

RECURSIVE SOLUTION OF CERTAIN STRUCTURED LINEAR SYSTEMS*

ANDRÉ KLEIN† AND PETER SPREIJ‡

Abstract. We provide explicit representations of the null space \mathcal{S} of adjoints of companion-related matrices and of certain rectangular generalized Vandermonde matrices of block Toeplitz type which are encountered in the Fisher information matrix of time series processes. A formula for the right-inverse of this class of matrices A is provided which allows one to express the solution of the system $Ax = b$ as $x = A^{-1}b + \mathcal{S}$. The formulas can be easily turned into solution algorithms.

Key words. linear systems, coefficient matrix, null space, generalized Vandermonde matrix, Toeplitz matrix

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1. Introduction. The subject of this paper is concerned with a recursive solution of new linear systems of equations. The following two linear systems of equations are investigated:

$$(1.1) \quad \mathcal{K}_\nu(\sigma)X = \mathcal{E}$$

and

$$(1.2) \quad \mathcal{M}_\tau(\rho)Y = \mathcal{R}.$$

The coefficient matrices in (1.1) and (1.2) have the form

$$\mathcal{K}_\nu(\sigma) = \left(\frac{d^\nu}{dz^\nu} (u_q(z)u_q^{*\top}(z)), \frac{d^{\nu-1}}{dz^{\nu-1}} (u_q(z)u_q^{*\top}(z)), \dots, u_q(z)u_q^{*\top}(z) \right)_{z=\sigma}$$

and

$$\mathcal{M}_\tau(\rho) = \left(\frac{d^\tau}{dz^\tau} (\text{adj}(zI - C_p)), \frac{d^{\tau-1}}{dz^{\tau-1}} (\text{adj}(zI - C_p)), \dots, \text{adj}(zI - C_p) \right)_{z=\rho},$$

where $\mathcal{K}_\nu(\sigma) \in \mathbb{R}^{q \times q(\nu+1)}$ and $\mathcal{M}_\tau(\rho) \in \mathbb{R}^{p \times p(\tau+1)}$. The companion matrix $C_p \in \mathbb{R}^{p \times p}$ is given by

$$(1.3) \quad C_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -c_p & & & -c_2 & -c_1 \end{pmatrix},$$

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†Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands (aklein@fee.uva.nl).

‡Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands (spreij@science.uva.nl).

where ρ is an eigenvalue of C_p with algebraic multiplicity $\tau + 1$, \top denotes the transpose, $\text{adj}(X)$ denotes the adjoint of matrix X , σ , q , and ν are arbitrary scalar values. Further, we have

$$(1.4) \quad u_q(z) = (1, z, \dots, z^{q-1})^\top \quad \text{and} \quad u_q^*(z) = (z^{q-1}, \dots, 1)^\top.$$

Here X and Y are matrices of size $q(\nu + 1) \times \ell$ and $p(\tau + 1) \times h$, respectively, while \mathcal{E} and \mathcal{R} have size $q \times \ell$ and $p \times h$, respectively. The coefficient matrices in (1.1) are rectangular generalized Vandermonde matrices of block Toeplitz type and in (1.2) they are adjoints of companion-related matrices. The linear equations studied in this paper are extracted from [5], where the Fisher information matrix of a stationary time series process is interconnected with a solution to a Stein equation. The matrix \mathcal{E} is the Fisher information matrix of a stationary time series process, whereas matrix \mathcal{R} is a solution to a Stein equation for an extended version of $\mathcal{M}_\tau(\rho)$. The matrices X and Y are equal and this enables the interconnections to be successfully implemented. In this paper, stationary processes do not play any role, contrary to [5]. However, it is worth noticing that the interconnection between Toeplitz forms and stationary processes has been extensively studied in [3].

In [5], q is the degree of a polynomial $d_q(z)$ in $z \in \mathbb{C}$, σ is a root of polynomial $d_q(z)$ with algebraic multiplicity $\nu + 1$. In other words, q , σ , and ν are interconnected through polynomial $d_q(z)$, whereas in this paper q , σ , and ν are arbitrary scalar values with no link to a common polynomial and $q, \nu > 0$. The algorithm derived in [5] constructs a vector belonging to the null space of $\mathcal{K}_\nu(\sigma)$, which requires matrix multiplications.

A property proved in [6] is used in [5] to derive an algorithm for the kernel of $\mathcal{M}_\tau(\rho)$, it concerns an interconnection between $\text{adj}(zI - C_p)$ and the basis vector $u_p(z)$, this holds for $p = q$, $\sigma = \rho$, $\nu = \tau$ and when ρ is an eigenvalue of C_p . The vectors $y \in \text{Ker}(\mathcal{M}_\tau(\rho))$ and $x \in \text{Ker}(\mathcal{K}_\tau(\rho))$, where $\text{Ker}(X)$ is the kernel of the matrix X , are then interconnected. Consequently, the algorithm of the null space of $\mathcal{M}_\tau(\rho)$ given by vector y is based on the algorithm of the null space of $\mathcal{K}_\tau(\rho)$ expressed by vector x . The computation of the vector y involves an inversion of a lower triangular and Toeplitz matrix. However, this is combined with $p\tau$ matrix multiplications of the inverted matrix with the corresponding vector $x \in \text{Ker}(\mathcal{K}_\tau(\rho))$. This is in agreement with the dimension of the null space of $\mathcal{M}_\tau(\rho)$.

In this paper the approach is different, (1.1) and (1.2) are two different linear systems of equations without a common matrix, and we develop a new algorithm for the null space of the coefficient matrices $\mathcal{K}_\nu(\sigma)$ and $\mathcal{M}_\tau(\rho)$ independently.

A solution of the linear systems of (1.1) and (1.2) is considered when $q = \nu + 1$ and $p = \tau + 1$. In this case, the newly developed algorithms for the null spaces and right-inverses are equivalent for both coefficient matrices. The appropriate right-inverse is expressed in terms of a generalized Vandermonde matrix. A new algorithm is also developed for the kernel of $\mathcal{K}_\nu(\sigma)$ for the case $q > \nu + 1$. The newly displayed algorithms for the null space do not require matrix multiplications and matrix inversions. The main computational exercise consists of evaluating factorials and binomial coefficients, the latter can be computed by applying the Pascal triangle, combined with recursions that consist of addition of two vectors. However, the problem set forth in this paper is algebraical. The purpose is to write a solution of new linear systems of equations as a function of z and the problem studied is therefore not numerical. For that purpose one will subsequently consider the coefficient matrix $\mathcal{K}_\nu(z)$. When we consider the coefficient matrix $\mathcal{M}_\tau(z)$, for technical reasons that shall be specified in section 4, we will then consider the case $z = \rho$.

When $q = \nu + 1$ and $p = \tau + 1$, the representation of the null space $\text{Ker}(\mathcal{M}_\tau(\rho))$ is obtained by simply transposing certain matrices in the representation of the null space $\text{Ker}(\mathcal{K}_\nu(\sigma))$. This means that when the algorithm of $\text{Ker}(\mathcal{M}_\tau(\rho))$ needs to be evaluated one can use the algorithm for the null space $\text{Ker}(\mathcal{K}_\nu(\sigma))$. Contrary to the corresponding algorithm displayed in [5], where a matrix inversion and matrix multiplications are involved, there is no need for a computational exercise of any kind when the algorithm set forth in this paper is applied.

Another fundamental difference with the approach in [5] is that the algorithms developed in this paper cover the entire span of the null spaces of $\mathcal{K}_\nu(\sigma)$ and $\mathcal{M}_\tau(\rho)$ and not just a vector as in [5].

Consequently, we may apply these results to provide explicit expressions of the solutions to the systems (1.1) and (1.2); more specifically, for $q = \nu + 1$ and $p = \tau + 1$ we have

$$(1.5) \quad X = (\mathcal{K}_\nu(\sigma))^{-1} \mathcal{E} + \mathcal{W}(\sigma) \text{ with } \mathcal{W}(\sigma) \in \text{Ker}(\mathcal{K}_\nu(\sigma)),$$

$$(1.6) \quad Y = (\mathcal{M}_\tau(\rho))^{-1} \mathcal{R} + \mathcal{L}(\rho) \text{ with } \mathcal{L}(\rho) \in \text{Ker}(\mathcal{M}_\tau(\rho)).$$

The similarity of the null spaces of the coefficient matrices in (1.1) and (1.2) is interesting. It implies a connection between adjoints of companion-related matrices and rectangular generalized Vandermonde matrices of the block Toeplitz type.

Solutions of linear systems of equations are also presented in, e.g., [1], [2], and [4], where the coefficient matrices are Toeplitz, Hankel, Hilbert-type, Cauchy, and Vandermonde-type matrices.

The paper is organized as follows. In section 2, a right-inverse representation of the coefficient matrices $\mathcal{K}_\nu(z)$ and $\mathcal{M}_\tau(\rho)$ is introduced. In sections 3 and 4, a corresponding algorithm for the kernel of the coefficient matrices $\mathcal{K}_\nu(z)$ and $\mathcal{M}_\tau(\rho)$ is developed for the case $q = \nu + 1$, respectively, $p = \tau + 1$. The main conclusions are formulated in section 5. An algorithm for the kernel of $\mathcal{K}_\nu(z)$, when $q > \nu + 1$, is displayed in section 6.

2. A right-inverse: Case $q = \nu + 1$. A right-inverse of $\mathcal{K}_\nu(z)$ is given for $q = \nu + 1$, which is a special form of the right-inverse presented in [5]. We introduce the $q \times q$ generalized Vandermonde matrix $\mathcal{T}_\nu^q(z)$ where

$$\mathcal{T}_\nu^q(z) = \left(\mathcal{T}_\nu^{(\nu)}(z), \mathcal{T}_\nu^{(\nu-1)}(z), \dots, \mathcal{T}_\nu^{(0)}(z) \right)$$

and

$$\mathcal{T}_\nu^{(\nu-k)}(z) = \frac{\partial^{\nu-k}}{\partial z^{\nu-k}} u_q(z), \quad k = 0, 1, \dots, \nu.$$

The following lemma can now be formulated.

LEMMA 2.1. *When $q = \nu + 1$ the relations*

$$\begin{aligned} \mathcal{K}_\nu(z) (I_q \otimes e_q) &= \mathcal{T}_\nu^q(z), \\ \mathcal{K}_\nu(z) \left((\mathcal{T}_\nu^q(z))^{-1} \otimes e_q \right) &= I_q \end{aligned}$$

hold true. Clearly, an appropriate right-inverse is then $(\mathcal{K}_\nu(z))_R^{-1} = (\mathcal{T}_\nu^q(z))^{-1} \otimes e_q$, where e_q is the last standard basis vector in \mathbb{R}^q .

Proof. Straightforward computation confirms the property. \square

Consider the matrices A and B of size $m \times n$ and $p \times q$, respectively; then the $mp \times nq$ Kronecker product of the two matrices is defined as $A \otimes B = (a_{ij})B$ for all i and j .

A choice for an appropriate right-inverse of $\mathcal{M}_\tau(\rho)$ when $p = \tau + 1$ is given in the following corollary.

COROLLARY 2.2. *When $p = \tau + 1$ a right-inverse of $\mathcal{M}_\tau(\rho)$ is given by*

$$(\mathcal{T}_\tau^p(\rho))^{-1} \otimes e_p,$$

where e_p is the last standard basis vector in \mathbb{R}^p . We then have

$$\mathcal{M}_\tau(\rho) \left((\mathcal{T}_\tau^p(\rho))^{-1} \otimes e_p \right) = I_p.$$

Proof. We have the property that the last column of $\text{adj}(zI - C_p)$ is $u_p(z)$; this can be shown by equality (4.4), and this coincides with the last column of the matrix $u_p(z)u_p^*(z)$. This implies equality of the last column of the blocks composing $\mathcal{K}_\nu(z)$ and $\mathcal{M}_\tau(z)$. Since the construction of the right-inverse displayed in Lemma 2.1 is based on the last column of the blocks in $\mathcal{K}_\nu(z)$, the right-inverse set forth in Lemma 2.1 then also holds for $\mathcal{M}_\tau(z)$. \square

In the next section an algorithm for the null space $\text{Ker}(\mathcal{K}_\nu(z))$ is displayed.

