We thus have

$$\inf_{f\in\text{cl }S}I_x(f)=\inf_{f\in S}I_x(f)=\inf_{a\geq b}\inf_{f\in S_a}I_x(f),\quad\text{and}\quad\inf_{f\in\text{int }S}I_x(f)=\inf_{a>b}\inf_{f\in S_a}I_x(f).$$

Using proposition 1 and the fact that $a\geq b>x(T)$,

$$\inf_{a\geq b}\inf_{f\in S_a}I_x(f)=\inf_{a>b}\inf_{f\in S_a}I_x(f)=\frac{\left[b-\left(\frac{\alpha}{\gamma}+\left(x-\frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{[1-e^{-2\gamma T}](\sigma^2/\gamma)}.$$

Consequently, S is an I_x-continuity set. □

Now, the decay rate under consideration can be determined.

Proposition 3. *Let $b>x(T)$. Then*

$$\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}_x\left(X_T^{\epsilon}\geq b\right)=\frac{-\left[b-\left(\frac{\alpha}{\gamma}+\left(x-\frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{[1-e^{-2\gamma T}](\sigma^2/\gamma)}.$$

Moreover, the minimal cost path is as given in proposition 1 (with a replaced by b).

Proof Apply 'Freidlin–Wentzell' to the I_x-continuity set S:

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_x\left(X_T^\epsilon \geq b\right) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{P}_x\left(X_\cdot^\epsilon \in S\right)$$
$$= -\inf_{f\in S} I_x(f) = -\inf_{a\geq b} \inf_{f\in S_a} I_x(f).$$

By the computations in the proof of proposition 2, we obtain the desired result. The minimal cost path is directly obtained from proposition 1. □

We mentioned in Section 1 that there is an alternative method to compute the decay rate under study. It follows relatively directly from the fact that X_T^ϵ (with $X_0^\epsilon = x$) is normally distributed with mean $\mu_T = x(T) = \frac{\alpha}{\gamma}(1 - e^{-\gamma T}) + xe^{-\gamma T}$ and variance $\sigma_T^2(\epsilon) = \frac{\epsilon\sigma^2}{2\gamma}(1 - e^{-2\gamma T})$, in conjunction with the standard inequality (Shwartz and Weiss, 1995, p. 19)

$$\frac{1}{y + y^{-1}} e^{-\frac{1}{2}y^2} \leq \int_y^\infty e^{-\frac{1}{2}t^2} \mathrm{d}t \leq \frac{1}{y} e^{-\frac{1}{2}y^2}.$$

We have followed our sample-path approach, though, for two reasons: (i) the resulting most likely path is interesting in itself, as it gives insight into the behavior of the system conditional on the rare event, but, more importantly, (ii) it is useful when studying the counterpart of the decay rate for ROU (rather than OU), which we pursue in Section 3.

We also note that

$$\lim_{T\to\infty} \lim_{\epsilon\to 0} \epsilon log \mathbb{P}_x\left(X_T^\epsilon \geq b\right) = \frac{-\left(b - \frac{\alpha}{\gamma}\right)^2}{\sigma^2/\gamma}.$$

It is known that the steady-state distribution of X_t^ϵ with $X_0^\epsilon = x$ is normally distributed with mean α/γ and variance $\epsilon\ \sigma^2/(2\gamma)$. We conclude that this shows that the result is invariant under changing the orders of taking limits ($T \to \infty$ and $\epsilon \to 0$).

3 Transient asymptotics for reflected Ornstein–Uhlenbeck

This section determines the decay rate (1) for ROU. For the moment, we consider a setting more general than OU and ROU, namely SDEs with reflecting boundary conditions.

