



# The Bezoutian, state space realizations and Fisher's information matrix of an ARMA process

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## Abstract

In this paper we derive some properties of the Bezout matrix and relate the Fisher information matrix for a stationary ARMA process to the Bezoutian. Some properties are explained via realizations in state space form of the derivatives of the white noise process with respect to the parameters. A factorization of the Fisher information matrix as a product in factors which involve the Bezout matrix of the associated AR and MA polynomials is derived. From this factorization we can characterize singularity of the Fisher information matrix.

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## 1. Introduction

The Cramér–Rao lower bound on the covariance matrix of an estimator is a classical result in statistics, see Cramér [6] and Rao [15]. This bound is given by the inverse of Fisher's information matrix. For regular statistical models, it is also known that the maximum likelihood estimator

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is asymptotically normal with this inverse as the asymptotic covariance matrix. Therefore it is natural to ask for conditions of an underlying statistical model that guarantee non-singularity of this matrix. In the present paper we are concerned with the Fisher information matrix for (stationary) autoregressive moving average (ARMA) models. The information matrix is singular in the presence of common roots of the AR and the MA polynomial and vice versa. This fact is considered to be well known in time series analysis, see [13] or [14] for an early discussion of this phenomenon, and [9] for the extension to ARMA models with an exogenous input (the ARMAX case).

In [10] properties of the Fisher information matrix for an ARMA process have been derived using contour integration in the complex plane and state space realizations of the ARMA process itself. In the present paper we study Fisher's information matrix by means of state space realizations for the score process and by linking Fisher's information matrix to the Sylvester resultant matrix and the Bezout matrix associated with the autoregressive and moving average polynomials.

The role of the resultant matrix has been discussed in various studies in the fields of time series and systems theory. For instance, in [1] this matrix shows up in a convergence analysis of maximum likelihood estimators of the ARMA parameters (more precisely in the study of the convergence of the criterion function), in Barnett [2] a relationship between Sylvester's resultant matrix and the companion matrix of a polynomial is given. Kalman [8] has investigated the concept of observability and controllability in terms of Sylvester's resultant matrix. Similar results can be found in Barnett [3], which contains further discussions and references on these topics. But, it seems that the use of the Bezout matrix has not been recognized yet. For ARMA models we will show that Fisher's information matrix can be factorized, where one of the factors is expressed in terms of the Bezout matrix. Also from this it follows that Fisher's information matrix is singular if and only if the AR and MA polynomials have a non-trivial greatest common divisor. Singularity of the information matrix can thus be interpreted as the result of overparametrization of the chosen ARMA model and of using a model of too high order. In Söderström and Stoica [17, p. 162 ff.] a discussion on overparametrization in terms of the transfer function of a system can be found. In a static context, Fisher's information matrix has already been studied in [16] for problems of local and global identifiability.

The paper is organized as follows. In Section 2 we relate Fisher's information matrix to the derivatives of the noise process. The main results are state space realizations for these derivatives and properties of these realizations are presented. In Section 3 we study some properties of the Bezout matrix as well as its kernel. Section 4 is devoted to further properties of the Bezout matrix, to be exploited in Sections 5 and 6. In the first of these sections, we study singularity of solutions to certain Stein equations with coefficients related to the AR and MA polynomials, whereas in Section 6 all previous results are assembled to characterize non-singularity of Fisher's information matrix.

## 2. Computations in state space

In this section we adopt the following notation. All random variables and processes used below are assumed to be defined on some measurable space  $(\Omega, \mathcal{F})$ . If  $x = (x_t, t \in \mathbb{Z})$  is a stochastic process and  $\pi$  a polynomial given by  $\pi(z) = \sum_{j=0}^k \pi_j z^j$ , then we define the stochastic process  $\pi(L)x = (\pi(L)x_t, t \in \mathbb{Z})$  by  $\pi(L)x_t = \sum_{j=0}^k \pi_j x_{t-j}$ , where  $t \in \mathbb{Z}$ . Thus  $L$  is the *lag operator*. We use similar notation if  $\pi$  is replaced with the power series expansion of an analytic function, assuming that the possibly infinite series is convergent in some sense. See [4, Chapter 3], for general theorems on convergence issues.