3. Ker ($\mathcal{K}_\nu(z)$) for the case $\nu + 1 = q$. We shall specify the dimension of the null space $\text{Ker}(\mathcal{K}_\nu(z))$ in the next proposition.

PROPOSITION 3.1. *The null space $\text{Ker}(\mathcal{K}_\nu(z))$ has dimension equal to $q\nu$ and the rank of the coefficient matrix $\mathcal{K}_\nu(z)$ is q , when $\nu + 1 = q$.*

Proof. In Lemma 2.1, a right-inverse of the coefficient matrix $\mathcal{K}_\nu(z)$ is set forth. This implies that the $q \times q(\nu + 1)$ coefficient matrix $\mathcal{K}_\nu(z)$ is surjective or has full row rank; its rank is then q . By virtue of the dimension rule it can be concluded that $\dim \text{Ker}(\mathcal{K}_\nu(z)) = q\nu$. \square

We are going to prove that a basis of the null space $\text{Ker}(\mathcal{K}_\nu(z))$ is formed by the columns of the matrix

$$(3.1) \quad \mathcal{N} = \begin{pmatrix} \mathcal{U}(z) \\ J_{q\nu} \end{pmatrix},$$

where $J_{q\nu}$ is the $q\nu$ rotation matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

and where the $q \times q\nu$ matrix $\mathcal{U}(z)$ will be specified later on.

Observe that \mathcal{N} has full rank $q\nu$ since $J_{q\nu}$ is a nonsingular submatrix of \mathcal{N} . Therefore the columns of \mathcal{N} form a basis of $\text{Ker}(\mathcal{K}_\nu(z))$.

The matrix $\mathcal{U}(z)$ is represented in the following form:

$$(3.2) \quad \mathcal{U}(z) = \frac{1}{\nu!} (\mathcal{U}_0(z), \mathcal{U}_1(z), \mathcal{U}_2(z), \dots, \mathcal{U}_{\nu-1}(z)).$$

The submatrices constituting (3.2) shall be specified in the next sections.

3.1. A representation of $\mathcal{U}_0(z)$. In Lemma 3.2 we prove that a column of the matrix $\mathcal{U}_0(z)$ which has the form

$$(3.3) \quad \mathcal{U}_0(z) = (\xi \odot u_q(z)) \otimes u_q^\top(z),$$

and where the vector ξ is given by

$$(3.4) \quad \xi = (\xi_i), \xi_i = \left((-1)^i \binom{\nu}{i-1} \right)_{i=1, \dots, \nu+1},$$

belongs to the null space $\text{Ker}(\mathcal{K}_\nu(z))$. The Hadamard product \odot is defined by $A \odot B = (a_{ij}b_{ij})$ for $A = (a_{ij})$ and $B = (b_{ij})$ which are matrices of the same size.

Recall that the m th row of $\mathcal{K}_\nu(z)$ is given by

$$(3.5) \quad \frac{d^\nu}{dz^\nu} (z^{q-2+m}, z^{q-3+m}, \dots, z^{m-1}) = \left(z^{q+m-i-\nu-1} \prod_{j=1}^{\nu} (n-i-j) \right)_{i=1, \dots, q}.$$

We have the following lemma.

LEMMA 3.2. *The $q(\nu + 1)$ column vector composed of an arbitrary column of $\mathcal{U}_0(z)$ and the corresponding standard basis vector in $\mathbb{R}^{q\nu}$ belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$.*

Proof. The k th column of $\mathcal{U}_0(z)$ has elements

$$(3.6) \quad \frac{1}{\nu!} \left((-1)^i z^{k+i-2} \binom{\nu}{i-1} \right), \quad i = 1, \dots, q.$$

The scalar product of (3.5) and (3.6) provides a monomial in $z^{n-\nu+k-3}$, where $n = q + m$, whose coefficient is given by

$$(3.7) \quad \frac{1}{\nu!} \sum_{i=0}^{\nu} (-1)^{i+1} \binom{\nu}{i} (n-2-i)(n-3-i) \cdots (n-\nu-1-i) \\ = \frac{1}{\nu!} \left\{ \frac{d^\nu}{dx^\nu} \sum_{i=0}^{\nu} (-1)^{i+1} \binom{\nu}{i} x^{n-2-i} \right\}_{x=1} = -\frac{1}{\nu!} \left\{ \frac{d^\nu}{dx^\nu} (x^{n-2-\nu}(x-1)^\nu) \right\}_{x=1}.$$

The application of the Leibnitz rule to ν -fold differentiation of a product of two functions yields the value $-\frac{1}{\nu!} \{x^{n-2-\nu}\nu!\}_{x=1} = -1$. Consequently, the scalar product of (3.5) and (3.6) is $-z^{n-\nu+k-3}$. This should be added to the product of the appropriate z -variable in the coefficient matrix $\mathcal{K}_\nu(z)$ by the nonzero element of the standard basis vector in the rotation matrix $J_{q\nu}$ which is $z^{n-\nu+k-3}$, so the sum is null. This completes the proof. \square

3.1.1. Summary of the construction of $\mathcal{U}_0(z)$. *Step 1.* Introduce the vector ξ according to (3.4).

Step 2. Define the columns of $\mathcal{U}_0(\sigma)$ according to (3.3).

3.2. A representation of $\mathcal{U}_j(z)$ when $j = 1, 2, \dots, \nu - 1$. We shall now describe the form of the matrices $\mathcal{U}_1(z), \mathcal{U}_2(z), \dots, \mathcal{U}_{\nu-1}(z)$ that consist of the following structural representation:

$$\mathcal{U}_j(\sigma) = \left(\mathcal{U}_j^{(1)}(z) \mathcal{U}_j^{(2)}(z) \right),$$

for $j = 1, 2, \dots, \nu - 1$.

3.2.1. A representation of $\mathcal{U}_j^{(1)}(z)$. In this section the matrix $\mathcal{U}_j^{(1)}(z)$ is displayed. It is Hankel type with the following configuration:

$$(3.8) \quad \mathcal{U}_j^{(1)}(z) = \left(\delta_j^1(z) \delta_j^2(z) \cdots \delta_j^{j+1}(z) \right),$$

where the $(\nu + 1)$ basis column vector $\delta_j^{\ell+1}(z)$ has components

$$(3.9) \quad [\delta_j^{\ell+1}(z)]_i = \begin{cases} 0 & \text{for } i \leq j - \ell \text{ or } \nu + 1 - i \leq \ell \\ -j!(-z)^{i+\ell-j-1} \binom{\nu-j}{i+\ell-j-1} & \text{otherwise} \end{cases}$$

for $\ell = 0, 1, \dots, j$. The following lemma is proved.

LEMMA 3.3. *The $q(\nu + 1)$ column vector composed of any arbitrary column of $\mathcal{U}_j^{(1)}(z)$ and the corresponding standard basis vector in $\mathbb{R}^{q\nu}$ belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$.*

Proof. Set $j = p$ and $\ell = g$ in (3.9). As can be seen from (3.5), the appropriate nonzero elements of the scalar product of (3.9) with (3.5) provide a monomial in $z^{f-p-2-\nu}$, where $f = q + m + g$. Its coefficient is given by

$$(3.10) \quad \begin{aligned} & \frac{p!}{\nu!} \sum_{i=0}^{\nu-p} (-1)^{i+1} \binom{\nu-p}{i} (f-p-2-i)(f-p-3-i) \cdots (f-p-1-i-\nu) \\ &= \frac{p!}{\nu!} \left\{ \frac{d^\nu}{dx^\nu} \sum_{i=0}^{\nu-p} (-1)^{i+1} \binom{\nu-p}{i} x^{f-p-2-i} \right\}_{x=1} \\ &= -\frac{p!}{\nu!} \left\{ \frac{d^\nu}{dx^\nu} \left[x^{f-2-\nu} \sum_{i=0}^{\nu-p} (-1)^i \binom{\nu-p}{i} x^{\nu-p-i} \right] \right\}_{x=1} \\ &= -\frac{p!}{\nu!} \left\{ \frac{d^\nu}{dx^\nu} x^{f-2-\nu} (x-1)^{\nu-p} \right\}_{x=1} \\ &= -\frac{p!}{\nu!} \left\{ 0 + 0 + \cdots + \binom{\nu}{\nu-p} \frac{d^p}{dx^p} x^{f-2-\nu} \frac{d^{\nu-p}}{dx^{\nu-p}} (x-1)^{\nu-p} + 0 + \cdots + 0 \right\}_{x=1} \\ &= -(f-2-\nu)(f-3-\nu) \cdots (f-p-1-\nu). \end{aligned}$$

The scalar product is then given by $-(f-2-\nu)(f-3-\nu) \cdots (f-p-1-\nu)z^{f-p-2-\nu}$. The appropriate element of the m th row of the coefficient matrix $\mathcal{K}_\nu(z)$ that is multiplied by the nonzero element of the corresponding standard basis vector in the rotation matrix $J_{q\nu}$ is $z^{q-w+m-1}$, where $w = \nu - g + 1$, and the appropriate derivative is p . We therefore have

$$(3.11) \quad (d^p/dz^p) z^{f-\nu-2} = (f-2-\nu)(f-3-\nu) \cdots (f-p-1-\nu) z^{f-p-2-\nu}.$$

Adding (3.10) to (3.11) confirms that the $q(\nu + 1)$ column vector composed of $\delta_j^{\ell+1}(z)$ given in (3.9) and the corresponding standard basis vector in the rotation matrix $J_{q\nu}$ belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$. This completes the proof. \square

3.2.2. Summary of construction of the matrix $\mathcal{U}_j^{(1)}(z)$. *Step 1.* Define vector $\delta_j^{\ell+1}(z)$ according to (3.9) for $\ell = 0, 1, \dots, j$.

Step 2. Derive the columns of matrix $\mathcal{U}_j^{(1)}(z)$ according to (3.8).

3.2.3. A representation of $\mathcal{U}_j^{(2)}(z)$. For $j = 1, 2, 3, \dots, \nu - 1$, the submatrix $\mathcal{U}_j^{(2)}(z)$ admits the structure

$$(3.12) \quad \mathcal{U}_j^{(2)}(z) = \left(\kappa_j^1(z) \kappa_j^2(z) \cdots \kappa_j^{\nu-j}(z) \right).$$

To specify the basis vectors $\kappa_j^1(z) \kappa_j^2(z) \cdots \kappa_j^{\nu-j}(z)$, we first compute recursively for $j = 1$ and $k = 2, 3, \dots, \nu - j$ the appropriate column vectors according to

$$(3.13) \quad \kappa_1^k = \kappa_1^{k-1} + \xi,$$

where ξ is given in (3.4). A solution to recursion (3.13) in terms of the initial vector κ_j^1 whose form shall be introduced below is

$$(3.14) \quad \kappa_1^k = \kappa_1^1 + (k - 1)\xi.$$

We can now compute recursively for $j = 2, 3, \dots, \nu - 1$, according to

$$(3.15) \quad \kappa_j^k = \kappa_j^{k-1} + j\kappa_{j-1}^k.$$

In the next proposition, an explicit solution to recursion equations (3.15) and (3.13) shall be displayed for $j = 1, 2, 3, \dots, \nu - 1$.