Let $D^\circ \in \mathbb{R}$ be an open interval, and ∂D and D denote its boundary and closure. Let $v(x)$ denote the function giving the inward normal at $x \in \partial D$, that is, $v(x) = 1$ if x is a finite left endpoint of D and $v(x) = -1$ if x is a finite right endpoint of D. The reflected diffusion H^ϵ w.r.t. D is defined as the unique strong solution to

$$\mathrm{d}H_t^\epsilon = b\left(H_t^\epsilon\right)\mathrm{d}t + \sqrt{\epsilon}\sigma \mathrm{d}B_t + \mathrm{d}\xi_t^\epsilon, \qquad H_0^\epsilon = x \in D, \tag{3}$$

where $|\xi^\epsilon|_t = \int_0^t 1_{\partial D}(H_s^\epsilon)\mathrm{d}|\xi^\epsilon|_s$ and $\xi_t^\epsilon = \int_0^t \nu(H_s)\mathrm{d}|\xi^\epsilon|_s$. Here, $|\xi^\epsilon|_t$ denotes the total variation of ξ^ϵ by time t. We assume that $b(\cdot)$ is uniformly Lipschitz continuous and grows no faster than linearly, and σ is a non-zero constant. The existence and uniqueness of the strong solution are proved in Tanaka (1979).

We now recall the sample-path LDP for the reflected diffusion process, as it is considerably less known than the (standard) Freidlin–Wentzell theorem for the non-reflected case. We denote by H_x^+ the non-negative functions in H_x and by ω a function from $[0,T]$ to $\mathbb{R}$.

Theorem 2. *(Doss and Priouret, 1983, Thm. 4.2) If $b(\cdot)$ is uniformly Lipschitz continuous and bounded, and σ is a non-zero constant, then H^ϵ satisfies the LDP in $C_{[0,T]}(D)$ with the rate function*

$$I(h) = \inf_{\omega\geq 0} \frac{1}{2\sigma^2}\int_0^T \left(h_t' - b(h_t) - \nu(h_t)\omega_t 1_{\partial D}(h_t)\right)^2 \mathrm{d}t.$$

if $h\in H_x^+$ and ∞ else.

For reflected diffusions with a single reflecting boundary at 0, we can identify $\omega(t)$ and have the following explicit expression of the rate function. As usual, we define $x^+ = \max\{x,0\}$ and $x^- = -\min\{x,0\}$.

Proposition 4. *Let $D = [0,\infty)$. When $b(0) \geq 0$, H^ϵ satisfies the LDP in $C_{[0,T]}([0,\infty))$ with the rate function*

$$I^+(h) = \frac{1}{2\sigma^2}\int_0^T \left(h_t' - b(h_t)\right)^2 \mathrm{d}t$$

if $h\in H_x^+$ and ∞ else. In short, for $h\in H_x^+$ and $b(0)\in\mathbb{R}$, we have

$$I^+(h) = \frac{1}{2\sigma^2}\int_0^T \left(h_t' - b(h_t)\right)^2 \mathrm{d}t - \frac{1}{2\sigma^2}(b(0)^-)^2\int_0^T 1_{\{0\}}(h_t)\,\mathrm{d}t.$$

Proof In this case, $D = [0,\infty)$, $\partial D = \{0\}$, and $\nu(0) = 1$. The rate function becomes

$$I^+(h) = \inf_{\{\omega_t\geq 0\}} \frac{1}{2\sigma^2}\int_0^T \left(h_t' - b(h_t) - \omega_t 1_{\{0\}}(h_t)\right)^2 \mathrm{d}t.$$

We minimize for each t separately under the integral. If $h_t' - b(h_t) < 0$, then $\omega_t = 0$ is optimal. If $h_t' - b(h_t) \geq 0$ and $h_t > 0$, then $1_{\{0\}}(h_t)\omega_t \equiv 0$, which means that any value of ω is optimal. If $h_t^{'} - b(h_t) \geq 0$ and $h_t = 0$, then $\omega_t = h_t^{'} - b(h_t)$ is optimal. Hence, $\omega_t^\star = \left(h_t^{'} - b(h_t)\right)^+$ is the optimizer. It gives the following explicit expression:

$$I^+(h) = \frac{1}{2\sigma^2}\int_0^T \left(h_t^{'} - b(h_t) - 1_{\{0\}}(h_t)\left(h_t^{'} - b(h_t)\right)^+\right)^2 \mathrm{d}t$$