Consider the following two scalar monic polynomials in the variable  $z$ .

$$\begin{aligned} \hat{a}(z) &= z^p + a_1 z^{p-1} + \dots + a_p, \\ \hat{c}(z) &= z^q + c_1 z^{q-1} + \dots + c_q. \end{aligned}$$

By  $a$  and  $c$  we denote the reciprocal polynomials, so  $a(z) = z^p \hat{a}(z^{-1})$  and  $c(z) = z^q \hat{c}(z^{-1})$ , and also write  $a^\top = (a_1, \dots, a_p)$  and  $c^\top = (c_1, \dots, c_q)$ . Usually no confusion between the notation  $a$  for the polynomial and vector will arise, but sometimes we will write  $a(\cdot)$  when a polynomial is considered. Let  $\theta = (a_1, \dots, a_p, c_1, \dots, c_q)$  be the *parameter vector*. The set  $\Theta$  of parameters  $\theta$  (the parameter space) will be such that all corresponding polynomials  $a$  and  $c$  have only zeros outside the unit circle (equivalently,  $\hat{a}$  and  $\hat{c}$  have only zeros inside the unit circle).

We will consider a stationary ARMA( $p, q$ ) process  $y = (y_t, t \in \mathbb{Z})$  that satisfies

$$a(L)y = c(L)\varepsilon \tag{2.1}$$

with  $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$  a white noise sequence with variance  $\sigma^2$ , to be further specified below. From the assumption on  $\Theta$  it then follows that  $y$  is *causal* and *invertible* and also that the infinite sums that we use below, have the right convergence properties.

The distribution of the process  $y$  satisfying (2.1) depends on  $\theta$  and will be denoted by  $\mathbb{P}_\theta$ . In the present paper  $\sigma^2$  is not taken as a parameter and henceforth it will be assumed to have some fixed, but otherwise arbitrary positive value. Let  $(\Omega, \mathcal{F}, \mathbb{P}_\theta)$  be the resulting probability space. Obviously, given the observed process  $y$ , the process  $\varepsilon$  then also depends on  $\theta$  and to emphasize this, we will write  $\varepsilon(\theta)$  for this process and  $\varepsilon_t(\theta)$  for  $\varepsilon_t$ . Indeed, let  $\pi(z, \theta) = \frac{a(z)}{c(z)} = \sum_{j=0}^\infty \pi_j(\theta)z^j$ , then

$$\varepsilon_t(\theta) = \sum_{j=0}^\infty \pi_j(\theta)y_{t-j}, \tag{2.2}$$

where for each  $\theta \in \Theta$  the sum converges a.s. and in  $L^2(\Omega, \mathcal{F}, \mathbb{P}_\theta)$ , see [4], Proposition 3.1.2 and Theorem 3.1.2. The distribution of  $y$  is further specified by assuming that  $y$  is a *Gaussian* process under each  $\mathbb{P}_\theta$ .

We denote by  $\varepsilon_t^{\theta_i}$  the partial derivative of  $\varepsilon_t$  with respect to  $\theta_i$  and by  $\dot{\varepsilon}_t$  the row vector with elements  $\varepsilon_t^{\theta_i}$ . Of course,  $\dot{\varepsilon}_t$  depends on the whole parameter vector  $\theta$  as well and so we also write  $\dot{\varepsilon}_t = \dot{\varepsilon}_t(\theta)$ . As we will explain now, the gradient vector  $\dot{\varepsilon}_t(\theta)$  is crucial for the computation of the *Fisher information matrix*.

Recall the assumption that the process  $y$  is Gaussian and that  $\sigma^2$  is some fixed constant. If we have observations  $y_1, \dots, y_n$ , the log likelihood  $\ell_n(\theta)$  is then essentially given by

$$\ell_n(\theta) = \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{t=1}^n \varepsilon_t(\theta)^2.$$

The *score function*, by definition the gradient of  $\ell_n(\theta)$  w.r.t.  $\theta$ , is then given by

$$\dot{\ell}_n(\theta) = -\frac{1}{\sigma^2} \sum_{t=1}^n \varepsilon_t(\theta) \dot{\varepsilon}_t(\theta). \tag{2.3}$$

To emphasize that the expectations below are taken w.r.t.  $\mathbb{P}_\theta$ , we use the notation  $\mathbb{E}_\theta$ . Traditionally in statistics, the Fisher information matrix, denoted by  $\mathcal{I}_n(\theta)$ , is defined as the covariance matrix of the score function. Since the expectation (taken w.r.t. the distribution under  $\theta$ ) of the score function is zero,  $\mathcal{I}_n(\theta)$  can then be computed as

$$\mathcal{I}_n(\theta) = \mathbb{E}_\theta \dot{\ell}_n(\theta)^\top \dot{\ell}_n(\theta).$$