PROPOSITION 3.4. *An explicit solution to the recursion equations (3.15) and (3.13), expressed in terms of the initial vectors $\kappa_j^1, \kappa_{j-1}^1, \dots, \kappa_2^1, \kappa_1^1$ and the known vector ξ , is given by*

$$(3.16) \quad \kappa_j^k = \sum_{i=0}^{j-1} i! \binom{j}{i} \binom{k-2+i}{k-2} \kappa_{j-i}^1 + j! \binom{k+j-2}{k-2} \xi.$$

Proof. The proof consists of using the recursion equations (3.15) and (3.14). Take $j = 2$, a combination of (3.15) and (3.14) yields for $k = 2, 3, 4, \dots$

$$(3.17) \quad \begin{aligned} \kappa_2^2 &= \kappa_2^1 + 2\kappa_1^1 + 2\xi \\ \kappa_2^3 &= \kappa_2^1 + 4\kappa_1^1 + 6\xi \\ \kappa_2^4 &= \kappa_2^1 + 6\kappa_1^1 + 12\xi \\ &\vdots \\ \kappa_2^k &= \kappa_2^1 + 2(k-1)\kappa_1^1 + k(k-1)\xi. \end{aligned}$$

Similarly when $j = 3, 4$, the recursion exercise yields for the k th column

$$(3.18) \quad \kappa_3^k = \kappa_3^1 + 3(k-1)\kappa_2^1 + 3k(k-1)\kappa_1^1 + k(k^2-1)\xi,$$

$$(3.19) \quad \kappa_4^k = \kappa_4^1 + 4(k-1)\kappa_3^1 + 6k(k-1)\kappa_2^1 + 4k(k^2-1)\kappa_1^1 + k(k^2-1)(k+2)\xi.$$

From (3.17), (3.18), and (3.19) can be concluded that for all values of j , the solution is then given by (3.16), where the case $j = 1$ is also included. When $j = 1$, (3.16) becomes (3.14). \square

The columns κ_j^k for $k = 1, 2, \dots, \nu - j$ and $j = 1, 2, 3, \dots, \nu - 1$ are essential for displaying the corresponding columns of the submatrix $\mathcal{U}_j^{(2)}(z)$ set forth in (3.12) and to obtain

$$(3.20) \quad \kappa_j^k(z) = \kappa_j^k \odot z^k u_{\nu+1}(z).$$

In order to start the recursions, the $(\nu + 1)$ initial column vector κ_j^1 shall be introduced. For $j = 1, 2, \dots, \nu - 1$, the components of the vector κ_j^1 are given by

$$(3.21) \quad \begin{cases} [\kappa_j^1]_1 = (j + 1)!, & [\kappa_j^1]_2 = ((j + 1)!/2) (2\nu - j), \\ [\kappa_j^1]_i = j! \binom{\nu+1}{i} - s_i, & i = 3, \dots, \nu - j, \\ [\kappa_j^1]_i = j! \binom{\nu+1}{i}, & i = \nu - j + 1, \dots, \nu + 1, \end{cases}$$

where the terms s_ℓ , encountered if $\nu \geq 5$, are defined by

$$(3.22) \quad s_\ell = \begin{cases} j! \binom{\nu - j}{\ell}, & \ell = 3, 4, \dots, \nu - j \quad \text{for } j = 1, 2, \dots, \nu - 3, \\ 0, & \alpha > \nu - 3 \quad \text{for } \kappa_\alpha^1. \end{cases}$$

From (3.22) it can be concluded that when $j = \nu - 2$ and $j = \nu - 1$, $s_\ell = 0$ for the corresponding initial vectors $\kappa_{\nu-2}^1$ and $\kappa_{\nu-1}^1$ of the submatrices $\mathcal{U}_{j=\nu-2}^{(2)}(z)$ and $\mathcal{U}_{j=\nu-1}^{(2)}(z)$, respectively. For the case $q \leq 5$, the initial vectors κ_j^1 do not contain the terms s_ℓ so the elements of κ_j^1 to be considered are the two first elements and then pursuing the reading upwards, starting from the last term at the bottom.

The first part of the right-hand side of (3.16) is displayed in order to better understand the development of the proof of Lemma 3.6 by setting $\vartheta = \sum_{i=0}^{j-1} \binom{k-2+i}{k-2}$,

$$(3.23) \quad j! \begin{pmatrix} \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} (j - i + 1) \\ \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} ((j - i + 1)/2) (2\nu - j + i) \\ \vartheta \binom{\nu+1}{3} - \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} \binom{\nu-j+i}{3} \\ \vartheta \binom{\nu+1}{4} - \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} \binom{\nu-j+i}{4} \\ \vdots \\ \vartheta \binom{\nu+1}{\nu-j} - \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} \binom{\nu-j+i}{\nu-j} \\ \vartheta \binom{\nu+1}{\nu-j+1} \\ \vdots \\ \vartheta \binom{\nu+1}{\nu+1} \end{pmatrix}.$$

The sign pattern of the elements in each column of $\mathcal{U}_j^{(2)}(z)$ is given by $(-1)^\ell$ with $\ell = 1, 2, \dots, \nu + 1$.

First some results which shall be used in the proof of Lemma 3.6 are set forth.

PROPOSITION 3.5. *The following equalities hold true:*

$$(3.24) \quad \sum_{i=0}^{j-1} \binom{k-2+i}{k-2} = \binom{k-2+j}{k-1},$$

$$(3.25) \quad \sum_{i=1}^{j-1} \binom{k-2+i}{k-2} i = \frac{j(j-1)}{k} \binom{k-2+j}{k-2},$$

$$(3.26) \quad \sum_{i=1}^{j-1} \binom{k-2+i}{k-2} i^2 = \frac{j(j-1)(1+(j-1)k)}{k(k+1)} \binom{k-2+j}{k-2}.$$

Proof. We shall prove the equalities (3.24), (3.25), and (3.26) by applying mathematical induction.

It is straightforward to see that the left-hand side of equality (3.24) yields the right-hand side when $j = 1$.

Assume for $j = p$ that

$$\sum_{i=0}^{p-1} \binom{k-2+i}{k-2} = \binom{k-2+p}{k-1}.$$

This implies that for $j = p + 1$,

$$\begin{aligned} \sum_{i=0}^p \binom{k-2+i}{k-2} &= \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} + \binom{k-2+p}{k-2} \\ &= \binom{k-2+p}{k-1} + \binom{k-2+p}{k-2} \\ &= \binom{k-1+p}{k-1}. \end{aligned}$$

The last equality is based on the elementary identity for integers n and j :

$$(3.27) \quad \binom{n}{j} + \binom{n}{j+1} = \binom{n+1}{j+1}.$$

The proof of (3.24) is completed. When $j = 2$, the left-hand side of equality (3.25) is $\binom{k-1}{k-2}$ and equals the right-hand side which becomes $\frac{2}{k} \binom{k}{k-2}$. Assume that (3.25) is true for $j = p$. Then

$$\sum_{i=0}^{p-1} \binom{k-2+i}{k-2} i = \frac{p(p-1)}{k} \binom{k-2+p}{k-2},$$

for $j = p + 1$,

$$\begin{aligned} \sum_{i=0}^p \binom{k-2+i}{k-2} i &= \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} i + p \binom{k-2+p}{k-2} \\ &= \frac{p(p-1)}{k} \binom{k-2+p}{k-2} + p \binom{k-2+p}{k-2} \\ &= \frac{(k+p-1)!}{(k-2)!(p-1)!k} = \frac{p(p+1)}{k} \binom{k-1+p}{k-2}. \end{aligned}$$

This confirms (3.25). Finally we prove (3.26). When $j = 2$, the left-hand side of equality (3.26) is $\binom{k-1}{k-2}$ and equals the right-hand side which becomes $\frac{2}{k} \binom{k}{k-2}$. Assume for $j = p$ that

$$\sum_{i=0}^{p-1} \binom{k-2+i}{k-2} i^2 = \frac{p(p-1)(1+(p-1)k)}{k(k+1)} \binom{k-2+p}{k-2}.$$

This implies that for $j = p + 1$,

$$\begin{aligned} \sum_{i=0}^p \binom{k-2+i}{k-2} i^2 &= \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} i^2 + p^2 \binom{k-2+p}{k-2} \\ &= \frac{p(p-1)(1+(p-1)k)}{k(k+1)} \binom{k-2+p}{k-2} + p^2 \binom{k-2+p}{k-2} \\ &= \frac{(1+pk)(k+p-1)(k+p-2)!}{(k-2)!(p-1)!k(k+1)} \\ &= \frac{p(p+1)(1+pk)}{k(k+1)} \binom{k-1+p}{k-2}. \end{aligned}$$

This completes the proof. \square

We shall now continue with the following lemma.

LEMMA 3.6. *The $q(\nu+1)$ column vector composed of $\kappa_j^k(z)$, described in (3.16) and (3.20), and the corresponding standard basis vector in $\mathbb{R}^{q\nu}$ belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$.*

Proof. The scalar product of (3.20) and (3.5) provides a monomial in $z^{n+k-\nu-2}$. The z -variables will be reintroduced at a later stage for typographical brevity. The scalar product is first computed for the last $\nu-1$ entries of the first column of (3.23), then sets $j = p$ in (3.16) and takes (3.24) into consideration yielding

$$\begin{aligned} (3.28) \quad & \frac{p!}{\nu!} \binom{k-2+p}{k-1} \sum_{i=0}^{\nu-2} (-1)^{i+1} \binom{\nu+1}{3+i} (n-4-i)(n-5-i) \cdots (n-\nu-3-i) \\ &= \frac{p!}{\nu!} \binom{k-2+p}{k-1} \left\{ \frac{d^\nu}{dx^\nu} \sum_{i=0}^{\nu-2} (-1)^{i+1} \binom{\nu+1}{3+i} x^{n-4-i} \right\}_{x=1} \\ &= -\frac{p!}{\nu!} \binom{k-2+p}{k-1} \left\{ \frac{d^\nu}{dx^\nu} \left[x^{n-2-\nu} \sum_{i=0}^{\nu-2} (-1)^i \binom{\nu+1}{3+i} x^{\nu-2-i} \right] \right\}_{x=1}. \end{aligned}$$

Then set $j = 3 + i$

$$(3.29) \quad = \frac{p!}{\nu!} \binom{k-2+p}{k-1} \left\{ \frac{d^\nu}{dx^\nu} \left[x^{n-2-\nu} \left(\sum_{j=3}^{\nu+1} (-1)^j \binom{\nu+1}{j} x^{\nu+1-j} \right) \right] \right\}_{x=1}.$$

The following holds:

$$\sum_{j=3}^{\nu+1} (-1)^j \binom{\nu+1}{j} x^{\nu+1-j} = \sum_{j=0}^{\nu+1} (-1)^j \binom{\nu+1}{j} x^{\nu+1-j} - \sum_{j=0}^2 (-1)^j \binom{\nu+1}{j} x^{\nu+1-j}.$$

Equation (3.29) becomes

$$\begin{aligned} (3.30) \quad & \frac{p!}{\nu!} \binom{k-2+p}{k-1} \frac{d^\nu}{dx^\nu} \left\{ x^{n-2-\nu} \left((x-1)^{\nu+1} - x^{\nu+1} + (\nu+1)x^\nu - \frac{\nu(\nu+1)}{2} x^{\nu-1} \right) \right\}_{x=1} \\ &= \frac{p!}{\nu!} \binom{k-2+p}{k-1} \left\{ \begin{array}{l} -(n-1)(n-2) \cdots (n-\nu) \\ +(\nu+1)(n-2)(n-3) \cdots (n-\nu-1) \\ -\frac{\nu(\nu+1)}{2} (n-3)(n-4) \cdots (n-\nu-2) \end{array} \right\}. \end{aligned}$$