if $h \in H_x^+$ and ∞ else. For any $h \in C_{[0,T]}([0,\infty))$, which is differentiable a.e., note that $h_t^{'} = 0$ if $h_t = 0$. Then we have

$$\begin{aligned} I^+(h) &= \frac{1}{2\sigma^2}\int_0^T \left(h_t^{'} - b(h_t) - 1_{\{0\}}(h_t)b(0)^-\right)^2 \mathrm{d}t \\ &= \frac{1}{2\sigma^2}\int_0^T \left(h_t^{'} - b(h_t)\right)^2 \mathrm{d}t + \frac{1}{2\sigma^2}\int_0^T 1_{\{0\}}(h_t)(b(0)^-)^2 \mathrm{d}t + \frac{1}{\sigma^2}\int_0^T 1_{\{0\}}(h_t)b(0)^- b(0)\ \mathrm{d}t. \end{aligned}$$

When $b(0) \geq 0$, the last two terms are zero, and for $b(0) < 0$, they sum to

$$-\frac{1}{2\sigma^2} b(0)^2 \int_0^T 1_{\{0\}}(h_t)\ \mathrm{d}t.$$

This completes our proof. □

Theorem 2 requires $b(\cdot)$ to be bounded, which is a condition that ROU does not satisfy. But the mapping in (3) from Brownian motion B_t to reflected diffusion process H_t^ϵ is continuous because σ is a constant and $b(\cdot)$ is uniformly continuous. One can directly apply the contraction principle (Dembo and Zeitouni, 1998) to the rate function of Brownian motion. Details can be found in the proof of Theorem 2 on page 10 in Dupuis (1987). As a result, Proposition 4 is valid for ROU.

Earlier, we observed (i) that the most likely path for OU was non-negative (Proposition 1), and (ii) the rate functions I and I^+ for OU and ROU are the same as long as their arguments are non-negative paths on $[0,T]$ (Proposition 4). This suggests that the decay rates for OU and ROU (and the corresponding most likely paths) coincide.

The idea is now that we find the decay rate (1) for ROU by using the sample-path results that we derived in the previous section for OU. Recall that the zeroth-order approximation of OU is $x(t) = \alpha/\gamma + (x - \alpha/\gamma)e^{-\gamma t}$. It is readily checked that $x(t) > 0$ when the starting point $x \geq 0$. So we still assume $b > x(T)$ in the decay rate for ROU. We define $S^+ := \{f \in C_{[0,T]}([0,\infty)) : f(0) = x, f(T) \geq b\}$, corresponding to the rare event $\left\{Y_\bullet^\epsilon \in S^+\right\}$, so as to compute the decay rate (1); the set S_a^+ is defined as $\{f \in C_{[0,T]}$ $([0,\infty)) : f(0) = x, f(T) = a\}$. Later, we keep the notation $\mathbb{P}_x$ for probabilities of events in terms of Y^ϵ when this process starts in x.

Theorem 3. *Let $b > x(T)$. Then, similar to the result of proposition* 3,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}_x\left(Y_T^\epsilon \geq b\right) = \frac{-\left[b - \left(\frac{\alpha}{\gamma} + \left(x - \frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{\left[1 - e^{-2\gamma T}\right](\sigma^2/\gamma)}.$$

Moreover, the minimal cost path is as given in proposition 1 (with a replaced by b).

Proof As $b(0)=\alpha>0$, by proposition 4, Y^ϵ satisfies the sample path LDP in $C_{[0,T]}([0,\infty))$ with the rate function

$$I_x^+(h)=\frac{1}{2\sigma^2}\int_0^T\left(h_t^{'}-\alpha+\gamma h_t)\right)^2\mathrm{d}t$$

if $h\in H_x^+$ and ∞ else. By an argument that is similar to the one used in the proof of Proposition 2, S^+ is an I_x^+-continuity set. Then

$$\mathbb{P}_x\left(Y_T^\epsilon\geq b\right)=\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}_x\left(Y_\cdot^\epsilon\in S^+\right)=-\inf_{h\in S^+}I_x^+(h)=-\inf_{a\geq b}\inf_{h\in S_a^+}I_x^+(h).$$