Given that  $\dot{\varepsilon}_t(\theta)$  is (stochastically) independent of  $\varepsilon_t(\theta)$  under  $\mathbb{P}_\theta$ , it then follows from (2.3) that

$$\begin{aligned} \mathcal{J}_n(\theta) &= \frac{1}{\sigma^4} \sum_{t=1}^n \mathbb{E}_\theta \varepsilon_t(\theta)^2 \mathbb{E}_\theta \dot{\varepsilon}_t(\theta)^\top \dot{\varepsilon}_t(\theta) \\ &= \frac{1}{\sigma^2} \sum_{t=1}^n \mathbb{E}_\theta \dot{\varepsilon}_t(\theta)^\top \dot{\varepsilon}_t(\theta). \end{aligned} \tag{2.4}$$

This motivates why  $\dot{\varepsilon}_t(\theta)$  is needed for the computation of the Fisher information matrix. In Section 6 we will come back to this issue. Now we continue to study the  $\dot{\varepsilon}(\theta)$  process. We obtain by differentiation of (2.1) the formal expressions

$$\varepsilon_t^{aj}(\theta) = \frac{1}{a(L)} \varepsilon_{t-j}(\theta), \tag{2.5}$$

$$\varepsilon_t^{cl}(\theta) = -\frac{1}{c(L)} \varepsilon_{t-l}(\theta). \tag{2.6}$$

We introduce some auxiliary notation. Write for each positive integer  $k$

$$\begin{aligned} u_k(z) &= (1, z, \dots, z^{k-1})^\top, \\ u_k^*(z) &= (z^{k-1}, \dots, 1)^\top = z^{k-1} u_k(z^{-1}). \end{aligned}$$

It is notationally convenient to represent the process  $\dot{\varepsilon}(\theta)$  by column vectors. So we write  $\xi(\theta)_t = \dot{\varepsilon}(\theta)_t^\top$ ,  $t \in \mathbb{Z}$ . Let us compute the transfer function  $\tau(z, \theta)$  that relates  $\xi(\theta)$  to  $\varepsilon(\theta)$  by replacing  $L$  with  $z^{-1}$  in Eqs. (2.5) and (2.6). Here  $z^{-1}$  represents the forward shift. One obtains from (2.5) and (2.6)

$$\tau(z, \theta) = \begin{pmatrix} \frac{1}{\hat{a}(z)} u_p^*(z) \\ -\frac{1}{\hat{c}(z)} u_q^*(z) \end{pmatrix}. \tag{2.7}$$

In [10] we have investigated certain controllable or observable realizations of the ARMA process  $y$ . There we have also briefly outlined a procedure without detailed proofs to obtain from these realizations also realizations for the process  $\dot{\varepsilon}(\theta)$ . We repeat the conclusions, but give in the present paper a short proof of them, based on transfer function considerations, without using the realizations of the ARMA process  $y$  itself. Let  $e_n$  be the first basis vector of the Euclidean space  $\mathbb{R}^n$ . When no confusion can arise (often in the proofs), we often simply write  $e$ , which we will also use as the notation for the first basis vector in Euclidean spaces of different dimensions. By  $J$  we denote the forward  $n \times n$  shift matrix,  $J_{ij} = 1$  if  $i = j + 1$  and zero else. Similarly, we denote by  $I$  the identity matrix of the appropriate size and  $0$  stands for the zero vector or matrix of appropriate dimensions. Occasionally these matrices and vectors will have a subscript, when it is necessary to indicate the sizes.

Let  $\hat{g}(z, \theta) = \hat{a}(z)\hat{c}(z) = z^{p+q} + \sum_{i=1}^{p+q} g_i(\theta)z^{p+q-i}$  and  $\hat{g}(\theta)$  the vector  $\hat{g}(\theta) = (g_{p+q}(\theta), \dots, g_1(\theta))^\top$ . Likewise we have  $g(z, \theta) = a(z)c(z)$  and the column vector  $g(\theta) = (g_1(\theta), \dots, g_{p+q}(\theta))^\top$ .

The Sylvester resultant matrix  $R$  of  $\hat{c}$  and  $-\hat{a}$  is defined as the  $(p + q) \times (p + q)$  matrix

$$R(c, -a) = \begin{pmatrix} R_p(c) \\ -R_q(a) \end{pmatrix},$$

where  $R_p(c)$  is the  $q \times (p + q)$  matrix

$$R_p(c) = \begin{pmatrix} 1 & c_1 & \cdots & c_q & 0 & \cdots & 0 \\ 0 & 1 & c_1 & \cdots & c_q & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & \cdots & 0 & 1 & c_1 & \cdots & c_q \end{pmatrix}$$

and  $R_q(a)$  is the  $q \times (p + q)$  matrix given by

$$R_q(a) = \begin{pmatrix} 1 & a_1 & \cdots & a_p & 0 & \cdots & 0 \\ 0 & 1 & a_1 & \cdots & a_p & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & a_p \end{pmatrix}.$$