We shall now focus on the part of (3.23) that contains s_ℓ . For that purpose an explicit representation is displayed,

$$p! \left\{ \binom{k-2}{k-2} \begin{pmatrix} \binom{\nu-p}{3} \\ \binom{\nu-p}{4} \\ \vdots \\ \binom{\nu-p}{\nu-p} \end{pmatrix} + \binom{k-1}{k-2} \begin{pmatrix} \binom{\nu-p+1}{3} \\ \binom{\nu-p+1}{4} \\ \vdots \\ \binom{\nu-p+1}{\nu-p+1} \end{pmatrix} \right. \\ \left. + \dots + \binom{k+p-3}{k-2} \begin{pmatrix} \binom{\nu-1}{3} \\ \binom{\nu-1}{4} \\ \vdots \\ \binom{\nu-1}{\nu-1} \end{pmatrix} \right\}.$$

The scalar product of (3.5) with each of the columns above can be expressed as follows, consider the index $\ell = 0, 1, 2, \dots, p-1$, to obtain

$$\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \sum_{i=0}^{\nu-p+\ell-3} (-1)^i \binom{\nu-p+\ell}{3+i} (n-4-i)(n-5-i) \dots (n-\nu-3-i) \\ = \frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \frac{d^\nu}{dx^\nu} \sum_{i=0}^{\nu-p+\ell-3} (-1)^i \binom{\nu-p+\ell}{3+i} x^{n-4-i} \right\}_{x=1}.$$

Set $j = 3 + i$; it then yields

$$\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \frac{d^\nu}{dx^\nu} \sum_{j=3}^{\nu-p+\ell} (-1)^{j-3} \binom{\nu-p+\ell}{j} x^{n-j-1} \right\}_{x=1} \\ = -\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \frac{d^\nu}{dx^\nu} x^{n-\nu+p-\ell-1} \left(\sum_{j=3}^{\nu-p+\ell} (-1)^j \binom{\nu-p+\ell}{j} x^{\nu-p+\ell-j} \right) \right\}_{x=1} \\ = -\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \frac{d^\nu}{dx^\nu} x^{n-\nu+p-\ell-1} \left(\sum_{j=0}^{\nu-p+\ell} (-1)^j \binom{\nu-p+\ell}{j} x^{\nu-p+\ell-j} \right. \right. \\ \left. \left. - \sum_{j=0}^2 (-1)^j \binom{\nu-p+\ell}{j} x^{\nu-p+\ell-j} \right) \right\}_{x=1} \\ = -\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \frac{d^\nu}{dx^\nu} \left(x^{n-\nu+p-\ell-1} (x-1)^{\nu-p+\ell} - x^{n-1} + (\nu-p+\ell)x^{n-2} \right. \right. \\ \left. \left. - \frac{(\nu-p+\ell-1)(\nu-p+\ell)}{2} x^{n-3} \right) \right\}_{x=1}.$$

The first term can be expanded according to Leibnitz rule for ν -fold differentiation of a product of two functions,

$$\left\{ 0 + 0 + \dots + \binom{\nu}{\nu-p+\ell} \frac{d^{p-\ell}}{dx^{p-\ell}} x^{n-\nu+p-\ell-1} \frac{d^{\nu-p+\ell}}{dx^{\nu-p+\ell}} (x-1)^{\nu-p+\ell} + 0 + \dots + 0 \right\}_{x=1}.$$

The result is then

(3.31)

$$\frac{p!}{\nu!} \binom{k+\ell-2}{k-2} \left\{ \begin{array}{l} (n-1)(n-2)\cdots(n-\nu) \\ -(\nu-p+\ell)(n-2)(n-3)\cdots(n-\nu-1) \\ +\frac{(\nu-p+\ell-1)(\nu-p+\ell)}{2}(n-3)(n-4)\cdots(n-\nu-2) \end{array} \right\}$$

(3.32) $-\frac{p!}{(p-\ell)!} \binom{k+\ell-2}{k-2} (n-\nu+p-\ell-1)(n-\nu+p-\ell-2)\cdots(n-\nu).$

Since $q = \nu + 1$, the terms $(n - \nu)$, $(n - \nu - 1)$, and $(n - \nu - 2)$ in (3.31) are positive and $(n - \nu - 2) \geq 0$.

The terms involving $\frac{p!}{\nu!}(n-1)(n-2)\cdots(n-\nu)$ appearing in (3.30) and (3.31), the latter for $\ell = 0, 1, 2, \dots, p-1$, when added yield

$$\begin{aligned} & -\binom{k-2+p}{k-1} + \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} \\ & = -\binom{k-2+p}{k-1} + \binom{k-2+p}{k-1} = 0. \end{aligned}$$

The last equality is established by virtue of (3.24). A more explicit expression for the first term in (3.23) is now considered, with the corresponding minus sign. By virtue of (3.24) and (3.25) it can be seen that

$$\begin{aligned} & -\frac{p!}{\nu!} \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} (p-i+1) = -\frac{p!}{\nu!} \left\{ (p+1) \binom{k-2+p}{k-1} \right. \\ & \left. - \frac{p(p-1)}{k} \binom{k-2+p}{k-2} \right\}. \end{aligned}$$

In the scalar product, the first term of (3.23) is multiplied by $(n-2)(n-3)\cdots(n-\nu-1)$. Summing up all of the terms involving this product, which also appears in (3.30) and (3.31), yields

$$\begin{aligned} & \left\{ -(p+1) \binom{k-2+p}{k-1} + \frac{p(p-1)}{k} \binom{k-2+p}{k-2} + \binom{k+p-2}{k-1} (\nu+1) \right. \\ & \left. - \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} (\nu-p+i) \right\} \\ & = -(p+1) \binom{k-2+p}{k-1} + \frac{p(p-1)}{k} \binom{k-2+p}{k-2} + \binom{k+p-2}{k-1} (\nu+1) \\ & \quad - (\nu-p) \binom{k+p-2}{k-1} - \frac{p(p-1)}{k} \binom{k-2+p}{k-2} = 0. \end{aligned}$$

The last equality is established by virtue of (3.24) and (3.25).

We focus now on an explicit form of the second term of (3.23). By virtue of (3.24), (3.25), and (3.26) we obtain

$$(3.33) \quad \begin{aligned} & \frac{p!}{2} \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} (p-i+1) (2\nu-p+i) \\ &= \frac{p!}{2} \left\{ \begin{aligned} & (p+1)(2\nu-p) \binom{k-2+p}{k-1} + (2p-2\nu+1) \frac{p(p-1)}{k} \binom{k-2+p}{k-2} \\ & - \frac{p(p-1)(1+(p-1)k)}{k(k+1)} \binom{k-2+p}{k-2} \end{aligned} \right\}. \end{aligned}$$

In the scalar product, the term (3.33) is multiplied by $(n-3)(n-4) \cdots (n-\nu-2)$. Summing up all of the terms involving this product, without $(p!/\nu!)$, which also appears in (3.30) and (3.31), yields next to (3.33),

$$(3.34) \quad \left\{ - \binom{k-2+p}{k-1} \frac{\nu(\nu+1)}{2} + \frac{1}{2} \sum_{i=0}^{p-1} \binom{k-2+i}{k-2} (\nu-p+i-1)(\nu-p+i) \right\}.$$

We now collect all of the terms involved to obtain

$$\begin{aligned} & \frac{(p+1)(2\nu-p)}{2} \binom{k-2+p}{k-1} + (2p-2\nu+1) \frac{p(p-1)}{2k} \binom{k-2+p}{k-2} \\ & - \frac{p(p-1)(1+(p-1)k)}{2k(k+1)} \binom{k-2+p}{k-2} - \frac{\nu(\nu+1)}{2} \binom{k-2+p}{k-1} \\ & + \frac{(\nu-p-1)(\nu-p)}{2} \binom{k-2+p}{k-1} + (2\nu-2p-1) \frac{p(p-1)}{2k} \binom{k-2+p}{k-2} \\ & + \frac{p(p-1)(1+(p-1)k)}{2k(k+1)} \binom{k-2+p}{k-2} = 0; \end{aligned}$$

as in the other cases, this result is obtained by using (3.24), (3.25), and (3.26).

Consequently, the remaining terms are now collected—it concerns the term involving ξ in (3.16), the appropriate scalar product is by virtue of (3.7) $-p! \binom{k-2+p}{k-2}$, and the terms derived from (3.32), for $\ell = 0, 1, 2, \dots, p-1$, to obtain

$$- \sum_{i=0}^{p-1} \frac{p!}{(p-i)!} \binom{k+i-2}{k-2} (n-\nu+p-i-1)(n-\nu+p-i-2) \cdots (n-\nu).$$

The remaining terms can be summarized according to

$$(3.35) \quad -p! \sum_{i=0}^{p-1} \binom{k+i-2}{k-2} \binom{n-\nu+p-i-1}{n-\nu-1} - p! \binom{k-2+p}{k-2}.$$

Concerning (3.35), the following property will be proved:

$$(3.36) \quad \sum_{i=0}^p \binom{k+i-2}{k-2} \binom{n-\nu+p-i-1}{n-\nu-1} = \binom{n-\nu+k+p-2}{n-\nu+k-2}.$$

For proving (3.36), we consider, for all nonnegative integers l, p and $n \geq p$,

$$(3.37) \quad \binom{n+l+1}{p} = \sum_{i=0}^p \binom{l+i}{i} \binom{n-i}{p-i}.$$

The proof is based on (3.27), which we rewrite as

$$(3.38) \quad \binom{m}{j} + \binom{m}{j+1} = \binom{m+1}{j+1}.$$

First we prove the formula for $n = p$. In this case the identity (3.37) reduces to

$$\binom{l+p+1}{p} = \sum_{i=0}^p \binom{l+i}{i}.$$

We use induction w.r.t. the variable p . The case $p = 0$ is a triviality. Assume that (3.37) holds true for a certain value of p . Then

$$\sum_{i=0}^{p+1} \binom{l+i}{i} = \sum_{i=0}^p \binom{l+i}{i} + \binom{l+p+1}{p+1}.$$

The first term on the right-hand side is equal to $\binom{l+p+1}{p}$ by hypothesis. Then adding the second term gives $\binom{l+p+2}{p+1}$ by virtue of (3.38).

The rest of the proof is by induction w.r.t. the variable n , $n \geq p$, since we have settled the case $n = p$. Consider the right-hand side of (3.37) with $n+1$ instead of n and compute using the induction hypothesis two times and repeatedly the identity (3.38),

$$\begin{aligned} \sum_{i=0}^p \binom{l+i}{i} \binom{n+1-i}{p-i} &= \sum_{i=0}^p \binom{l+i}{i} \binom{n-i}{p-i} + \sum_{i=0}^{p-1} \binom{l+i}{i} \binom{n-i}{p-1-i} \\ &= \binom{n+l+1}{p} + \binom{n+l+1}{p-1} \\ &= \binom{n+l+2}{p}. \end{aligned}$$

From (3.35) and (3.36) it can be concluded that the scalar product is equal to

$$(3.39) \quad -(n-\nu+k+p-2)(n-\nu+k+p-3)\cdots(n-\nu+k-1)z^{n+k-\nu-2}.$$

The corresponding nonzero element of the standard basis vector in the rotation matrix $J_{q\nu}$ is multiplied by z^{n-w-1} for $w = \nu - p + 1 - k$, and the appropriate derivative is

$$(3.40) \quad (d^p/dz^p) z^{n-\nu+k+p-2} = (n-\nu+k+p-2)(n-\nu+k+p-3)\cdots(n-\nu+k-1)z^{n+k-\nu-2}.$$

Adding (3.39) to (3.40) confirms that the $q(\nu+1)$ column vector composed of vector $\kappa_j^k(z)$, described in (3.16) and (3.20), and the corresponding standard basis vector in the rotation matrix $J_{q\nu}$ belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$ when $s_\ell \neq 0$ in (3.21).

