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Weak convergence of Markov-modulated diffusion processes with rapid switching

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ABSTRACT

In this paper, we study the weak convergence of a sequence of Markov-modulated diffusion processes when the modulating Markov chain is ergodic and rapidly switching. We prove, in particular, its tightness property based on Aldous' tightness criterion.

1. Introduction

The Markov-modulated diffusion process is defined as a two-component Markov process $\{X_t, M_t\}_{t\geqslant 0}$ on the space $\mathbb{S}\times\mathbb{R}$, where $\mathbb{S} := \{1, \ldots, d\}$, for some $d \in \mathbb{N}$. X_t is a finite-state Markov chain with transition rate matrix Q, and M_t is an X_t-modulated diffusion process. Such processes have been intensively studied in the literature. They have proven to be particularly useful in finance, as they are able to capture various relevant phenomena; see for instance the option pricing models of Yao et al. (2006) and Jobert and Rogers (2006) (where the focus is on Markov-modulated geometric Brownian motions), and the term structure model of Elliott and Siu (2009) (focusing on Markov-modulated versions of the Ornstein–Uhlenbeck and Cox–Ingersoll–Ross types). This kind of models allows the volatility and rate of returns to be random, and it turns out that parameters can be relatively easily calibrated on real data (where it is noted that this becomes harder when the modulating Markov chain has a large number of states, requiring sophisticated numerical methods).

There is also a large body of literature devoted to asymptotic behaviors of Markov-modulated (diffusion) processes. For example, Yin and Zhou (2004) and Nguyen and Yin (2010) studied the weak convergence of Markov-modulated random sequences where the modulating Markov chains and the modulated sequences are both discrete-time processes. Their continuous-time interpolations under time scaling are proved to converge weakly to switching diffusion processes. The result was applied to construct asymptotically optimal portfolios for discrete-time models of Markowitz's mean–variance portfolio selection problem in Zhou and Yin (2003) and Yin and Zhou (2004).

In this paper we consider the Markov-modulated diffusion process from another point of view, namely that we study its limiting behavior when the modulating Markov chain is rapidly switching and ergodic. We speed up the process X_t by a factor n, and consider the resulting system as $n \to \infty$. In more concrete terms, it means that we replace the transition rate matrix Q by nQ, and we denote by X_t^n the sequence of rapidly switching Markov chains. We let $\boldsymbol{\pi}$ be the ergodic distribution corresponding to X_t, and hence also to X_t^n for every $n \in \mathbb{N}$. Each process M_t^n is defined on a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n =$

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$\{\mathcal{F}_t^n\}_{t\geqslant 0}, \mathbb{P}^n)$ through the stochastic differential equation

$$M_t^n = m_0 + \int_0^t b(X_s^n, M_s^n)\mathrm{d}s + \int_0^t \sigma(X_s^n, M_s^n)\mathrm{d}B_s^n, \quad X_0^n = x_0, \tag{1}$$

with $x_0 \in \mathbb{S}$ and B^n a Brownian motion. Processes with rapid switching have been studied in great detail by Skorokhod (1989), where it is noted that the analysis covers models that are more general than the one introduced above. In particular, Theorem 2.8 in Skorokhod (1989) implies that, as $n \to \infty$, the sequence of the Markov-modulated diffusion processes $\{M_t^n\}_{t\in[0,T]}$ converges, in terms of finite-dimensional distributions, to a (non-modulated) diffusion process $\{\hat{M}_t\}_{t\in[0,T]}$ whose coefficients are obtained by averaging the coefficients of M_t^n with respect to the ergodic distribution $\boldsymbol{\pi}$.

The main result in this paper is a proof of the tightness of $\{M_t^n\}_{t\in[0,T]}$. That, combined with convergence of finite-dimensional distributions, implies the weak convergence of $\{M_t^n\}_{t\in[0,T]}$ to $\{\hat{M}_t\}_{t\in[0,T]}$. In our proof, Aldous' tightness criterion in Aldous (1978) is the main ingredient, in combination with a technique which is used to prove the exponential tightness of two scaled diffusions in Liptser (1996).

In order to make the notation in the sequel not unnecessarily heavy, we drop the dependence on n everywhere where it does not lead to confusion, e.g. we simply use $\mathbb{P}$ for probability, $\mathbb{E}$ for expectation and B for Brownian motion.