In the presence of common roots of  $\hat{a}$  and  $\hat{c}$  the matrix  $R(c, -a)$  becomes singular. Moreover it is known (see e.g. [18, p. 106]) that

$$\det R(c, -a) = (-1)^p \prod_{i=1}^p \prod_{j=1}^q (\gamma_j - \alpha_i),$$

where the  $\alpha_i$  and the  $\gamma_j$  are the roots of  $\hat{a}$  and  $\hat{c}$ , respectively.

Next we introduce the matrices  $F(\theta)$  and  $G(\theta)$  defined by

$$F(\theta) = \begin{pmatrix} J - e_p a^\top & 0 \\ 0 & J - e_q c^\top \end{pmatrix} \tag{2.8}$$

and

$$G(\theta) = J - e_{p+q} g(\theta)^\top. \tag{2.9}$$

**Lemma 2.1.** *Let  $F(\theta)$  and  $G(\theta)$  be as in (2.8) and (2.9). Then the following relation holds:*

$$R(c, -a)G(\theta) = F(\theta)R(c, -a).$$

**Proof.** The easiest way to see this, is to multiply both sides of this equation on the right with  $u_{p+q}(z)$ . Then we compute on the left hand side the product

$$\begin{aligned} R(c, -a)(J - e_{p+q} g(\theta)^\top)u_{p+q}(z) &= R(c, -a)(u_{p+q}(z) - g(z, \theta)e_{p+q}) \\ &= \begin{pmatrix} c(z)u_p(z) - g(z, \theta)e_p \\ -a(z)u_q(z) + g(z, \theta)e_q \end{pmatrix}. \end{aligned}$$

The computations on the right hand side are of a similar nature and an easy comparison yields the result.  $\square$

We now present the first realization of the process  $\dot{\varepsilon}(\theta)$ .

**Proposition 2.2.** *The process  $\xi(\theta) = \dot{\varepsilon}(\theta)^\top$  can be realized by the following stable and controllable system:*

$$\begin{aligned} Z_{t+1} &= G(\theta)Z_t + e\varepsilon(\theta)_t, \\ \xi(\theta)_t &= R(c, -a)Z_t, \end{aligned} \tag{2.10}$$

where  $G(\theta)$  is as in (2.9). This system is observable iff the polynomials  $a$  and  $c$  have no common zeros.

**Proof.** Let us compute the transfer function  $\tau(\cdot, \theta)$  of the above system. Standard computations show that  $(z - G(\theta))^{-1}e = \frac{1}{\hat{g}(z, \theta)}u_{p+q}^*(z)$ . The trivial observations  $R_p(c)u_{p+q}^*(z) = \hat{c}(z)u_p^*(z)$  and  $R_q(a)u_{p+q}^*(z) = \hat{a}(z)u_q^*(z)$  then lead to the conclusion that  $\tau(z, \theta) = R(c, -a)(z - G(\theta))^{-1}e$  is exactly the same as in (2.7). The system is obviously controllable. The observability matrix of the system involves products of the form  $R(c, -a)G(\theta)^k$  ( $k = 0, \dots, p + q - 1$ ). In view of Lemma 2.1 these can be written as  $F(\theta)^k R(c, -a)$ , from which the assertion follows. Stability is an immediate consequence of the assumptions on the polynomials  $a$  and  $c$ . Indeed, the characteristic polynomial of  $G(\theta)$  is  $\hat{g}(\cdot, \theta) = \hat{a}(\cdot)\hat{c}(\cdot)$ , which has its zeros inside the unit circle.  $\square$

An alternative (observable) realization of the process  $\dot{\varepsilon}(\theta)$  is given in the next proposition.

**Proposition 2.3.** *The process  $\xi(\theta) = \dot{\varepsilon}(\theta)^\top$  is the state process of the stable system given by*

$$\xi(\theta)_{t+1} = F(\theta)\xi(\theta)_t + B\varepsilon(\theta)_t, \tag{2.11}$$

where  $F(\theta)$  is as in (2.8) and  $B = \begin{pmatrix} e_p \\ -e_q \end{pmatrix}$ . This system is controllable iff  $\hat{a}$  and  $\hat{c}$  have no common zeros.