We now proceed with the proof when $s_\ell = 0$ in (3.21), the cases $j = \nu - 1$ and $j = \nu - 2$ are therefore considered. The initial vector (3.21) for the former case is

$$(3.41) \quad \kappa_{\nu-1}^1 = (\nu - 1)! \left(\nu, \binom{\nu + 1}{2}, \binom{\nu + 1}{3}, \dots, \binom{\nu + 1}{\nu + 1} \right)^\top.$$

The scalar product involving the first $\nu + 1$ elements is displayed, and the last ν entries of (3.41) are first considered to obtain

$$\frac{(\nu - 1)!}{\nu!} \sum_{i=0}^{\nu-1} (-1)^i \binom{\nu + 1}{2 + i} (n - 3 - i)(n - 4 - i) \cdots (n - \nu - 2 - i).$$

The same approach as we used to derive (3.29) yields

$$\frac{(\nu - 1)!}{\nu!} \left\{ \frac{-(n - 1)(n - 2) \cdots (n - \nu)}{+(\nu + 1)(n - 2)(n - 3) \cdots (n - \nu - 1)} \right\}.$$

Adding the scalar product involving the first element of (3.41) and (3.5) yields

$$\begin{aligned} & \left\{ \begin{array}{l} \frac{-(n - 2)(n - 3) \cdots (n - \nu - 1)}{+(\nu - 1)!(\nu + 1)(n - 2)(n - 3) \cdots (n - \nu - 1)} \\ - \frac{(\nu - 1)!}{\nu!} (n - 1)(n - 2) \cdots (n - \nu) \end{array} \right\} \\ & = -(n - 2)(n - 3) \cdots (n - \nu). \end{aligned}$$

This result is obtained through straightforward calculation. It can now be concluded that the scalar product is

$$(3.42) \quad -(n - 2)(n - 3) \cdots (n - \nu)z^{n-\nu-1}.$$

Note for the case under study, $k = 1$ (it concerns the initial vector $\kappa_{\nu-1}^1$). The corresponding nonzero element of the standard basis vector in the rotation matrix $J_{q\nu}$ is multiplied by z^{n-w-1} for $w = 2 - k$, $w = \nu - p + 1 - k$ in the general case. The appropriate derivative is then

$$(3.43) \quad (d^{\nu-1}/dz^{\nu-1}) z^{n-2} = (n - 2)(n - 3) \cdots (n - \nu)z^{n-\nu-1}.$$

Adding (3.42) to (3.43) confirms that when in (3.21) $s_\ell = 0$ and $j = \nu - 1$, the $q(\nu + 1)$ column vector, composed of vector $\kappa_{\nu-1}^1$, given in (3.41), and the corresponding standard basis vector in the rotation matrix $J_{q\nu}$, belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$.

The case $j = \nu - 2$ is considered next. The initial vector (3.21) is then

$$(3.44) \quad \kappa_{\nu-2}^1 = (\nu - 2)! \left((\nu - 1), ((\nu - 1)/2)(\nu + 2), \binom{\nu + 1}{3}, \dots, \binom{\nu + 1}{\nu + 1} \right)^\top.$$

The scalar product involving the first $\nu + 1$ elements is displayed, and the last $(\nu - 1)$ entries of (3.44) are first considered to obtain

$$\frac{(\nu - 2)!}{\nu!} \sum_{i=0}^{\nu-2} (-1)^{i+1} \binom{\nu + 1}{3 + i} (n - 4 - i)(n - 5 - i) \cdots (n - \nu - 3 - i).$$

According to (3.28) we have

$$\frac{(\nu-2)!}{\nu!} \left\{ \begin{array}{l} -(n-1)(n-2)\cdots(n-\nu) \\ +(\nu+1)(n-2)(n-3)\cdots(n-\nu-1) \\ -\frac{\nu(\nu+1)}{2}(n-3)(n-4)\cdots(n-\nu-2) \end{array} \right\}.$$

The scalar product involving the first and second elements of (3.44) and (3.5) are

$$-\frac{(\nu-1)!}{\nu!}(n-2)(n-3)\cdots(n-\nu-1)$$

and $\frac{((\nu-1)!/2)(\nu+2)}{\nu!}(n-3)(n-4)\cdots(n-\nu-2),$

respectively. Summing all of the terms yields

$$\left\{ \begin{array}{l} \frac{(\nu-2)!(\nu+1)-(\nu-1)!}{\nu!}(n-2)(n-3)\cdots(n-\nu-1) \\ +\frac{(\nu-1)!(\nu+2)-(\nu-2)!\nu(\nu+1)}{2(\nu!)}(n-3)(n-4)\cdots(n-\nu-2) \\ -\frac{(\nu-2)!}{\nu!}(n-1)(n-2)\cdots(n-\nu) \end{array} \right\}$$

$$= -(n-3)(n-4)\cdots(n-\nu).$$

This result is obtained through straightforward computation. It can now be concluded that the scalar product is

$$(3.45) \quad -(n-3)(n-4)\cdots(n-\nu)z^{n-\nu-1}.$$

Note for the case under study, $k = 1$ (it concerns the initial vector $\kappa_{\nu-2}^1$).

The corresponding nonzero element of the standard basis vector in the rotation matrix $J_{q\nu}$ is multiplied by z^{n-w-1} for $w = 3 - k$ and $w = \nu - p + 1 - k$ in the general case. The appropriate derivative is then

$$(3.46) \quad (d^{\nu-2}/dz^{\nu-2}) z^{n-3} = (n-3)(n-4)\cdots(n-\nu)z^{n-\nu-1}.$$

Adding (3.45) to (3.46) confirms that when in (3.21) $s_\ell = 0$ and $j = \nu - 2$, the $q(\nu + 1)$ column vector, composed of vector $\kappa_{\nu-2}^1$, given in (3.44), and the corresponding standard basis vector in the rotation matrix $J_{q\nu}$, belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$.

It can be concluded that the $q(\nu + 1)$ column vector, composed of vector $\kappa_j^k(z)$, described in (3.16) and (3.20), and the corresponding standard basis vector in the rotation matrix $J_{q\nu}$, belongs to the null space of the coefficient matrix $\mathcal{K}_\nu(z)$. The proof of Lemma 3.6 is now complete. \square

3.2.4. Summary of the construction of matrix $\mathcal{U}_j^{(2)}(z)$. *Step 1.* Define the initial vectors κ_j^1 given in (3.21) for the values of $j = 1, 2, 3, \dots, \nu - 1$.

Step 2. Expand (3.16) for the corresponding values of $j = 1, 2, 3, \dots, \nu - 1$.

Step 3. Compute the columns of $\mathcal{U}_j^{(2)}(z)$ according to (3.20) for the corresponding values of $j = 1, 2, 3, \dots, \nu - 1$.

In the next section an example will illustrate the results set forth in previous sections.

3.3. Example Ker $(\mathcal{K}_\nu(z))$ for the case $\nu + 1 = 6$. This case will be illustrated for $q = 6$ and $\nu = 5$. The first submatrix contained in the null space of $\mathcal{K}_\nu(z)$ is then

$$\mathcal{U}_0(z) = \begin{pmatrix} -1 & -z & -z^2 & -z^3 & -z^4 & -z^5 \\ 5z & 5z^2 & 5z^3 & 5z^4 & 5z^5 & 5z^6 \\ -10z^2 & -10z^3 & -10z^4 & -10z^5 & -10z^6 & -10z^7 \\ 10z^3 & 10z^4 & 10z^5 & 10z^6 & 10z^7 & 10z^8 \\ -5z^4 & -5z^5 & -5z^6 & -5z^7 & -5z^8 & -5z^9 \\ z^5 & z^6 & z^7 & z^8 & z^9 & z^{10} \end{pmatrix}.$$

This is followed by the second class of submatrices $\mathcal{U}_j(z)$ when $j = 1, 2, 3, 4$,

$$\mathcal{U}_{j=1}^{(1)}(z) = \begin{pmatrix} 0 & -1 \\ -1 & 4z \\ 4z & -6z^2 \\ -6z^2 & 4z^3 \\ 4z^3 & -z^4 \\ -z^4 & 0 \end{pmatrix}, \quad \mathcal{U}_{j=2}^{(1)}(z) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 6z \\ -2 & 6z & -6z^2 \\ 6z & -6z^2 & 2z^3 \\ -6z^2 & 2z^3 & 0 \\ 2z^3 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_{j=3}^{(1)}(z) = \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 0 & -6 & 12z \\ 0 & -6 & 12z & -6z^2 \\ -6 & 12z & -6z^2 & 0 \\ 12z & -6z^2 & 0 & 0 \\ -6z^2 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{U}_{j=4}^{(1)}(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & -24 \\ 0 & 0 & 0 & -24 & 24z \\ 0 & 0 & -24 & 24z & 0 \\ 0 & -24 & 24z & 0 & 0 \\ -24 & 24z & 0 & 0 & 0 \\ 24z & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is then followed by a class of submatrices $\mathcal{U}_j^{(2)}(z)$ when $j = 1, 2, 3, 4$,

$$\mathcal{U}_{j=1}^{(2)}(z) = \begin{pmatrix} -2z & -3z^2 & -4z^3 & -5z^4 \\ 9z^2 & 14z^3 & 19z^4 & 24z^5 \\ -16z^3 & -26z^4 & -36z^5 & -46z^6 \\ 14z^4 & 24z^5 & 34z^6 & 44z^7 \\ -6z^5 & -11z^6 & -16z^7 & -21z^8 \\ z^6 & 2z^7 & 3z^8 & 4z^9 \end{pmatrix},$$

$$\mathcal{U}_{j=2}^{(2)}(z) = \begin{pmatrix} -6z & -12z^2 & -20z^3 \\ 24z^2 & 52z^3 & 90z^4 \\ -38z^3 & -90z^4 & -162z^5 \\ 30z^4 & 78z^5 & 146z^6 \\ -12z^5 & -34z^6 & -66z^7 \\ 2z^6 & 6z^7 & 12z^8 \end{pmatrix},$$

$$\mathcal{U}_{j=3}^{(2)}(z) = \begin{pmatrix} -24z & -60z^2 \\ 84z^2 & 240z^3 \\ -120z^3 & -390z^4 \\ 90z^4 & 324z^5 \\ -36z^5 & -138z^6 \\ 6z^6 & 24z^7 \end{pmatrix}, \text{ and}$$

$$\mathcal{U}_{j=4}^{(2)}(z) = \begin{pmatrix} -120z \\ 360z^2 \\ -480z^3 \\ 360z^4 \\ -144z^5 \\ 24z^6 \end{pmatrix}.$$

Insertion of $\mathcal{U}_0(z)$ and the matrices

$$\begin{aligned} \mathcal{U}_1(z) &= \left(\mathcal{U}_{j=1}^{(1)}(z) \mathcal{U}_{j=1}^{(2)}(z) \right), \\ \mathcal{U}_2(z) &= \left(\mathcal{U}_{j=2}^{(1)}(z) \mathcal{U}_{j=2}^{(2)}(z) \right), \\ \mathcal{U}_3(z) &= \left(\mathcal{U}_{j=3}^{(1)}(z) \mathcal{U}_{j=3}^{(2)}(z) \right), \\ \mathcal{U}_4(z) &= \left(\mathcal{U}_{j=4}^{(1)}(z) \mathcal{U}_{j=4}^{(2)}(z) \right), \end{aligned}$$

in (3.2) yields the form

$$\mathcal{U}(z) = \frac{1}{5!} \left(\mathcal{U}_0(z), \mathcal{U}_1(z), \mathcal{U}_2(z), \mathcal{U}_3(z), \mathcal{U}_4(z) \right).$$

The columns that compose $\left(\mathcal{U}_{j=0}^{(z)} \right)$ span $\text{Ker}(\mathcal{K}_\nu(z))$ when $q = 6$ and $\nu = 5$.

