Weak convergence of Markov-modulated diffusion processes with rapid switching

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1. Introduction

The Markov-modulated diffusion process is defined as a two-component Markov process \( \{X_t, M_t\}_{t \geq 0} \) on the space \( S \times \mathbb{R} \), where \( S := \{1, \ldots, d\} \), for some \( d \in \mathbb{N} \). \( X_t \) is a finite-state Markov chain with transition rate matrix \( Q \), and \( M_t \) is an \( X_t \)-modulated diffusion process. Such processes have been intensively studied in the literature. They have proven to be particularly useful in finance, as they are able to capture various relevant phenomena; see for instance the option pricing models of Yao et al. (2006) and Jobert and Rogers (2006) (where the focus is on Markov-modulated geometric Brownian motions), and the term structure model of Elliott and Siu (2009) (focusing on Markov-modulated versions of the Ornstein–Uhlenbeck and Cox–Ingersoll–Ross types). This kind of models allows the volatility and rate of returns to be random, and it turns out that parameters can be relatively easily calibrated on real data (where it is noted that this becomes harder when the modulating Markov chain has a large number of states, requiring sophisticated numerical methods).

There is also a large body of literature devoted to asymptotic behaviors of Markov-modulated (diffusion) processes. For example, Yin and Zhou (2004) and Nguyen and Yin (2010) studied the weak convergence of Markov-modulated random sequences where the modulating Markov chains and the modulated sequences are both discrete-time processes. Their continuous-time interpolations under time scaling are proved to converge weakly to switching diffusion processes. The result was applied to construct asymptotically optimal portfolios for discrete-time models of Markowitz’s mean–variance portfolio selection problem in Zhou and Yin (2003) and Yin and Zhou (2004).

In this paper we consider the Markov-modulated diffusion process from another point of view, namely that we study its limiting behavior when the modulating Markov chain is rapidly switching and ergodic. We speed up the process \( X_t \) by a factor \( n \), and consider the resulting system as \( n \to \infty \). In more concrete terms, it means that we replace the transition rate matrix \( Q \) by \( nQ \), and we denote by \( X^n_t \) the sequence of rapidly switching Markov chains. We let \( \pi \) be the ergodic distribution corresponding to \( X_t \), and hence also to \( X^n_t \) for every \( n \in \mathbb{N} \). Each process \( M^n_t \) is defined on a probability space \( (\Omega^n, \mathcal{F}^n, \mathbb{P}^n) \).
Theorem 3.1.

The proof of tightness relies on the following theorem, to which we refer as Aldous' tightness criterion, see Aldous (1978), Liptser and Shiryaev (1989, Theorem 6.3.1).

Let $H^n = (H^n_t)_{t \geq 0}$ be a sequence in $D[0, \infty)$ defined on some stochastic basis, in particular we assume that each $H^n$ is adapted to a filtration $\mathbb{F}^n$. Let, for $T > 0$, $\mathcal{I}^n_T$ denote the family of stopping times adapted to $\mathbb{F}^n$ taking values in $[0, T]$.

Theorem 3.1. Let $H^n = (H^n_t)_{t \geq 0}$ be a sequence in $D[0, \infty)$ defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}^n, \mathbb{P})$. If (i) for all $T > 0$,

$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{t \leq T} |H^n_t| > K \right) = 0,$$

Proofs of this type of weak convergence results typically consist of two parts: (i) proving that all the finite-dimensional convergence of $\{M^n_t\}_{t \in [0, T]}$ to $\{\hat{M}_t\}_{t \in [0, T]}$, and (ii) proving the tightness of $M^n_t$ in $C[0, T]$. As mentioned in the introduction, the first part is a consequence of Theorem 2.8 in Skorokhod (1989); its proof is based on the method of generators. It is noted that, confusingly, in Skorokhod (1989) the term convergence in distribution is to be understood as finite-dimensional convergence; see p. 77 of Skorokhod (1989). Our contribution, as given in the next section, is the proof of the tightness part.

3. Proof of tightness

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$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{t \leq T} |H^n_t| > K \right) = 0,$$
and (ii) for all $T > 0$ and $\eta > 0$,  
\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{t \in F_T(\mathbb{F})} \mathbb{P} \left( \sup_{t \in \delta} |H^n_{t+\delta} - H^n_{t}| \geq \eta \right) = 0, 
\]
then the sequence $H^n$ is tight. 

Since Skorokhod's $J_1$-topology is equivalent to the uniform topology in the space of continuous functions and $M^n_t$ is a sequence of continuous processes, $(M^n_t)_{t \in [0, T]}$ is tight in $C[0, T]$ if we prove that it satisfies the above two conditions for $T$. We start with introducing some additional notation. Let $\tau \in F_T(\mathbb{F})$. Then 

\[
C^n_t := \int_0^t \sigma(X^n_s, M^n_s)dB_s, \\
Y^n_t := \int_\tau^{t+\tau} \sigma(X^n_s, M^n_s)dB_s, \\
Q^n_t := \int_\tau^{t+\tau} b(X^n_s, M^n_s)ds.
\]

We first establish an auxiliary result. 