We have

$$\begin{aligned}\inf_{h\in S_a^+}I_x^+(h)&=\inf\left\{\frac{1}{2\sigma^2}\int_0^T\left(h_t^{'}+\gamma h_t-\alpha\right)^2\mathrm{d}t,h\in H_x\cap S_a^+\right\}\\&\geqslant\inf\left\{\frac{1}{2\sigma^2}\int_0^T\left(h_t^{'}+\gamma h_t-\alpha\right)^2\mathrm{d}t,h\in H_x\cap S_a\right\}\\&=\inf_{f\in S_a}I_x(f).\end{aligned}$$

The optimizer $f^\star$ of $\inf_{f\in S_a}I_x(f)$ is always positive, for any starting point $x\geq 0$, because of Proposition . That is, $f^\star\in S_a^+$. Conclude that $\inf_{h\in S_a^+}I_x^+(h)=\inf_{f\in S_a}I_x(f)$. Then the results follows immediately from Proposition and $a\geq b>x(T)$,

$$\begin{aligned}\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}_x\left(Y_T^\epsilon\geq b\right)&=-\inf_{a\geq b}\frac{\left[a-\left(\frac{\alpha}{\gamma}+\left(x-\frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{\left[1-e^{-2\gamma T}\right](\sigma^2/\gamma)}\\&=-\frac{\left(b-\left(\frac{\alpha}{\gamma}+\left(x-\frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{\left[1-e^{-2\gamma T}\right](\sigma^2/\gamma)}.\end{aligned}$$

This proves the claim. □

4 Transient asymptotics for doubly reflected Ornstein–Uhlenbeck

This section computes the decay rate (1), but now for DROU. The case of DROU corresponds to choose the set $D=[0,d]$ in thm 2. We can still derive an explicit expression for the optimal $\omega_t^\star$ and hence have the following simplified rate function.

Proposition 5. *Given $D=[0,d]$, the rate function $I(h)$ in theorem 2 can be rewritten as follows:*

$$I^{++}(h)=\frac{1}{2\sigma^2}\int_0^T\left(h_t^{'}-b(h_t)-1_{\{0\}}(h_t)b(0)^-+1_{\{d\}}(h_t)b(d)^+\right)^2\mathrm{d}t$$

if $h\in H_x^{++}:=\{f\in H_x:0\leq f\leq d\}$ and ∞ else.

Proof As $\partial D = \{0,d\}$, the rate function becomes

$$I^{++}(h) = \inf_{\omega \geq 0} \frac{1}{2\sigma^2} \int_0^T \left(h_t^{'} - b(h_t) - \left[1_{\{0\}}(h_t) - 1_{\{d\}}(h_t)\right]\omega_t\right)^2 \mathrm{d}t.$$

Let $\omega^\star$ denote the optimizer of the preceding problem. We discuss the value of $\omega_t^\star$ in two cases.

- *Case 1*: $h_t^{'} - b(h_t) < 0$. If $h_t \in (0,d)$, then $\omega_t^\star$ can be any value; if $h_t = 0$, then $\omega_t^\star = 0$; if $h_t = d$, then $\omega_t^\star = -\left(h_t^{'} - b(h_t)\right)$.
- *Case 2*: $h_t^{'} - b(h_t) \geq 0$. If $h_t \in (0,d)$, then $\omega_t^\star$ can be any value; if $h_t = d$, then $\omega_t^\star = 0$; if $h_t = 0$, then $\omega_t^\star = h_t^{'} - b(h_t)$.