In the next section the null space of the coefficient matrix $\mathcal{M}_\tau(\rho)$ is set forth.

4. A representation of $\text{Ker}(\mathcal{M}_\tau(\rho))$. In this section a representation of the subspace $\text{Ker}(\mathcal{M}_\tau(\rho))$ is displayed for the case $\tau + 1 = p$. The coefficient matrix $\mathcal{M}_\tau(z)$ is considered for $z = \rho$, and a motivation is formulated below. We shall first focus on the dimension of the null space $\text{Ker}(\mathcal{M}_\tau(\rho))$.

PROPOSITION 4.1. *The null space $\text{Ker}(\mathcal{M}_\tau(\rho))$ has dimension equal to $p\tau$ and the rank of the coefficient matrix $\mathcal{M}_\tau(\rho)$ is p , when $\tau + 1 = p$.*

Proof. By virtue of Corollary 2.2, a similar argument as in Proposition 3.1 holds for the coefficient matrix $\mathcal{M}_\tau(\rho)$; see also Lemma 2.4 in [5]. It can be concluded that the $p \times p(\tau + 1)$ coefficient matrix $\mathcal{M}_\tau(\rho)$ is surjective or has full row rank; its rank is then p . By virtue of the dimension rule, it can be concluded that $\dim \text{Ker}(\mathcal{K}_\nu(z)) = p\tau$. \square

We can essentially reduce the problem of computing the null space $\text{Ker}(\mathcal{M}_\tau(\rho))$ to the computation of the kernel of the matrix $\mathcal{K}_\tau(\rho)$. The vectors contained in

$$(4.1) \quad \mathcal{G} = \begin{pmatrix} \mathcal{Y}(\rho) \\ J_{p\tau} \end{pmatrix}$$

span the null space of $\mathcal{M}_\tau(\rho)$, where $J_{p\tau}$ is the $p\tau$ rotation matrix.

Observe that \mathcal{G} has full rank $p\tau$ since $J_{p\tau}$ is a nonsingular submatrix of \mathcal{G} . Therefore the columns of \mathcal{G} form a basis of $\text{Ker}(\mathcal{M}_\tau(\rho))$.

Write

$$(4.2) \quad \mathcal{Y}(\rho) = \frac{1}{\tau!} (\mathcal{Y}_0(\rho), \mathcal{Y}_1(\rho), \mathcal{Y}_2(\rho), \dots, \mathcal{Y}_{\tau-1}(\rho)),$$

where

$$\mathcal{Y}_0(\rho) = \mathcal{U}_0^\top(\rho)$$

and

$$\mathcal{Y}_j(\rho) = \begin{pmatrix} \mathcal{Y}_j^{(1)}(\rho) \\ \mathcal{Y}_j^{(2)}(\rho) \end{pmatrix} = \begin{pmatrix} (\mathcal{U}_j^{(1)}(\rho))^\top \\ (\mathcal{U}_j^{(2)}(\rho))^\top \end{pmatrix} = \mathcal{U}_j^\top(\rho) \quad \text{for } j = 1, 2, \dots, \tau - 1.$$

The matrices $\mathcal{U}_0(\rho)$, $\mathcal{U}_j^{(1)}(\rho)$, and $\mathcal{U}_j^{(2)}(\rho)$ are given in section 3.

In section 3.2 of [5], the vector $y \in \text{Ker}(\mathcal{M}_\tau(\rho))$ is computed according to

$$(4.3) \quad y = (I_{\tau+1} \otimes S(f))^{-1}x,$$

where $x \in \text{Ker}(\mathcal{K}_\tau(\rho))$ and the $p \times p$ symmetrizer $S(f)$ is associated with a polynomial $f(z)$ of degree p . Consider $f(z) = z^p + a_1z^{p-1} + a_2z^{p-2} + \dots + a_p$, then the $p \times p$ matrix $S(f)$ is

$$S(f) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 1 & 0 \\ a_{p-1} & & & a_1 & 1 \end{pmatrix}.$$

Formula (4.3) is derived from an equality which connects the matrices $\text{adj}(zI - C_p)$ and $u_p(z)u_p^*(z)$, where $u_p(z)$ and $u_p^*(z)$ are defined in (1.4). From [6], we take Proposition 3.1 which gives the identity

$$(4.4) \quad \text{adj}(zI - C_p) = u_p(z)a^\top(z)J_p - \pi(z)\sum_{i=0}^{p-1} z^i S^{i+1}.$$

The vector $a(z)$ is the p -vector $(a_0(z), \dots, a_{p-1}(z))$, where $a_k(z)$ is the Hörner polynomial defined by $a_0(z) = 1$ and $a_k(z) = za_{k-1}(z) + a_k$, and a_k is an entry of C_p . Note that $a_p(z)$ is the characteristic polynomial of C_p . We further have that the rotation matrix $J_p \in \mathbb{R}^{p \times p}$, $\pi(z)$ is the characteristic polynomial of C_p and S denotes the shift matrix, so $S_{ij} = \delta_{i,j+1}$. Observe that the property $a^\top(z)J_p = u_p^*(z)^\top S(f)$ is used in (4.4) to obtain (4.3).

If $z = \rho$, where ρ is an eigenvalue of the companion matrix C_p , then the second term in the right-hand side of (4.4) vanishes. It is then possible to derive from (4.3) (see [5]), and this is the reason why in this section one chooses working with $z = \rho$ instead of z .

A relation between the submatrices $\mathcal{Y}(\rho)$ in (4.2) and $\mathcal{U}(\rho)$ in (3.2) can now be displayed through equality (4.3). For that purpose we denote the vectors $v_0(\rho)$, $v_1(\rho)$, $v_2(\rho), \dots, v_{\tau-1}(\rho)$ as being the first columns of the submatrices $\mathcal{U}_0(\rho)$, $\mathcal{U}_1(\rho)$,

$\mathcal{U}_2(\rho), \dots, \mathcal{U}_{\tau-1}(\rho)$, given in (3.2). Whereas the vectors $w_0(\rho), w_1(\rho), w_2(\rho), \dots, w_{\tau-1}(\rho)$ represent the first rows of the same submatrices. The following property is now summarized in the lemma.

LEMMA 4.2. *By virtue of (4.3), the following equalities hold true for $i = 0, 1, 2, \dots, \tau - 1$:*

$$y_i(\rho) = S^{-1}(f)v_i(\rho) = w_i^\top(\rho),$$

where $y_0(\rho), y_1(\rho), y_2(\rho), \dots, y_{\tau-1}(\rho)$ are the first columns of the submatrices $\mathcal{Y}_0(\rho), \mathcal{Y}_1(\rho), \mathcal{Y}_2(\rho), \dots, \mathcal{Y}_{\tau-1}(\rho)$ given in (4.2).

Proof. Straightforward matrix multiplications $S^{-1}(f)v_i(\rho)$ confirm the property. \square

This leads to the main result of this section.

COROLLARY 4.3. *For the case $\tau + 1 = p$, the span of the null space of $\mathcal{M}_\tau(\rho)$ is*

$$\begin{pmatrix} \mathcal{Y}(\rho) \\ J_{p\tau} \end{pmatrix},$$

where $\mathcal{Y}(\rho)$ is given by (4.2).

Proof. It can be verified through matrix multiplications that

$$\mathcal{M}_\tau(\rho) \begin{pmatrix} \mathcal{Y}(\rho) \\ J_{p\tau} \end{pmatrix} = 0$$

holds. This is in agreement with the appropriate dimensions specified above. \square

It can be seen from (4.3) that for every vector $y \in \text{Ker}(\mathcal{M}_\tau(\rho))$ computed according to the approach suggested in [5], the symmetrizer $S(f)$, a lower triangular and Toeplitz matrix has to be inverted once. However, this is combined with $p\tau$ matrix multiplications by the corresponding vector $x \in \text{Ker}(\mathcal{K}_\tau(\rho))$. This is in agreement with the dimension of the null space of $\mathcal{M}_\tau(\rho)$. In this paper there are neither matrix multiplications nor inversions involved in the construction of the span of the null spaces of $\mathcal{K}_\tau(\rho)$ and $\mathcal{M}_\tau(\rho)$. The null space of $\mathcal{M}_\tau(\rho)$ is obtained by transposing the submatrices contained in the null space of $\mathcal{K}_\tau(\rho)$. Consequently, when the algorithm of the null space of $\mathcal{K}_\tau(\rho)$ is available, the new approach does not require any computational exercise for displaying the span of the null space of $\mathcal{M}_\tau(\rho)$. In the next section an example of the null space of $\mathcal{M}_\tau(\rho)$ is set forth so that the property emphasized in this section will be illustrated.

4.1. Example $\text{Ker}(\mathcal{M}_\tau(\rho))$ when $\tau + 1 = 7$. This case will be illustrated for $p = 7$ and $\tau = 6$. The first matrix is then

$$\mathcal{Y}_0(\rho) = \begin{pmatrix} -1 & 6\rho & -15\rho^2 & 20\rho^3 & -15\rho^4 & 6\rho^5 & -\rho^6 \\ -\rho & 6\rho^2 & -15\rho^3 & 20\rho^4 & -15\rho^5 & 6\rho^6 & -\rho^7 \\ -\rho^2 & 6\rho^3 & -15\rho^4 & 20\rho^5 & -15\rho^6 & 6\rho^7 & -\rho^8 \\ -\rho^3 & 6\rho^4 & -15\rho^5 & 20\rho^6 & -15\rho^7 & 6\rho^8 & -\rho^9 \\ -\rho^4 & 6\rho^5 & -15\rho^6 & 20\rho^7 & -15\rho^8 & 6\rho^9 & -\rho^{10} \\ -\rho^5 & 6\rho^6 & -15\rho^7 & 20\rho^8 & -15\rho^9 & 6\rho^{10} & -\rho^{11} \\ -\rho^6 & 6\rho^7 & -15\rho^8 & 20\rho^9 & -15\rho^{10} & 6\rho^{11} & -\rho^{12} \end{pmatrix}.$$