**Lemma 3.2.** For all $T > 0$, $\eta > 0$, and $K > 0$, 

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{t \in F_T(\mathbb{F})} \mathbb{P} \left( \sup_{t \in \delta} |Y^n_t| \geq \eta, M^n_{T+1} \leq K \right) = 0.
\]

**Proof.** By (A.2) and estimate (2), for any fixed $n$ and all $T > 0$, 

\[
\mathbb{E} \int_0^T \sigma^2(X^n_s, M^n_s)ds \leq \mathbb{E} \int_0^T C^2 \left( 2 + 2|M^n_s|^2 \right) ds \leq 2C^2T + 2T (K(T, 2))^2 < \infty.
\]

We can conclude that $C^n_t$ is a square-integrable continuous martingale. Also, $Y^n_t$ is a continuous martingale adapted to $(F^n_{t+\tau})_{t \geq 0}$ due to Theorem 4.7.1 in Liptser and Shiryayev (1989). We therefore find that 

\[
Z^n_t := \exp \left( \lambda Y^n_t - \frac{\lambda^2}{2} \langle Y^n \rangle_t \right), \quad \lambda \in \mathbb{R},
\]

is a positive local martingale, and hence a supermartingale. So $EZ^n_\sigma \leq 1$ for any stopping time $\sigma$. Take $\sigma := \inf\{t \geq 0 : |Y^n_t| \geq \eta\}$. It follows evidently that 

\[
\mathbb{P} \left( \sup_{t \in \delta} |Y^n_t| \geq \eta, M^n_{T+1} \leq K \right) = \mathbb{P} \left( |Y^n_\sigma| \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K \right) = \mathbb{P}(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) + \mathbb{P}(Y^n_\sigma \leq -\eta, \sigma \leq \delta, M^n_{T+1} \leq K).
\]

When $M^n_{T+1} \leq K$, $0 \leq \sigma \leq \delta \leq 1$ and $\tau \in F_T(\mathbb{F})$, Assumption (A.2) implies 

\[
\langle Y^n \rangle_\sigma = \int_\tau^{\tau+\sigma} \sigma^2(X^n_s, M^n_s)ds \leq C^2(1 + K)^2 \delta.
\]

Denote $C_K := C^2(1 + K)^2$, we have for any $\lambda > 0$, 

\[
\exp \left( \lambda \eta - \frac{\lambda^2}{2} C_K \delta \right) 1(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) \leq Z^n_\sigma 1(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K),
\]

almost surely. As this holds for any $\lambda > 0$, we take the supremum of the left-hand side in the previous display. Recalling $E Z^n_\sigma \leq 1$, taking expectations yields 

\[
\exp \left( \sup_{\lambda > 0} \left( \lambda \eta - \frac{\lambda^2}{2} C_K \delta \right) \right) \mathbb{P}(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) \leq EZ^n_\sigma 1(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) \leq 1.
\]

Upon rewriting the above equation, we obtain 

\[
\mathbb{P}(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) \leq \exp \left( - \sup_{\lambda > 0} \left( \lambda \eta - \frac{\lambda^2}{2} C_K \delta \right) \right) = \exp \left( - \frac{\eta^2}{2C_K \delta} \right).
\]

It thus follows that, for all $T > 0$, $\eta > 0$, and $K > 0$, 

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{t \in F_T(\mathbb{F})} \mathbb{P}(Y^n_\sigma \geq \eta, \sigma \leq \delta, M^n_{T+1} \leq K) = 0.
\]

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Moreover,

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{t \in T \cap \{\delta^n\}} \mathbb{P}(Y_n^\delta \leq -\eta, \sigma \leq \delta, M_{T+1}^{n\delta} \leq K) = 0$$

is proved in the same way by considering \( \lambda < 0 \). We have thus established the claim. \(\Box\)

We are now ready to prove the tightness claim.

**Proposition 3.3.** For all \( T > 0 \), the sequence \( \{M_t^n\}_{t \in [0, T]} \) is tight in \( C[0, T] \).

**Proof.** Firstly, we verify the condition (i) of Theorem 3.1. For any \( T > 0 \), evidently,

$$\sup_{t \leq T} |M_t^n| \leq |m_0| + \sup_{t \leq T} \int_0^t |b(X_s^n, M_s^n)|\,ds + \sup_{t \leq T} \int_0^t \sigma(X_s^n, M_s^n)\,dB_s,$$

almost surely. By (A.2),

$$M_t^n \leq |m_0| + C \int_0^T (1 + M_s^{n\delta})\,ds + C_t^{n\delta},$$

almost surely. Since \( |m_0| + CT + C_t^{n\delta} \) is nonnegative and non-decreasing in \( T \), Gronwall’s inequality implies

$$M_t^n \leq e^{CT} \left[ |m_0| + CT + C_t^{n\delta} \right], \quad \text{(3)}$$