As a consequence, we have the following explicit expression:

$$I^{++}(h) = \frac{1}{2\sigma^2} \int_0^T \left(h_t^{'} - b(h_t) - 1_{\{0\}}(h_t)\left(h_t^{'} - b(h_t)\right)^+ + 1_{\{d\}}(h_t)\left(h_t^{'} - b(h_t)\right)^-\right)^2 \mathrm{d}t$$

if $h \in H_x^{++}$ and ∞ else. Also, $h \in [0,d]$ and h is differentiable a.e. imply that $\forall\, t \in (0,T)$, $h_t^{'} = 0$ when $h_t = 0$ or $h_t = d$. So the preceding expression can be further simplified to

$$I^{++}(h) = \frac{1}{2\sigma^2} \int_0^T \left(h_t^{'} - b(h_t) - 1_{\{0\}}(h_t)(-b(0))^+ + 1_{\{d\}}(h_t)(-b(d))^-\right)^2 \mathrm{d}t$$

if $h \in H_x^{++}$ and ∞ else. □

For DROU, $b(\cdot)$ is bounded, and as a consequence, it fulfills all requirements in theorem 2, and its rate function can be obtained by proposition 5 directly. Now, by a similar argument as employed in the last section, we prove that the decay rates (1) for OU and DROU (and the corresponding most likely paths) coincide. Recall from Section 1 that we have assumed $\alpha/\gamma < d$ for DROU throughout this paper. Under this assumption, the zeroth-order approximation $x(t)$ belongs to $(0,b)$ when the starting point x is in $[0,d]$.

We consider crossing levels $d \geq a \geq b > x(T)$. Define

$$S^{++} := \left\{f \in C_{[0,T]}([0,d]) : f(0) = x, f(T) \geq b\right\},$$

so that our rare event corresponds to $Z_\cdot^\epsilon \in S^{++}$; the set S_a^{++} is defined as $\{f \in C_{[0,T]}([0,d]) : f(0) = x, f(T) = a\}$. Finally, we arrive at the main result for DROU.

Theorem 4. *Let* $d \geq b > x(T)$. *Then*

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}_x\left(Z_T^\epsilon \geq b\right) = \frac{-\left[b - \left(\frac{\alpha}{\gamma} + \left(x - \frac{\alpha}{\gamma}\right)e^{-\gamma T}\right)\right]^2}{[1 - e^{-2\gamma T}]\sigma^2/\gamma}.$$

Moreover, the minimal cost path is the one given in proposition 3.

Proof *As* $b(0) = \alpha > 0$ and $b(d) = \alpha - \gamma d < 0$, by proposition 4, Z^{ϵ} satisfies the sample-path LDP in $C_{[0,T]}([0,d])$ with the rate function

$$I_x^{++}(h) = \frac{1}{2\sigma^2}\int_0^T \left(h_t^{'} - \alpha + \gamma h_t\right)^2 \mathrm{d}t$$

if $h \in H_x^{++}$ and ∞ else. Because of Proposition , it holds that $f^{\star} = \mathrm{arginf}_{f\in S_a} I_x(f)$ is in $[0, d]$ on $t \in [0,T]$, when the starting point $x \in [0,d]$, $a \in [0,d]$, and $\alpha/\gamma < d$, that is, $f^{\star} \in S_a^{++}$. As an immediate consequence, $f^{\star} = \mathrm{arginf}_{\varphi \in S_a^{++}} I_x^{++}(\varphi)$, and $\inf_{h\in S_a^{++}} I_x^{++}(h) = \inf_{f\in S_a} I_x(f)$. The rest of proof is similar to that of theorem 3. □

5 Central limit theorem and weak convergence of the loss process

The main objective of this section is to derive a CLT for the loss process U_t, for t large. We do so by relying on martingale techniques. A similar procedure can be followed for the idleness process L_t.

To prepare for the main result of this section, we start with some preliminary calculations. Let h be a twice continuously differentiable function on $\mathbb{R}$ and Z be the DROU process defined earlier. By Itô's formula, we have

$$\mathrm{d}h(Z_t) = \left((\alpha - \gamma Z_t)h^{'}(Z_t) + \frac{\sigma^2}{2}h^{''}(Z_t)\right)\mathrm{d}t + \sigma h^{'}(Z_t)\mathrm{d}B_t + h^{'}(Z_t)\mathrm{d}L_t - h^{'}(Z_t)\mathrm{d}U_t.$$