2. Main result

We first introduce some additional notation. For an arbitrary stochastic process H_t, we denote $H_t^* := \sup_{s\leqslant t}|H_s|$. For $T \geqslant 0$, let $C[0,T]$ denote the space of real-valued continuous functions on $[0,T]$ equipped with the uniform topology; also, $D[0,\infty)$ is the space of càdlàg functions on $[0,\infty)$, equipped with Skorokhod's J_1-topology. We then impose some assumptions on the coefficients of (1).

(A.1) Local Lipschitz continuity: for any $r < \infty$, there is a positive constant K_r such that

$$|b(i,x) - b(i,y)| + |\sigma(i,x) - \sigma(i,y)| \leqslant K_r|x-y|, \quad \forall i \in \mathbb{S},\ |x|, |y| \leqslant r.$$

(A.2) Linear growth: there exists a positive constant C such that

$$|b(i,x)| + |\sigma(i,x)| \leqslant C(1+|x|), \quad \forall i \in \mathbb{S},\ x \in \mathbb{R}.$$

(A.3) X_t^n is ergodic with the ergodic distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$ for every $n \in \mathbb{N}$.

Assumptions (A.1) and (A.2) guarantee a unique strong solution (X_t^n, M_t^n) of (1), and the following estimate for fixed $T > 0$ and $n \in \mathbb{N}$:

$$\mathbb{E}(M_T^{n*})^\gamma \leqslant K(T,\gamma) < \infty, \quad \forall \gamma > 0; \tag{2}$$

cf. Yin and Zhu (2010). The following result is our main result, stating the weak convergence of the solution of the Markov-modulated stochastic differential equation to that of a *non-modulated* stochastic differential equation.

Theorem 2.1. *If Assumptions* (A.1)–(A.3) *hold, then as $n \to \infty$, M_t^n converges weakly in $C[0,T]$ to $\hat{M}_t$, which is the solution of the stochastic differential equation*

$$\hat{M}_t = m_0 + \int_0^t \hat{b}(\hat{M}_s)\mathrm{d}s + \int_0^t \hat{\sigma}(\hat{M}_s)\mathrm{d}B_s,$$

where

$$\hat{b}(x) := \sum_{i=1}^d b(i,x)\pi_i, \qquad \hat{\sigma}(x) := \left(\sum_{i=1}^d \sigma^2(i,x)\pi_i\right)^{1/2}.$$

Proofs of this type of weak convergence results typically consist of two parts: (i) proving that all the finite-dimensional convergence of $\{M_t^n\}_{t\in[0,T]}$ to $\{\hat{M}_t\}_{t\in[0,T]}$, and (ii) proving the tightness of M_t^n in $C[0,T]$. As mentioned in the introduction, the first part is a consequence of Theorem 2.8 in Skorokhod (1989); its proof is based on the method of generators. It is noted that, confusingly, in Skorokhod (1989) the term *convergence in distribution* is to be understood as finite-dimensional convergence; see p. 77 of Skorokhod (1989). Our contribution, as given in the next section, is the proof of the tightness part.

3. Proof of tightness

The proof of tightness relies on the following theorem, to which we refer as *Aldous' tightness criterion*, see Aldous (1978), Liptser and Shiryayev (1989, Theorem 6.3.1).

Let $H^n = \{H_t^n\}_{t\geqslant 0}$ be a sequence in $D[0,\infty)$ defined on some stochastic basis, in particular we assume that each H^n is adapted to a filtration $\mathbb{F}^n$. Let, for $T > 0$, $\Gamma_T(\mathbb{F}^n)$ denote the family of stopping times adapted to $\mathbb{F}^n$ taking values in $[0,T]$.

Theorem 3.1. *Let $H^n = \{H_t^n\}_{t\geqslant 0}$ be a sequence in $D[0,\infty)$ defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}^n, \mathbb{P})$. If* (i) *for all $T > 0$,*

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\leqslant T}|H_t^n| \geqslant K\right) = 0,$$

and (ii) *for all* $T > 0$ *and* $\eta > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}\left(\sup_{t\leqslant\delta}|H^n_{\tau+t}-H^n_\tau|\geqslant\eta\right)=0,$$

then the sequence H^n *is tight.*

Since Skorokhod's J_1-topology is equivalent to the uniform topology in the space of continuous functions and M^n_t is a sequence of continuous processes, $\{M^n_t\}_{t\in[0,T]}$ is tight in $C[0,T]$ if we prove that it satisfies the above two conditions for T. We start with introducing some additional notation. Let $\tau\in\Gamma_T(\mathbb{F}^n)$. Then

$$C^n_t := \int_0^t \sigma(X^n_s, M^n_s)\mathrm{d}B_s,$$

$$Y^n_t := \int_\tau^{\tau+t} \sigma(X^n_s, M^n_s)\mathrm{d}B_s,$$

$$Q^n_t := \int_\tau^{\tau+t} b(X^n_s, M^n_s)\mathrm{d}s.$$

We first establish an auxiliary result.