The following class of matrices are for $j = 1, 2, 3, 4, 5$:

$$\mathcal{Y}_{j=1}^{(1)}(\rho) = \begin{pmatrix} 0 & -1 & 5\rho & -10\rho^2 & 10\rho^3 & -5\rho^4 & \rho^5 \\ -1 & 5\rho & -10\rho^2 & 10\rho^3 & -5\rho^4 & \rho^5 & 0 \end{pmatrix},$$

$$\mathcal{Y}_{j=2}^{(1)}(\rho) = \begin{pmatrix} 0 & 0 & -2 & 8\rho & -12\rho^2 & 8\rho^3 & -2\rho^4 \\ 0 & -2 & 8\rho & -12\rho^2 & 8\rho^3 & -2\rho^4 & 0 \\ -2 & 8\rho & -12\rho^2 & 8\rho^3 & -2\rho^4 & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y}_{j=3}^{(1)}(\rho) = \begin{pmatrix} 0 & 0 & 0 & -6 & 18\rho & -18\rho^2 & 6\rho^3 \\ 0 & 0 & -6 & 18\rho & -18\rho^2 & 6\rho^3 & 0 \\ 0 & -6 & 18\rho & -18\rho^2 & 6\rho^3 & 0 & 0 \\ -6 & 18\rho & -18\rho^2 & 6\rho^3 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y}_{j=4}^{(1)}(\rho) = \begin{pmatrix} 0 & 0 & 0 & 0 & -24 & 48\rho & -24\rho^2 \\ 0 & 0 & 0 & -24 & 48\rho & -24\rho^2 & 0 \\ 0 & 0 & -24 & 48\rho & -24\rho^2 & 0 & 0 \\ 0 & -24 & 48\rho & -24\rho^2 & 0 & 0 & 0 \\ -24 & 48\rho & -24\rho^2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y}_{j=5}^{(1)}(\rho) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -120 & 120\rho \\ 0 & 0 & 0 & 0 & -120 & 120\rho & 0 \\ 0 & 0 & 0 & -120 & 120\rho & 0 & 0 \\ 0 & 0 & -120 & 120\rho & 0 & 0 & 0 \\ 0 & -120 & 120\rho & 0 & 0 & 0 & 0 \\ -120 & 120\rho & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices $\mathcal{Y}_j^{(2)}(\rho)$ with $j = 1, 2, 3, 4, 5$ are now displayed:

$$\mathcal{Y}_{j=1}^{(2)}(\rho) = \begin{pmatrix} -2\rho & 11\rho^2 & -25\rho^3 & 30\rho^4 & -20\rho^5 & 7\rho^6 & -\rho^7 \\ -3\rho^2 & 17\rho^3 & -40\rho^4 & 50\rho^5 & -35\rho^6 & 13\rho^7 & -2\rho^8 \\ -4\rho^3 & 23\rho^4 & -55\rho^5 & 70\rho^6 & -50\rho^7 & 19\rho^8 & -3\rho^9 \\ -5\rho^4 & 29\rho^5 & -70\rho^6 & 90\rho^7 & -65\rho^8 & 25\rho^9 & -4\rho^{10} \\ -6\rho^5 & 35\rho^6 & -85\rho^7 & 110\rho^8 & -80\rho^9 & 31\rho^{10} & -5\rho^{11} \end{pmatrix},$$

$$\mathcal{Y}_{j=2}^{(2)}(\rho) = \begin{pmatrix} -6\rho & 30\rho^2 & -62\rho^3 & 68\rho^4 & -42\rho^5 & 14\rho^6 & -2\rho^7 \\ -12\rho^2 & 64\rho^3 & -142\rho^4 & 168\rho^5 & -112\rho^6 & 40\rho^7 & -6\rho^8 \\ -20\rho^3 & 110\rho^4 & -252\rho^5 & 308\rho^6 & -212\rho^7 & 78\rho^8 & -12\rho^9 \\ -30\rho^4 & 168\rho^5 & -392\rho^6 & 488\rho^7 & -342\rho^8 & 128\rho^9 & -20\rho^{10} \end{pmatrix},$$

$$\mathcal{Y}_{j=3}^{(2)}(\rho) = \begin{pmatrix} -24\rho & 108\rho^2 & -204\rho^3 & 210\rho^4 & -126\rho^5 & 42\rho^6 & -6\rho^7 \\ -60\rho^2 & 300\rho^3 & -630\rho^4 & 714\rho^5 & -462\rho^6 & 162\rho^7 & -24\rho^8 \\ -120\rho^3 & 630\rho^4 & -1386\rho^5 & 1638\rho^6 & -1098\rho^7 & 396\rho^8 & -60\rho^9 \end{pmatrix},$$

$$\mathcal{Y}_{j=4}^{(2)}(\rho) = \begin{pmatrix} -120\rho & 480\rho^2 & -840\rho^3 & 840\rho^4 & -504\rho^5 & 168\rho^6 & -24\rho^7 \\ -360\rho^2 & 1680\rho^3 & -3360\rho^4 & 3696\rho^5 & -2352\rho^6 & 816\rho^7 & -120\rho^8 \end{pmatrix},$$

$$\mathcal{Y}_{j=5}^{(2)}(\rho) = (-720\rho \quad 2520\rho^2 \quad -4200\rho^3 \quad 4200\rho^4 \quad -2520\rho^5 \quad 840\rho^6 \quad -120\rho^7).$$

Insertion of the matrix $\mathcal{Y}_0(\rho)$ in (4.2) followed by

$$\mathcal{Y}_1(\rho) = \begin{pmatrix} \mathcal{Y}_{j=1}^{(1)}(\rho) \\ \mathcal{Y}_{j=1}^{(2)}(\rho) \end{pmatrix}, \quad \mathcal{Y}_2(\rho) = \begin{pmatrix} \mathcal{Y}_{j=2}^{(1)}(\rho) \\ \mathcal{Y}_{j=2}^{(2)}(\rho) \end{pmatrix},$$

$$\mathcal{Y}_3(\rho) = \begin{pmatrix} \mathcal{Y}_{j=3}^{(1)}(\rho) \\ \mathcal{Y}_{j=3}^{(2)}(\rho) \end{pmatrix}, \quad \mathcal{Y}_4(\rho) = \begin{pmatrix} \mathcal{Y}_{j=4}^{(1)}(\rho) \\ \mathcal{Y}_{j=4}^{(2)}(\rho) \end{pmatrix}, \quad \mathcal{Y}_5(\rho) = \begin{pmatrix} \mathcal{Y}_{j=5}^{(1)}(\rho) \\ \mathcal{Y}_{j=5}^{(2)}(\rho) \end{pmatrix}$$

yields the representation

$$\mathcal{Y}(\rho) = \frac{1}{6!} (\mathcal{Y}_0(\rho), \mathcal{Y}_1(\rho), \mathcal{Y}_2(\rho), \mathcal{Y}_3(\rho), \mathcal{Y}_4(\rho), \mathcal{Y}_5(\rho)).$$

The vectors contained in $(\mathcal{Y}_{J_{42}}^{(\rho)})$ span the null space of $\mathcal{M}_\tau(\rho)$ when $p = 7$ and $\tau = 6$. It is straightforward to verify that when the matrices $\mathcal{Y}_0(\rho)$, $\mathcal{Y}_j^{(1)}(\rho)$, and $\mathcal{Y}_j^{(2)}(\rho)$, with $j = 1, 2, 3, 4, 5$, are transposed and inserted in (3.2) accordingly, one obtains the null space of $\mathcal{K}_\tau(\rho)$.

A summary of the results will be given in the next section.

5. Main conclusions. The results displayed in sections 2–4 allow us to present an explicit representation of the solutions to the linear systems of equations introduced in this paper. The solutions, (1.5) and (1.6), to the linear system of (1.1) and (1.2) are given by

$$\begin{aligned} X &= (\mathcal{K}_\nu(z))^- \mathcal{E} + \mathcal{W}(z) \text{ with } \mathcal{W}(z) \in \text{Ker}(\mathcal{K}_\nu(z)), \\ Y &= (\mathcal{M}_\tau(\rho))^- \mathcal{R} + \mathcal{L}(\rho) \text{ with } \mathcal{L}(\rho) \in \text{Ker}(\mathcal{M}_\tau(\rho)). \end{aligned}$$

An explicit expression for $(\mathcal{K}_\nu(z))^-$ and $\mathcal{W}(z)$ has been developed in sections 2 and 3, respectively, and a solution to the linear system of equations (1.1) is implementable. Analogously for the expressions $(\mathcal{M}_\tau(\rho))^-$ and $\mathcal{L}(\rho)$, constructed in sections 2 and 4, respectively, a solution to the linear system of (1.2) is implementable.

In the next section an algorithm for the null space $\text{Ker}(\mathcal{K}_\nu(z))$, for the case $\nu + 1 < q$, is presented. It is a variant of the algorithm displayed in section 3.

6. Ker $(\mathcal{K}_\nu(z))$ for the case $\nu + 1 < q$. In this section the case $\nu + 1 < q$ is considered for the null space $\text{Ker}(\mathcal{K}_\nu(z))$. We then have $\text{rank}(\mathcal{K}_\nu(z)) = \nu + 1$ so that $\dim \text{Ker}(\mathcal{K}_\nu(z)) = (q - 1)(\nu + 1)$. In this case the coefficient matrix $\mathcal{K}_\nu(z)$ is not surjective, so a Moore–Penrose generalized inverse should be used when one is interested in a solution of (1.1). This can be a subject for future research. Consider the null space of the coefficient matrix $\mathcal{K}_\nu(z)$,

$$\text{Ker } \mathcal{K}_\nu(z) = \text{span} \left(\begin{array}{c} \mathcal{U}(z) \\ J_{(q-1)(\nu+1)} \end{array} \right),$$

where $J_{(q-1)(\nu+1)}$ is the $(q - 1)(\nu + 1)$ rotation matrix. An algorithm of the matrix $\mathcal{U}(z)$ contained in $\text{Ker}(\mathcal{K}_\nu(z))$ will be set forth to obtain

$$\mathcal{U}(z) = \frac{1}{\nu!} (\mathcal{U}_0(z), \mathcal{U}_1(z), \mathcal{U}_2(z), \dots, \mathcal{U}_{\nu-1}(z)).$$

In this section no proofs are provided since they are similar to the proofs done in section 3.

6.1. A representation for $\mathcal{U}_0(z)$. An appropriate partition is $\mathcal{U}_0(z) = (\mathcal{U}_0^{(1)}(z) \mathcal{U}_0^{(2)}(z))$. For evaluating $\mathcal{U}_0^{(1)}(z)$ we introduce the $(\nu + 1) \times q$ matrix

$$(6.1) \quad \Omega = (\xi, \xi, \dots, \xi),$$

where the vector ξ is given in (3.4), and we put

$$(6.2) \quad \mathcal{U}_0^{(1)}(z) = \Omega \odot z^{-(q-\nu-1)} (u_{\nu+1}(z) u_q^\top(z)),$$

for $\nu = 1, 2, \dots, q - 2$. The signs of the elements of each column vector of $\mathcal{U}_0^{(1)}(z)$ follow the same pattern as for $\mathcal{U}_0(z)$ in section 3. The second part of $\mathcal{U}_0(z)$ is

$$(6.3) \quad \mathcal{U}_0^{(2)}(z) = \chi \odot \mathcal{U}_{1,2}^*(z),$$

where

$$\mathcal{U}_{1,2}^*(z) = \begin{cases} \mathcal{U}_1^*(z) & \text{for } \nu = 2, 3, \dots, q - 2 \\ \mathcal{U}_2^*(z) & \text{for } \nu = 1 \end{cases}$$

and

$$\begin{cases} \mathcal{U}_1^*(z) = u_{q-\nu-1}^{*\top}(z^{-1}) \otimes \begin{pmatrix} u_3^*(z^{-1}) \\ zu_{\nu-2}(z) \end{pmatrix} & \text{for } \nu = 2, 3, \dots, q - 2 \\ \mathcal{U}_2^*(z) = u_{q-\nu-1}^{*\top}(z^{-1}) \otimes z^{-1}u_2^*(z^{-1}) & \text{for } \nu = 1. \end{cases}$$

The matrix χ has the form $\chi = (\chi_{q-\nu-1}, \chi_{q-\nu-2}, \dots, \chi_2, \chi_1)$, where the columns are computed recursively for $k = 2, 3, \dots, q - \nu - 1$:

$$(6.4) \quad \chi_k = \chi_{k-1} + \xi.$$

The $(\nu + 1)$ column vector χ_1 is for $\nu = 1, 2, \dots, q - 2$

$$(6.5) \quad \chi_1 = \begin{pmatrix} \binom{\nu}{0} \\ \binom{\nu}{1} + \binom{\nu-1}{0} \\ \binom{\nu}{2} + \binom{\nu-1}{1} \\ \vdots \\ \binom{\nu}{\nu} + \binom{\nu-1}{\nu-1} \end{pmatrix}.$$

The sign pattern of each column of $\mathcal{U}_0^{(2)}(z)$ is $(-1)^\ell$ with $\ell = 0, 1, \dots, \nu$. In the next section we shall summarize the construction of $\mathcal{U}_0(z)$.