Based on the key properties of L and U, this reduces to

$$\mathrm{d}h(Z_t) = (\mathcal{L}h)(Z_t)\mathrm{d}t + \sigma h^{'}(Z_t)\mathrm{d}B_t + h^{'}(0)\mathrm{d}L_t - h^{'}(\mathrm{d})\mathrm{d}U_t, \tag{4}$$

where the operator $\mathcal{L}$ is defined through

$$\mathcal{L} := (\alpha - \gamma x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

The following lemma provides a key ingredient of the proof of proposition 6. It gives the solution to a linear ODE whose inhomogeneous term is considered as a variable as well, under initial and boundary conditions.

Lemma 1. *Consider the ODE with real variable right-hand side* $q \in \mathbb{R}$

$$(\mathcal{L}h) = q, \qquad 0 \leqslant x \leqslant d,$$

under the mixed initial/boundary conditions $h(0) = 0$, $h'(0) = 0$, *and* $h'(d) = 1$. *It has the unique solution* $(h,q) \in C^2(\mathbb{R}) \times \mathbb{R}$ *given by*

$$q = q_U := \frac{\sigma^2}{2}\frac{W(d)}{\int_0^d W(v)\mathrm{d}v}, \qquad h(x) = \frac{2q_U}{\sigma^2}\int_0^x\int_0^u \frac{W(v)}{W(u)}\mathrm{d}v\ \mathrm{d}v,$$

where

$$W(v) := exp\left(\frac{2\alpha v}{\sigma^2} - \frac{\gamma v^2}{\sigma^2}\right).$$

Proof By applying reduction of order, the ODE can be written as a system of first-order ODEs:

$$h^{'}(x) = f(x), \qquad f^{'}(x) + \frac{2\alpha - 2\gamma x}{\sigma^2} f(x) = \frac{2q}{\sigma^2}.$$

The integrating factor of the second first-order ODE is $W(x)$. Hence,

$$f(x) = \frac{C_1}{W(x)} + \frac{2q}{\sigma^2}\int_0^x \frac{W(u)}{W(x)}\mathrm{d}u.$$

Then the general solution is

$$h(x) = C_2 + \int_0^x f(u)\mathrm{d}u = C_2 + C_1\int_0^x \frac{1}{W(u)}\mathrm{d}u + \frac{2q}{\sigma^2}\int_0^x\int_0^u \frac{W(v)}{W(u)}\mathrm{d}v\ \mathrm{d}u.$$

Then the initial conditions $h(0) = 0, h'(0) = 0$ uniquely determine the values of C_1, C_2, while $h'(d) = 1$ uniquely determines q_U. Hence, we obtained the desired unique solution. □

We now present some additional observations. Let X be distributed according to the stationary distribution of the OU process, that is, X has a $N\left(\frac{\alpha}{\gamma}, \frac{\sigma^2}{2\gamma}\right)$ distribution, and denote by p_X its continuous density. We see that up to a multiplicative constant, the integrating factor W in the preceding proof is equal to p_X. It then follows that

$$q_U = \frac{\sigma^2}{2}\frac{p_X(d)}{\mathbb{P}(0 \le X \le d)}.$$

We will use the stationary distribution π_Z of the DROU process Z. Let, with a light abuse of notation, Z also denote a random variable with that distribution and let p_Z denote its density and F_Z its cumulative distribution function. The density p_Z of π_Z is obtained in Ward and Glynn (2003, Prop. 1) (see also Linetsky, 2005, Equation 31) as, with $N(m,s^2)$ denoting a Normal random variable with mean m and variance s^2,

$$\begin{aligned} p_Z(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}\left(N\left(\frac{\alpha}{\gamma}, \frac{\sigma^2}{2\gamma}\right) \le z \,\middle|\, 0 \leqslant N\left(\frac{\alpha}{\gamma}, \frac{\sigma^2}{2\gamma}\right) \leqslant d\right) \\ &= \sqrt{\frac{2\gamma}{\sigma^2}} \frac{\varphi\left(\left(z - \frac{\alpha}{\gamma}\right)\sqrt{\frac{2\gamma}{\sigma^2}}\right)}{\Phi\left(\left(d - \frac{\alpha}{\gamma}\right)\sqrt{\frac{2\gamma}{\sigma^2}}\right) - \Phi\left(\left(-\frac{\alpha}{\gamma}\right)\sqrt{\frac{2\gamma}{\sigma^2}}\right)}, \end{aligned}$$