**Proof.** Again, the proof that this realization produces  $\dot{\varepsilon}(\theta)$  boils down to computing the transfer function, like we did in the proof of Proposition 2.2. The computations needed for this have been encountered there, so we skip them. To explain the statement on controllability, we consider the equation ( $u, v$  are row vectors and  $\lambda$  is an arbitrary complex number)

$$(u, v) \begin{pmatrix} F_a - \lambda & 0 & e \\ 0 & F_c - \lambda & -e \end{pmatrix} = 0,$$

where  $F_a = J - e_p a^\top$  and  $F_c = J - e_q c^\top$ . This equation is equivalent to  $u(F_a - \lambda) = 0$ ,  $v(F_c - \lambda) = 0$  and  $(u - v)e = 0$ . We first consider the case where  $\hat{a}$  and  $\hat{c}$  have no common zeros. Suppose that  $u = 0$ . Then we have that  $v(F_c - \lambda) = 0$  and  $ve = 0$ . Since  $(F_c, e)$  is controllable,  $v$  must be zero as well. Therefore we will assume that there is a nonzero solution  $u$ . Then  $\lambda$  must be a root of  $\hat{a}(z) = 0$ . If  $v = 0$ , then we also have  $ue = 0$ . This situation cannot happen since  $(F_a, e)$  is a controllable pair. So we have to assume that  $v$  is not zero, but then  $\lambda$  is also a root of  $\hat{c}(z) = 0$ . It then follows from the above that this cannot happen. Hence  $(F(\theta), B)$  is controllable. In the other case  $\hat{a}$  and  $\hat{c}$  have a common zero  $\lambda$ . In this case  $u$  is the row vector  $(1, \hat{a}_1(\lambda), \dots, \hat{a}_{p-1}(\lambda))$ , where the  $\hat{a}_i$  are the Hörner polynomials, defined by  $\hat{a}_0(z) = 1, \hat{a}_k(z) = z\hat{a}_{k-1}(z) + a_k$  and we have a similar expression for  $v$ . One obviously then also has  $(u - v)e = 0$  and hence the system is not controllable. Stability follows upon noting that the characteristic polynomial of  $F(\theta)$  is equal to  $\hat{g}(\theta)$ .  $\square$

**Remark 2.4.** By Lemma 2.1, the realization of Proposition 2.3 is connected to the one in Proposition 2.2 in a very simple way. Starting from Eq. (2.10), one obtains

$$\begin{aligned} \xi(\theta)_{t+1} &= R(c, -a)Z_{t+1} \\ &= R(c, -a)(G(\theta)Z_t + e\varepsilon_t(\theta)) \\ &= F(\theta)\xi_t + B\varepsilon_t(\theta). \end{aligned}$$

### 3. The Bezoutian

We follow the notation of Lancaster and Tismenetsky [12]. Recall the following definitions. In this section and henceforth we assume that  $p$  and  $q$  are taken to have a common value, denoted by  $n$ , to yield many of the subsequent expression meaningful. We consider polynomials  $a$  and  $b$  given by  $a(z) = \sum_{k=0}^n a_k z^k$  and  $b(z) = \sum_{k=0}^n b_k z^k$ . We will always assume that the constant term  $a_0 = 1$  and likewise for  $b$  and other polynomials.

Consider the Bezout matrix  $B(a, b)$  of the polynomials  $a$  and  $b$ . It is defined by the relation

$$a(z)b(w) - a(w)b(z) = (z - w)u_n(z)^\top B(a, b)u_n(w).$$

We often refer to this matrix as the Bezoutian. For a given complex number  $\phi$ , we introduce the matrix  $U_\phi$  as follows:

$$U_\phi = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\phi & 1 & & & \vdots \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & 0 & -\phi & 1 \end{pmatrix}.$$

We also need the inverses  $T_\phi$  of the matrices  $U_\phi$ . These take the form

$$T_\phi = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \phi & 1 & 0 & & \vdots \\ \phi^2 & \phi & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \phi^{n-1} & \dots & \phi^2 & \phi & 1 \end{pmatrix}.$$

Observe that matrices  $U_\phi$  and  $U_\psi$  commute, as well as  $T_\phi$  and  $T_\psi$ .