Lemma 3.2. *For all* $T > 0$, $\eta > 0$, *and* $K > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}\left(\sup_{t\leqslant\delta}|Y^n_t|\geqslant\eta, M^{n*}_{T+1}\leqslant K\right)=0.$$

Proof. By (A.2) and estimate (2), for any fixed n and all $T > 0$,

$$\mathbb{E}\int_0^T\sigma^2(X^n_s,M^n_s)\mathrm{d}s\leqslant\mathbb{E}\int_0^T C^2\left[2+2|M^n_s|^2\right]\mathrm{d}s\leqslant 2C^2T+2T\,(K(T,2))^2<\infty.$$

We can conclude that C^n_t is a square-integrable continuous martingale. Also, Y^n_t is a continuous martingale adapted to $(\mathcal{F}^n_{\tau+t})_{t\geqslant 0}$ due to Theorem 4.7.1 in Liptser and Shiryayev (1989). We therefore find that

$$Z^n_t := \exp\left(\lambda Y^n_t-\frac{\lambda^2}{2}\langle Y^n\rangle_t\right),\quad \lambda\in\mathbb{R},$$

is a positive local martingale, and hence a supermartingale. So $\mathbb{E}Z^n_\sigma\leqslant 1$ for any stopping time σ. Take $\sigma := \inf\{t\geqslant 0 : |Y^n_t|\geqslant\eta\}$. It follows evidently that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\leqslant\delta}|Y^n_t|\geqslant\eta, M^{n*}_{T+1}\leqslant K\right)&=\mathbb{P}\left(|Y^n_\sigma|\geqslant\eta,\sigma\leqslant\delta, M^{n*}_{T+1}\leqslant K\right)\\&=\mathbb{P}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)+\mathbb{P}(Y^n_\sigma\leqslant-\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K).\end{aligned}$$

When $M^{n*}_{T+1}\leqslant K$, $0\leqslant\sigma\leqslant\delta\leqslant 1$ and $\tau\in\Gamma_T(\mathbb{F}^n)$, Assumption (A.2) implies

$$\langle Y^n\rangle_\sigma=\int_\tau^{\tau+\sigma}\sigma^2(X^n_s,M^n_s)\mathrm{d}s\leqslant C^2(1+K)^2\delta.$$

Denote $C_K := C^2(1+K)^2$, we have for any $\lambda > 0$,

$$\exp\left(\lambda\eta-\frac{\lambda^2}{2}C_K\delta\right)\mathbf{1}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)\leqslant Z^n_\sigma\,\mathbf{1}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K),$$

almost surely. As this holds for any $\lambda > 0$, we take the supremum of the left-hand side in the previous display. Recalling $\mathbb{E}Z^n_\sigma\leqslant 1$, taking expectations yields

$$\exp\left(\sup_{\lambda>0}\left(\lambda\eta-\frac{\lambda^2}{2}C_K\delta\right)\right)\mathbb{P}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)\leqslant\mathbb{E}Z^n_\sigma\mathbf{1}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)\leqslant 1.$$

Upon rewriting the above equation, we obtain

$$\mathbb{P}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)\leqslant\exp\left(-\sup_{\lambda>0}\left(\lambda\eta-\frac{\lambda^2}{2}C_K\delta\right)\right)=\exp\left(-\frac{\eta^2}{2C_K\delta}\right).$$

It thus follows that, for all $T > 0$, $\eta > 0$, and $K > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}(Y^n_\sigma\geqslant\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)=0.$$

Moreover,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}(Y^n_\sigma\leqslant-\eta,\sigma\leqslant\delta,M^{n*}_{T+1}\leqslant K)=0$$

is proved in the same way by considering $\lambda<0$. We have thus established the claim. □

We are now ready to prove the tightness claim.