almost surely. Now define \( j_K := K \exp(-CT) - |m_0| - CT \). Then (3) entails that for sufficiently large \( K \) such that \( j_K > 0 \),

$$\mathbb{P}(M_t^n \geq K) \leq \mathbb{P}(C_t^n \geq j_K) \leq j_K^{-2} \mathbb{E}(C_t^n)^2,$$

using Chebyshev's inequality. By (A.2) again, we have

$$\sigma^2(X_t^n, M_t^n) \leq C^2 (1 + M_s^{n\delta})^2.$$

Since (3) remains valid with replacing \( T \) by \( s \) for any \( s \leq T \), we find

$$\sigma^2(X_t^n, M_t^n) \leq C^2 \left[ 1 + e^{CT}(|m_0| + Cs + C_t^{n\delta}) \right]^2 \leq C^2 \left[ 1 + e^{CT}(|m_0| + CT + C_t^{n\delta}) \right]^2 \leq C^2 \left[ 2 (1 + e^{CT}|m_0| + e^{CT}CT)^2 + 2e^{2CT}(C_t^{n\delta})^2 \right] \leq L_T (1 + (C_t^{n\delta})^2),$$

where \( L_T \) is a positive constant not depending on \( K \). Applying Doob’s maximal inequality on \( C_T^n \) and plugging in the above estimate, we obtain

$$\mathbb{E}(C_T^n)^2 \leq 4 \mathbb{E} \left( \int_0^T \sigma^2(X_t^n, M_t^n)\,ds \right) \leq 4L_T T + 4L_T \int_0^T \mathbb{E}(C_s^n)^2\,ds \leq 4L_T T e^{4L_T T};$$

the last inequality follows from Gronwall’s inequality. Hence, for all \( T > 0 \),

$$\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{P}(M_t^n \geq K) = 0. \quad \text{(4)}$$

Secondly, we verify part (ii) of Theorem 3.1. To this end, note that for arbitrary \( T > 0, \delta \leq 1 \), and stopping time \( \tau \in T(\mathbb{F}^n) \),

$$\mathbb{P} \left( \sup_{t \leq \delta} \left| M_{T+t}^n - M_T^n \right| \geq \eta \right) \leq \mathbb{P} \left( \sup_{t \leq \delta} |M_{T+t}^n - M_T^n| \geq \eta, M_{T+1}^{n\delta} \leq K \right) + \mathbb{P}(M_{T+1}^{n\delta} > K) \leq \mathbb{P} \left( Q_{n\delta}^\delta \geq \frac{\eta}{2}, M_{T+1}^{n\delta} \leq K \right) + \mathbb{P} \left( Y_{n\delta}^\delta \geq \frac{\eta}{2}, M_{T+1}^{n\delta} \leq K \right) + \mathbb{P}(M_{T+1}^{n\delta} > K). \quad \text{(5)}$$

We show that the three terms in the right-hand side of (5) converge to 0 in the right way. We start with the first term. By virtue of Chebyshev’s inequality and (A.2),

$$\mathbb{P} \left( Q_{n\delta}^\delta \geq \frac{\eta}{2}, M_{T+1}^{n\delta} \leq K \right) \leq \mathbb{P} \left( \int_{T}^{T+\delta} |b(X_s^n, M_s^n)|\,ds \geq \frac{\eta}{2}, M_{T+1}^{n\delta} \leq K \right) \leq \frac{4}{\eta^2} \mathbb{E} \left( \int_{T}^{T+\delta} |b(X_s^n, M_s^n)|\,ds \right)^2 1(M_{T+1}^{n\delta} \leq K).$$
4. Examples

We have proven the claim.

The second example is that of a sequence of Markov-modulated geometric Brownian motions under rapid switching. The processes \{S^n_t\}_{t \geq 0} are defined by, for \( n \in \mathbb{N} \),

\[
S^n_t = s_0 + \int_0^t \mu(X^n_s)S^n_s ds + \int_0^t \sigma(X^n_s)S^n_s dB_s.
\]

Then Theorem 2.1 implies that the weak limit of the \{S^n_t\}_{t \geq 0}, as \( n \to \infty \), is a (non-modulated) geometric Brownian motion \{\hat{S}_t\}_{t \geq 0}, which is defined as

\[
\hat{S}_t = s_0 + \int_0^t \hat{\mu} \hat{S}_s ds + \int_0^t \hat{\sigma} \hat{S}_s dB_s,
\]

where

\[
\hat{\mu} := \sum_{i=1}^d \mu(i) \pi_i, \quad \hat{\sigma} := \left( \sum_{i=1}^d \sigma^2(i) \pi_i \right)^{1/2}.
\]

This limiting process is a well-studied object as well; it is for instance known that the marginal distributions of \( \hat{S}_t \) are of the Lognormal type with parameters that can be explicitly expressed in terms of \( \hat{\mu} \) and \( \hat{\sigma} \).

References