where ϕ and Φ are the density function and cumulative density function of a standard Normal random variable. In short, for $0 \le z \le d$, we have

$$p_Z(z) = \frac{p_X(z)}{\mathbb{P}(0 \le X \le d)},$$

and hence,

$$q_U = \frac{\sigma^2}{2} p_Z(d).$$

For values of u, v in $[0,d]$, one also has

$$\frac{W(v)}{W(u)} = \frac{p_X(v)}{p_X(u)} = \frac{p_Z(v)}{p_Z(u)}.$$

Hence, the function h in lemma 1 has for $x \in [0,d]$ the alternative expressions

$$h(x) = p_Z(d) \int_0^x \frac{\mathbb{P}(0 \le X \le u)}{p_X(u)} \, \mathrm{d}u = p_Z(d) \int_0^x \frac{\mathbb{P}(0 \le Z \le u)}{p_Z(u)} \, \mathrm{d}u = p_Z(d) \int_0^x \frac{F_Z(u)}{p_Z(u)} \, \mathrm{d}u.$$

For the derivative $h'(x)$, we have for $x \in [0,d]$

$$h'(x) = \frac{p_Z(d)}{p_Z(x)} F_Z(x).$$

Proposition 6. *The loss process U satisfies the CLT, with η_U^2 defined in (6),*

$$\frac{U_t - q_U t}{\sqrt{t}} \Rightarrow \mathcal{N}\left(0, \eta_U^2\right), \text{ as } t \to \infty.$$

Proof We insert the unique solution $h(x)$ of lemma 1 into (4). As $h'(0) = 0, h'(d) = 1$, and $(\mathscr{L}h)(Z_t) = q_U$, we have the following integral expression:

$$U_t - q_U t + h(Z_t) - h(Z_0) = \sigma \int_0^t h'(Z_s) \mathrm{d}B_s. \tag{5}$$

We then observe that $M_t := U_t - q_U t + h(Z_t) - h(Z_0)$ is a zero-mean square integrable martingale. As usual, $< M >$ denotes the quadratic variation process of M. By the ergodic theorem (Gīhman and Skorohod, 1972, p. 134),

$$t^{-1} < M >_t = t^{-1} \sigma^2 \int_0^t h'(Z_s)^2 \mathrm{d}s \xrightarrow{\mathbb{P}} \sigma^2 \int_0^d h'(x)^2 \pi_Z(\mathrm{d}x) =: \eta_U^2, \tag{6}$$

where π_Z is the stationary distribution of Z_t.

Then we obtain from lemma 1

$$\eta_U^2 = \frac{4 q_U^2}{\sigma^2} \int_0^d \left(\int_0^x \frac{W(v)}{W(x)} \mathrm{d}v \right)^2 p_Z(x) \, \mathrm{d}x.$$

By the martingale CLT (Komorowski et al., 2012, Thm. 2.1), $t^{-\frac{1}{2}}M_t \Rightarrow \mathcal{N}\left(0, \eta_U^2\right)$ as $t \to \infty$. As $Z \in [0,d]$ and h is continuous, $h(Z)$ is bounded. So

$$\frac{h(Z_t) - h(Z_0)}{\sqrt{t}} \to 0$$

a.s. as $t \to \infty$, which implies the claim. □

The loss process at 0 can be treated analogously. Define

$$q_L := \frac{\sigma^2}{2} \frac{1}{\int_0^d W(v)\mathrm{d}v}, \qquad \eta_L^2 := \sigma^2 \int_0^d \left(-\frac{1}{W(x)} + \frac{2q_L}{\sigma^2} \int_0^x \frac{W(v)}{W(x)} \mathrm{d}x \right)^2 \pi(x) \, \mathrm{d}x. \tag{7}$$