Proposition 3.3. *For all $T>0$, the sequence $\{M^n_t\}_{t\in[0,T]}$ is tight in $C[0,T]$.*

Proof. Firstly, we verify the condition (i) of Theorem 3.1. For any $T>0$, evidently,

$$\sup_{t\leqslant T}|M^n_t|\leqslant|m_0|+\sup_{t\leqslant T}\int_0^t|b(X^n_s,M^n_s)|\mathrm{d}s+\sup_{t\leqslant T}\left|\int_0^t\sigma(X^n_s,M^n_s)\mathrm{d}B_s\right|,$$

almost surely. By (A.2),

$$M^{n*}_T\leqslant|m_0|+C\int_0^T(1+M^{n*}_s)\mathrm{d}s+C^{n*}_T,$$

almost surely. Since $|m_0|+CT+C^{n*}_T$ is nonnegative and non-decreasing in T, Gronwall's inequality implies

$$M^{n*}_T\leqslant e^{CT}\left[|m_0|+CT+C^{n*}_T\right],\tag{3}$$

almost surely. Now define $j_K:=K\exp(-CT)-|m_0|-CT$. Then (3) entails that for sufficiently large K such that $j_K>0$,

$$\mathbb{P}(M^{n*}_T\geqslant K)\leqslant\mathbb{P}(C^{n*}_T\geqslant j_K)\leqslant j_K^{-2}\,\mathbb{E}(C^{n*}_T)^2,$$

using Chebyshev's inequality. By (A.2) again, we have

$$\sigma^2(X^n_s,M^n_s)\leqslant C^2(1+M^{n*}_s)^2.$$

Since (3) remains valid with replacing T by s for any $s\leqslant T$, we find

$$\begin{aligned}\sigma^2(X^n_s,M^n_s)&\leqslant C^2[1+e^{Cs}(|m_0|+Cs+C^{n*}_s)]^2\\&\leqslant C^2[1+e^{CT}(|m_0|+CT+C^{n*}_s)]^2\\&\leqslant C^2[2\left(1+e^{CT}|m_0|+e^{CT}CT\right)^2+2e^{2CT}(C^{n*}_s)^2]\\&\leqslant L_T[1+(C^{n*}_s)^2],\end{aligned}$$

where L_T is a positive constant not depending on K. Applying Doob's maximal inequality on C^n_t and plugging in the above estimate, we obtain

$$\mathbb{E}(C^{n*}_T)^2\leqslant 4\mathbb{E}\left(\int_0^T\sigma^2(X^n_s,M^n_s)\mathrm{d}s\right)\leqslant 4L_TT+4L_T\int_0^T\mathbb{E}(C^{n*}_s)^2\mathrm{d}s\leqslant 4L_TT\,e^{4L_TT};$$

the last inequality follows from Gronwall's inequality. Hence, for all $T>0$,

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(M^{n*}_T\geqslant K\right)=0.\tag{4}$$

Secondly, we verify part (ii) of Theorem 3.1. To this end, note that for arbitrary $T>0$, $\delta\leqslant 1$, and stopping time $\tau\in\Gamma_T(\mathbb{F}^n)$,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\leqslant\delta}|M^n_{\tau+t}-M^n_\tau|\geqslant\eta\right)&\leqslant\mathbb{P}\left(\sup_{t\leqslant\delta}|M^n_{\tau+t}-M^n_\tau|\geqslant\eta,M^{n*}_{T+1}\leqslant K\right)+\mathbb{P}(M^{n*}_{T+1}>K)\\&\leqslant\mathbb{P}\left(Q^{n*}_\delta\geqslant\frac{\eta}{2},M^{n*}_{T+1}\leqslant K\right)+\mathbb{P}\left(Y^{n*}_\delta\geqslant\frac{\eta}{2},M^{n*}_{T+1}\leqslant K\right)+\mathbb{P}(M^{n*}_{T+1}>K).\end{aligned}\tag{5}$$