6.1.1. Summary of the construction of $\mathcal{U}_0(z)$. *Step 1.* Introduce the vector ξ according to (3.4).

Step 2. Define matrix Ω according to (6.1).

Step 3. Define the columns of $\mathcal{U}_0^{(1)}(z)$ according to (6.2).

Step 4. Introduce the vector χ_1 given in (6.5).

Step 5. Compute the vectors $\chi_2, \chi_3, \dots, \chi_{q-\nu-1}$ by means of the recursions (6.4).

Step 6. Compute the columns of $\mathcal{U}_0^{(2)}(z)$ according to (6.3).

6.2. Example for $\mathcal{U}_0(z)$ when $q = 6, \nu = 4$. An example is chosen when $q = 6$ and $\nu = 4$ so the first matrices to consider are $\mathcal{U}_0^{(1)}(\sigma)$ and $\mathcal{U}_0^{(2)}(\sigma)$ to obtain

$$\mathcal{U}_0^{(1)}(z) = \begin{pmatrix} -\frac{1}{z} & -1 & -z & -z^2 & -z^3 & -z^4 \\ 4 & 4z & 4z^2 & 4z^3 & 4z^4 & 4z^5 \\ -6z & -6z^2 & -6z^3 & -6z^4 & -6z^5 & -6z^6 \\ 4z^2 & 4z^3 & 4z^4 & 4z^5 & 4z^6 & 4z^7 \\ -z^3 & -z^4 & -z^5 & -z^6 & -z^7 & -z^8 \end{pmatrix}$$

and

$$\mathcal{U}_0^{(2)}(z) = \begin{pmatrix} \frac{1}{z^2} \\ -\frac{5}{z} \\ 9 \\ -7z \\ 2z^2 \end{pmatrix}.$$

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6.3. A representation of $\mathcal{U}_j(z)$ when $j = 1, 2, \dots, \nu - 1$. The matrices $\mathcal{U}_1(z), \mathcal{U}_2(z), \dots, \mathcal{U}_{\nu-1}(z)$ are now considered to obtain for $j = 1, 2, \dots, \nu - 1$

$$\mathcal{U}_j(z) = \left(\mathcal{U}_j^{(1)}(z) \mathcal{U}_j^{(2)}(z) \mathcal{U}_j^{(3)}(z) \right).$$

Since the submatrices $\mathcal{U}_j^{(1)}(z)$ and $\mathcal{U}_j^{(2)}(z)$ have the same structure as the corresponding submatrices in section 3, the case $q = \nu + 1$, we therefore omit the description of $\mathcal{U}_j^{(1)}(z)$ and $\mathcal{U}_j^{(2)}(z)$.

6.3.1. A representation of $\mathcal{U}_j^{(3)}(z)$. We shall now focus on matrix $\mathcal{U}_j^{(3)}(z)$ and for that purpose the following matrix is considered for $j = 1, 2, \dots, \nu - 1$:

$$(6.6) \quad \mu_j = \left(\mu_j^{q-\nu-1} \mu_j^{q-\nu-2} \dots \mu_j^2 \mu_j^1 \right).$$

The first recursion to consider is when $j = 1$ and $k = 2, 3, \dots, q - \nu - 1$, to obtain

$$(6.7) \quad \mu_1^k = \mu_1^{k-1} + 2\chi_k.$$

The vectors $\chi_2, \chi_3, \dots, \chi_{q-\nu-1}$ are obtained recursively for $\mathcal{U}_0^{(2)}(z)$; see (6.4). The solution to (6.4) is

$$\chi_k = \chi_1 + (k - 1)\xi,$$

where χ_1 is given in (6.5). A solution to (6.7) is then given by

$$\mu_1^k = \mu_1^1 + 2(k - 1)\chi_1 + k(k - 1)\xi.$$

A generalization can now be given for $j = 2, \dots, \nu - 1$ and $k = 1, 2, 3, \dots, q - \nu - 1$. The column vectors are computed recursively as follows:

$$(6.8) \quad \mu_j^k = \mu_j^{k-1} + (j + 1) \mu_{j-1}^k.$$

A solution to recursion (6.8) in terms of initial vectors $\mu_j^1, \mu_{j-1}^1, \dots, \mu_2^1, \mu_1^1$, specified in (6.10), and the known vectors χ_1 and ξ , is given by

$$(6.9) \quad \begin{aligned} \mu_j^k &= \sum_{i=0}^{j-1} i! \binom{j+1}{i} \binom{k-2+i}{k-2} \mu_{j-i}^1 \\ &+ (j+1)! \binom{k+j-2}{k-2} \chi_1 + (j+1)! \binom{k+j-1}{k-2} \xi. \end{aligned}$$

The explicit solution (6.10) is derived in a similar manner as in Proposition 3.4. For $j = 1, 2, \dots, \nu - 1$, the components of the vector μ_j^1 are given by

$$(6.10) \quad \begin{cases} [\mu_j^1]_i = (j+1)! \binom{\nu+1}{i}, & i = 0, 1, \dots, j+1, \\ [\mu_j^1]_i = (j+1)! \binom{\nu+1}{i} - r_{i-j-2}, & i = j+2, \dots, \nu-1, \\ [\mu_j^1]_{\nu+1} = (j+2)!, & \end{cases}$$

where the terms r_ℓ , are defined by

$$r_\ell = \begin{cases} (j+1)! \binom{\nu-j-1}{\ell} & \text{for } \ell = 0, 1, \dots, \nu-j-3 \\ 0 & \text{for } \nu-j < 3. \end{cases}$$

The submatrix $\mathcal{U}_j^{(3)}(z)$ can now be given according to

$$(6.11) \quad \mathcal{U}_j^{(3)}(z) = \mu_j \odot z^{-j} \mathcal{U}_1^*(z) \quad \text{for } j = 1, 2, \dots, \nu - 1.$$

The matrix $\mathcal{U}_1^*(z)$ has also been used for specifying $\mathcal{U}_0^{(2)}(z)$. The sign pattern of the elements of each column of $\mathcal{U}_j^{(3)}(\sigma)$ follows the ordering $(-1)^{\ell+j}$ with $\ell = 0, 1, \dots, \nu$.

6.3.2. Summary of the construction of matrix $\mathcal{U}_j^{(3)}(\sigma)$. *Step 1.* Define the initial vector μ_j^1 displayed in (6.10) for $j = 1, 2, \dots, \nu - 1$.

Step 2. Compute the columns of matrix (6.6) by applying recursions (6.8) for the corresponding values of $j = 1, 2, \dots, \nu - 1$.

Step 3. Compute the columns of matrix $\mathcal{U}_j^{(3)}(z)$ according to (6.11) for the corresponding values of $j = 1, 2, \dots, \nu - 1$.

6.4. Example $\mathcal{U}_j(z)$ when $q = 6, \nu = 4$ and $j = 1, 2, 3$. The matrix $\mathcal{U}_j^{(1)}(z) = (\delta_j^1(z)\delta_j^2(z) \cdots \delta_j^{j+1}(z))$ will be illustrated for $j = 1, 2, 3$, to obtain

$$\mathcal{U}_{j=1}^{(1)}(z) = \begin{pmatrix} 0 & -1 \\ -1 & 3z \\ 3z & -3z^2 \\ -3z^2 & z^3 \\ z^3 & 0 \end{pmatrix}, \quad \mathcal{U}_{j=2}^{(1)}(z) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 4z \\ -2 & 4z & -2z^2 \\ 4z & -2z^2 & 0 \\ -2z^2 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{U}_{j=3}^{(1)}(z) = \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 0 & -6 & 6z \\ 0 & -6 & 6z & 0 \\ -6 & 6z & 0 & 0 \\ 6z & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\mathcal{U}_j^{(2)}(z)$ is, for $j = 1, 2, 3$,

$$\mathcal{U}_{j=1}^{(2)}(z) = \begin{pmatrix} -2z & -3z^2 & -4z^3 \\ 7z^2 & 11z^3 & 15z^4 \\ -9z^3 & -15z^4 & -21z^5 \\ 5z^4 & 9z^5 & 13z^6 \\ -z^5 & -2z^6 & -3z^7 \end{pmatrix}, \quad \mathcal{U}_{j=2}^{(2)}(z) = \begin{pmatrix} -6z & -12z^2 \\ 18z^2 & 40z^3 \\ -20z^3 & -50z^4 \\ 10z^4 & 28z^5 \\ -2z^5 & -6z^6 \end{pmatrix},$$

$$\mathcal{U}_{j=3}^{(2)}(z) = \begin{pmatrix} -24z \\ 60z^2 \\ -60z^3 \\ 30z^4 \\ -6z^5 \end{pmatrix}.$$

The matrix $\mathcal{U}_j^{(3)}(z)$ is, for $j = 1, 2, 3$,

$$\mathcal{U}_{j=1}^{(3)}(z) = \begin{pmatrix} -\frac{2}{z^3} \\ \frac{10}{z^2} \\ -\frac{20}{z} \\ 18 \\ -6z \end{pmatrix}, \quad \mathcal{U}_{j=2}^{(3)}(z) = \begin{pmatrix} \frac{6}{z^4} \\ -\frac{30}{z^3} \\ \frac{60}{z^2} \\ -\frac{60}{z} \\ 24 \end{pmatrix},$$

and

$$\mathcal{U}_{j=3}^{(3)}(z) = \begin{pmatrix} -\frac{24}{z^5} \\ \frac{120}{z^4} \\ -\frac{240}{z^3} \\ \frac{240}{z^2} \\ -\frac{120}{z} \end{pmatrix}.$$

Insertion in (3.2) of the matrices $\mathcal{U}_0^{(1)}(z)$ and $\mathcal{U}_0^{(2)}(z)$, followed by the matrices

$$\mathcal{U}_1(z) = \left(\mathcal{U}_{j=1}^{(1)}(z) \mathcal{U}_{j=1}^{(2)}(z) \mathcal{U}_{j=1}^{(3)}(z) \right),$$

$$\mathcal{U}_2(z) = \left(\mathcal{U}_{j=2}^{(1)}(z) \mathcal{U}_{j=2}^{(2)}(z) \mathcal{U}_{j=2}^{(3)}(z) \right),$$

$$\mathcal{U}_3(z) = \left(\mathcal{U}_{j=3}^{(1)}(z) \mathcal{U}_{j=3}^{(2)}(z) \mathcal{U}_{j=3}^{(3)}(z) \right),$$

results in the scheme

$$\mathcal{U}(z) = \frac{1}{4!} \left(\mathcal{U}_0^{(1)}(z), \mathcal{U}_0^{(2)}(z), \mathcal{U}_1(z), \mathcal{U}_2(z), \mathcal{U}_3(z) \right).$$

The columns that compose the matrix $\binom{\mathcal{U}(z)}{J_{25}}$ span the null space of $\mathcal{K}_\nu(z)$ when $q = 6$ and $\nu = 4$.

7. Conclusions. In this paper a solution to new linear systems of equations is displayed. This is done when $q = \nu + 1$ and $p = \tau + 1$. The newly developed algorithms for the null space and right-inverse are then equivalent for both coefficient matrices. Explicit solutions to both linear system of equations can then be straightforwardly implemented by using the same algorithms. The algorithms for the null space do not require matrix multiplications and matrix inversions. The main computational exercise consists of evaluating factorials and binomial coefficients combined with recursions that consist of the addition of two vectors. The binomial coefficients can be computed by applying the Pascal triangle.

A connection between adjoints of companion-related matrices and rectangular generalized Vandermonde matrices of the block Toeplitz type is then confirmed through the corresponding null spaces.

An algorithm for the null space for $\mathcal{K}_\nu(z)$ is also set forth when $q > \nu + 1$. To compute a solution to the linear systems of (1.1) and (1.2) under these conditions can be considered for future research.

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