Proposition 7. *The loss process L satisfies the CLT*

$$\frac{L_t - q_L t}{\sqrt{t}} \Rightarrow \mathcal{N}\left(0, \eta_L^2\right), \text{ as } t \to \infty.$$

Let us add some additional observations. The considerations just before proposition 6 enable us to write an alternative expression for η_U^2 as well. Using these, we obtain from (6)

$$\eta_U^2 = \sigma^2 p_Z(d)^2 \int_0^d \frac{F_Z(x)^2}{p_Z(x)} \, \mathrm{d}x.$$

Analogously, one derives that the function h in lemma 1 satisfying the *alternative* boundary conditions $h'(0) = 1$, $h'(d) = 0$ is, for $x \in [0,d]$, given by

$$h'(x) = \frac{p_Z(0)}{p_Z(x)} \mathbb{P}(Z \geq x) =: \frac{p_Z(0)}{p_Z(x)} \overline{F}(x),$$

and for q_L and η_L^2 in proposition 7

$$q_L = \frac{\sigma^2}{2} p_Z(0)$$
$$\eta_L^2 = \sigma^2 p_Z(0)^2 \int_0^d \frac{\overline{F}_Z(x)^2}{p_Z(x)} \, \mathrm{d}x.$$

The duality between the expressions for q_U, η_U^2 and q_L, η_L^2 can be completely explained by looking at the OU process $X' := d - X$, its doubly reflected version, and the associated increasing processes U' and L'. One easily shows that Z', denoting a random variable having the stationary distribution of this doubly reflected process, has the property $\mathbb{P}(Z' \geq x) = \mathbb{P}(Z \leq d - x)$, for all $x \in [0,d]$, so that $p_{Z'}(0) = p_Z(d)$. Similarly, the processes U' and L' can be interpreted as U and L swapped.

We proceed to refine the assertions of Propositions 6 and 7 to weak convergence results. Let C denote the space of real-valued continuous functions on $[0,\infty)$, endowed with the topology of uniform convergence on compacts and its Borel σ-algebra.

Proposition 8. *Let η_U^2 and q_U be defined in (6) and lemma (1), respectively. The scaled loss process*

$$U_t^n := \frac{U_{nt} - q_U nt}{\eta_U \sqrt{n}} \text{ converges weakly in } C \text{ to } B_t, \quad \text{as } n \to \infty,$$

where B_t is a standard Brownian motion. Likewise, the scaled loss process $L_t^n := \frac{L_{nt} - q_L nt}{\eta_L \sqrt{n}}$, with η_L^2 and q_L as in (7), *converges weakly in C to a standard Brownian motion.*

Proof It suffices to give the proof for U_t^n. By speeding up the time and scaling down the space, the expression (5) in the proof of proposition 6 becomes

$$\frac{U_{nt} - q_U nt}{\sqrt{n}} = \frac{h(Z_0) - h(Z_{nt})}{\sqrt{n}} + \frac{\sigma}{\sqrt{n}} \int_0^{nt} h'(Z_s) \mathrm{d}B_s.$$

By the ergodic theorem (Gīhman and Skorohod, 1972, p. 134), for arbitrary $t \in [0, \infty)$,

$$< \frac{\sigma}{\sqrt{n}} \int_0^{n\cdot} \mathrm{h}'(Z_s) \mathrm{d}B_s >_t = \frac{\sigma^2 t}{nt} \int_0^{nt} \mathrm{h}'(Z_s)^2 \mathrm{d}s \xrightarrow{\mathbb{P}} \eta_U^2 t, \quad \text{as } n \to \infty.$$

As $Z_{nt} \in [0,d]$ and h is continuous, $h(Z_{nt})$ is bounded. So

$$\sup_{t \geq 0} \frac{h(Z_0) - h(Z_{nt})}{\sqrt{n}} \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Then the claim is proved by applying the functional limit theorem for semi-martingales (Shiryayev, 1981) to U_t^n. □

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