Consider again  $a$  and  $b$ ,  $n$ th order polynomials with constant term equal to 1. Let  $(1 - \alpha_1 z)$  be a factor of  $a(z)$  and  $(1 - \beta_1 z)$  be a factor of  $b(z)$ . Of course,  $\alpha_1$  and  $\beta_1$  are zeros of  $\hat{a}$  and  $\hat{b}$  respectively. Write  $a(z) = (1 - \alpha_1 z)a_{-1}(z)$  and  $b(z) = (1 - \beta_1 z)b_{-1}(z)$ . Continuing this way, for  $\alpha_1, \dots, \alpha_n$  we define recursively  $a_{-(k-1)}(z) = (1 - \alpha_k z)a_{-k}(z)$  and polynomials  $b_{-k}$  similarly. We also put  $a_0(z) = a(z)$  and  $b_0(z) = b(z)$ . The following proposition is not completely necessary for what follows, but may be of independent interest. Its proof, as well as those of the two corollaries that follow have been relegated to Appendix A.

**Proposition 3.1.** *With the above introduced notation we have*

$$\frac{a(z)b(w) - a(w)b(z)}{z - w} = (1 - \alpha_1 z)(1 - \beta_1 w) \frac{a_{-1}(z)b_{-1}(w) - a_{-1}(w)b_{-1}(z)}{z - w} + (\beta_1 - \alpha_1)a_{-1}(w)b_{-1}(z). \tag{3.1}$$

*In terms of the Bezoutian this is equivalent to the (non-symmetric) decomposition*

$$B(a, b) = U_{\alpha_1} \begin{pmatrix} B(a_{-1}, b_{-1}) & 0 \\ 0 & 0 \end{pmatrix} U_{\beta_1}^\top + (\beta_1 - \alpha_1)b_{\beta_1}a_{\alpha_1}^\top \tag{3.2}$$

*with  $a_{\alpha_1}$  such that  $a_{\alpha_1}^\top u_n(z) = a_{-1}(z)$  and  $b_{\beta_1}$  likewise.*

Iteration of this procedure gives

$$\frac{a(z)b(w) - a(w)b(z)}{z - w} = a(z)b(w) \sum_{k=1}^n (\beta_k - \alpha_k) \frac{a_{-k}(w)b_{-k}(z)}{a_{-(k-1)}(z)b_{-(k-1)}(w)}, \tag{3.3}$$

which gives the following expansion for the Bezout matrix

$$B(a, b) = \sum_{k=1}^n (\beta_k - \alpha_k) U_{\alpha_1} \cdots U_{\alpha_{k-1}} U_{\beta_{k+1}} \cdots U_{\beta_n} e e^\top U_{\beta_1}^\top \cdots U_{\beta_{k-1}}^\top U_{\alpha_{k+1}}^\top \cdots U_{\alpha_n}^\top.$$

**Corollary 3.2.** Let  $\phi$  be a common zero of  $\hat{a}$  and  $\hat{b}$ . Then  $a(z) = (1 - \phi z)a_{-1}(z)$  and  $b(z) = (1 - \phi z)b_{-1}(z)$  and

$$B(a, b) = U_\phi \begin{pmatrix} B(a_{-1}, b_{-1}) & 0 \\ 0 & 0 \end{pmatrix} U_\phi^\top. \tag{3.4}$$

It is a well known fact (see [12, Theorem 13.1] or [7, Theorem 8.4.3]) that the Bezout matrix  $B(a, b)$  is non-singular iff  $a$  and  $b$  have no common factors. This also follows from Corollary 3.3 below in which we give a description of the kernel of the Bezout matrix.

**Corollary 3.3.** Let  $\phi_1, \dots, \phi_m$  be all the common zeros of  $\hat{a}$  and  $\hat{b}$ , with multiplicities  $n_1, \dots, n_m$ . Let  $\ell$  be the last basis vector of  $\mathbb{R}^n$  and put

$$v_k^j = (T_{\phi_k}^j J^{j-1})^\top \ell$$

for  $k = 1, \dots, m$  and  $j = 1, \dots, n_k$ . Then  $\ker B(a, b)$  is the linear span of the vectors  $v_k^j$ .

**Remark 3.4.** For more applications of confluent Vandermonde matrices to the analysis of stationary ARMA processes, we refer to [11].

### 4. The Bezoutian and the ARMA polynomials

In this section we continue to study some properties related to the Bezout matrix, which (aimed at applications in Section 6) we express in terms of the ARMA polynomials  $a$  and  $c$  that define the process  $y$  of Eq. (2.1). For a polynomial  $a(z) = \sum_{k=0}^n a_k z^k$  the matrix  $S(a)$  is given by

$$S(a) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & \cdots & a_n & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

and  $S(\hat{a})$  is given by

$$S(\hat{a}) = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_0 \\ a_{n-2} & \cdots & a_0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_0 & 0 & \cdots & 0 \end{pmatrix}.$$

As before we will work with polynomials  $a$  whose constant term  $a_0 = 1$ . Notice that  $S(\hat{a})$  is connected to the reciprocal polynomial  $\hat{a}, \hat{a}(z) = \sum_{k=0}^n a_{n-k} z^k$ , as is  $S(a)$  to  $a$ . Let  $P$  be the



‘anti-diagonal identity’ matrix in  $\mathbb{R}^{n \times n}$ , so with elements  $P_{ij} = \delta_{i,n+1-j}$ . On a Toeplitz matrix  $M$  pre- and postmultiplication by  $P$  results in the same as transposition:  $PM P = M^\top$ . We will use this property mainly for the choice  $M = J$ , the shift matrix.