We show that the three terms in the right-hand side of (5) converge to 0 in the right way. We start with the first term. By virtue of Chebyshev's inequality and (A.2),

$$\begin{aligned}\mathbb{P}\left(Q^{n*}_\delta\geqslant\frac{\eta}{2},M^{n*}_{T+1}\leqslant K\right)&\leqslant\mathbb{P}\left(\int_\tau^{\tau+\delta}|b(X^n_s,M^n_s)|\mathrm{d}s\geqslant\frac{\eta}{2},M^{n*}_{T+1}\leqslant K\right)\\&\leqslant\frac{4}{\eta^2}\mathbb{E}\left(\int_\tau^{\tau+\delta}|b(X^n_s,M^n_s)|\mathrm{d}s\right)^2\mathbf{1}(M^{n*}_{T+1}\leqslant K)\end{aligned}$$

$$\leqslant \frac{4}{\eta^2}\mathbb{E}\left(\int_\tau^{\tau+\delta} C(1+|M_s^n|)\mathrm{d}s\right)^2 \mathbf{1}(M_{T+1}^{n*}\leqslant K)$$
$$\leqslant \frac{4}{\eta^2}C^2(1+K)^2\delta^2.$$

As a result,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}\left(Q_\delta^{n*}\geqslant\frac{\eta}{2}, M_{T+1}^{n*}\leqslant K\right)=0.$$

The second term in the right-hand side of (5) converges to 0 in the desired way as well; by Lemma 3.2,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}\left(Y_\delta^{n*}\geqslant\frac{\eta}{2}, M_{T+1}^{n*}\leqslant K\right)=0.$$

The desired convergence of the third term on the right-hand side of (5) follows directly from (4). Upon combining the above, we have found that, for all $T>0$ and $\eta>0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\tau\in\Gamma_T(\mathbb{F}^n)}\mathbb{P}\left(\sup_{t\leqslant\delta}|M_{\tau+t}^n-M_\tau^n|\geqslant\eta\right)=0.$$

We have proven the claim. □

4. Examples

The sequence of Markov-modulated Ornstein–Uhlenbeck processes $\{M_t^n\}_{t\geqslant 0}$ with rapid switching is defined as

$$M_t^n = m_0+\int_0^t\left(\alpha(X_s^n)-\gamma(X_s^n)M_s^n\right)\mathrm{d}s+\int_0^t\sigma(X_s^n)\mathrm{d}B_s,$$

with $n\in\mathbb{N}$. By Theorem 2.1, it converges weakly in $C[0,T]$ to the (non-modulated) Ornstein–Uhlenbeck process $\{\hat{M}_t\}_{t\geqslant 0}$, characterized by

$$\hat{M}_t = m_0+\int_0^t\left(\hat{\alpha}-\hat{\gamma}\hat{M}_s\right)\mathrm{d}s+\int_0^t\hat{\sigma}\,\mathrm{d}B_s,$$

where

$$\hat{\alpha}:=\sum_{i=1}^d\alpha(i)\pi_i,\qquad \hat{\gamma}:=\sum_{i=1}^d\gamma(i)\pi_i,\qquad \hat{\sigma}=\left(\sum_{i=1}^d\sigma^2(i)\pi_i\right)^{1/2}.$$

This limiting process has been intensively studied in the literature. For instance, its marginals correspond to normal distributions, with a mean and variance that are known in closed form.

The second example is that of a sequence of Markov-modulated geometric Brownian motions under rapid switching. The processes $\{S_t^n\}_{t\geqslant 0}$ are defined by, for $n\in\mathbb{N}$,

$$S_t^n = s_0+\int_0^t\mu(X_s^n)S_s^n\mathrm{d}s+\int_0^t\sigma(X_s^n)S_s^n\mathrm{d}B_s.$$

Then Theorem 2.1 implies that the weak limit of the $\{S_t^n\}_{t\geqslant 0}$, as $n\to\infty$, is a (non-modulated) geometric Brownian motion $\{\hat{S}_t\}_{t\geqslant 0}$, which is defined as

$$\hat{S}_t = s_0+\int_0^t\hat{\mu}\hat{S}_s\mathrm{d}s+\int_0^t\hat{\sigma}\hat{S}_s\mathrm{d}B_s,$$

where

$$\hat{\mu}:=\sum_{i=1}^d\mu(i)\pi_i,\qquad \hat{\sigma}:=\left(\sum_{i=1}^d\sigma^2(i)\pi_i\right)^{1/2}.$$

This limiting process is a well-studied object as well; it is for instance known that the marginal distributions of $\hat{S}_t$ are of the Lognormal type with parameters that can be explicitly expressed in terms of $\hat{\mu}$ and $\hat{\sigma}$.

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