We continue under the assumption that the polynomials  $\hat{a}$  and  $\hat{c}$  have common degree  $n$  and consistently write  $e$  for  $e_n$ , the first basis vector of  $\mathbb{R}^n$ . One of the possible relations between the Sylvester matrix  $R(c, -a)$  and the Bezoutian  $B(c, a)$  is given below.

**Proposition 4.1.** *The matrices  $R(c, -a)$  and  $B(c, a)$  satisfy*

$$\begin{pmatrix} P & 0 \\ PS(\hat{a})P & PS(\hat{c})P \end{pmatrix} R(c, -a) = \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix} \begin{pmatrix} PS(\hat{c})P & S(c) \\ 0 & I \end{pmatrix}.$$

**Proof.** This relation is just a variant on Eq. (21) on [12, p. 460] and can be proven similarly.  $\square$

We will use the short hand notation

$$M(c, a) = \begin{pmatrix} P & 0 \\ PS(\hat{a})P & PS(\hat{c})P \end{pmatrix} \tag{4.1}$$

and

$$N(c) = \begin{pmatrix} PS(\hat{c})P & S(c) \\ 0 & I \end{pmatrix}. \tag{4.2}$$

Notice that both  $M(c, a)$  and  $N(c)$  are nonsingular if  $a_0 \neq 0$  and  $c_0 \neq 0$  (which is our case, since we always work with  $a_0 = c_0 = 1$ ).

**Theorem 4.2.** *Let  $F(\theta)$ ,  $G(\theta)$ ,  $M(c, a)$  and  $N(c)$  be as in Eqs. (2.8), (2.9), (4.1) and (4.2). The following identities hold true:*

$$M(c, a)G(\theta)M(c, a)^{-1} = \begin{pmatrix} P(J - ea^\top)P & 0 \\ (c - a)e^\top & PJP - ce^\top \end{pmatrix} =: G_M(\theta) \tag{4.3}$$

and

$$N(c)F(\theta)N(c)^{-1} = \begin{pmatrix} P(J - ea^\top)P & 0 \\ ee^\top & J - ec^\top \end{pmatrix} =: F_N(\theta). \tag{4.4}$$

Moreover we have the relation

$$G_M(\theta) \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix} F_N(\theta). \tag{4.5}$$

Before giving the proof of this theorem we formulate a few technical lemmas that will be of use in this proof.

**Lemma 4.3.** *The following two equalities hold true:*

$$PS(\hat{c})P(J - ec^\top) = (PJP - ce^\top)PS(\hat{c})P = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 \\ \vdots & & & c_1 & \vdots \\ 0 & & & \vdots & \vdots \\ 1 & c_1 & \cdots & c_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -c_n \end{pmatrix}.$$

**Proof.** Compare to the analogous statement in [12, p. 455].  $\square$

**Lemma 4.4.** Let  $g(z, \theta) = a(z)c(z) = \sum_{k=0}^{2n} g(\theta)_k z^k$  and  $g(\theta) = (g^1(\theta), g^2(\theta))$ , with  $g^1(\theta) = (g_1(\theta), \dots, g_n(\theta))$  and  $g^2(\theta) = (g_{n+1}(\theta), \dots, g_{2n}(\theta))$ . Then the following identities hold true:

$$\begin{aligned} S(\hat{c})Pe &= e, \\ a^\top S(\hat{c})P &= (g^1(\theta) - c)^\top, \\ S(c)Pa &= g^2(\theta). \end{aligned} \tag{4.6}$$

**Proof.** This is a straightforward verification.  $\square$

Along with the matrices  $S(c)$  and  $S(\hat{c})$  we also use the Hankel matrix  $\tilde{S}(\hat{c}) \in \mathbb{R}^{(n+1) \times (n+1)}$  defined by

$$\tilde{S}(\hat{c}) = \begin{pmatrix} c_n & \cdots & \cdots & c_1 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ c_1 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

**Lemma 4.5.** One has

$$JS(\hat{c}) + ec^\top P = (I \quad 0)\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

In particular the matrix  $JS(\hat{c}) + ec^\top P$  is symmetric.

**Proof.** The following relations are immediate:

$$S(\hat{c}) = (0 \quad I)\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{4.7}$$

and

$$ec^\top P = (e \quad 0)\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{4.8}$$

Use Eqs. (4.7) and (4.8) to write

$$\begin{aligned} JS(\hat{c}) + ec^\top P &= J(0 \quad I)\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix} + (e \quad 0)\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= ((0 \quad J) + (e \quad 0))\tilde{S}(\hat{c}) \begin{pmatrix} I \\ 0 \end{pmatrix}, \end{aligned}$$

from which the result follow.  $\square$

**Proof of Theorem 4.2.** Compute the products  $M(c, a)G(\theta)$  and  $G_M(\theta)M(c, a)$  to get respectively

$$\begin{pmatrix} P(J - ea^\top) & 0 \\ PS(\hat{a})P(J - ea^\top) & PS(\hat{c})P(J - ec^\top) \end{pmatrix} \quad (4.9)$$

and

$$\begin{pmatrix} P(J - ea^\top) & 0 \\ (c - a)e^\top + (PJP - ce^\top)PS(\hat{a})P & (PJP - ce^\top)PS(\hat{c})P \end{pmatrix}. \quad (4.10)$$

Clearly we only have to look at the (2, 1) and (2, 2) blocks in (4.9) and (4.10). Comparing the two (2,2) blocks is just the content of Lemma 4.3, so we now focus on the (2, 1) blocks. Use Lemma 4.3 again to write  $PS(\hat{a})P(J - ea^\top)$  the (2, 1) block of (4.9) as

$$\begin{aligned} (PJP - ae^\top)PS(\hat{a})P &= PJS(\hat{a}) - ae^\top P = PJS(\hat{a}) - ce^\top P + (c - a)e^\top P \\ &= PJS(\hat{a}) - ce^\top PS(\hat{a})P + (c - a)e^\top P, \end{aligned}$$

which is just the (2, 1) block of (4.10). This proves the identity (4.3).

Next we prove (4.4). Write  $K(\theta) = \begin{pmatrix} J - eg^1(\theta)^\top & -eg^2(\theta)^\top \\ ee^\top P & J \end{pmatrix}$ . Work out the products  $N(c)K(\theta)$  and  $F_N(\theta)N(c)$  to get respectively

$$\begin{pmatrix} PS(\hat{c})P(J - eg^1(\theta)^\top) + S(c)ee^\top P & -PS(\hat{c})Peg^2(\theta)^\top + S(c)J \\ ee^\top P & J \end{pmatrix} \quad (4.11)$$

and

$$\begin{pmatrix} P(J - ea^\top)S(\hat{c})P & P(J - ea^\top)PS(c) \\ ee^\top PS(\hat{c})P & ee^\top S(c) + J - ec^\top \end{pmatrix}. \quad (4.12)$$

Compare now the corresponding blocks in these two matrices. We start with the (1, 1) block of (4.11). Write it as

$$P(S(\hat{c})PJP + Pce^\top)P - Peg^1(\theta)^\top = P(S(\hat{c})J^\top + Pce^\top)P - Peg^1(\theta)^\top$$

and use the symmetry asserted in Lemma 4.5 to get

$$\begin{aligned} P(JS(\hat{c}) + ec^\top)P - Peg^1(\theta)^\top &= PJS(\hat{c})P + e(c^\top - g^1(\theta)^\top) \\ &= PJS(\hat{c})P - Pea^\top S(\hat{c})P, \end{aligned}$$

which equals the (1, 1) block of (4.12).

Next we consider the (1, 2) blocks. Start with (4.11):

$$-PS(\hat{c})Peg^2(\theta)^\top + S(c)J = -Peg^2(\theta)^\top + S(c)J = -Pea^\top PS(c) + S(c)J,$$

where the last equality just follows from (4.6). Since  $S(c)J$  is symmetric it is equal to  $J^\top S(c) = PJPS(c)$ . Hence

$$-Pea^\top PS(c) + S(c)J = P(J - ea^\top)PS(c),$$

which is equal to the (1,2) block of (4.12). Comparison of the other blocks is trivial.

Finally we prove (4.5). Recall from Lemma 2.1 that  $G(\theta)R(c, -a) = R(c, -a)F(\theta)$ . Write  $\bar{B}(\theta)$  for  $\begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix}$ . Then we have the string of equalities

$$\begin{aligned} G_M(\theta)\overline{B}(\theta) &= G_M(\theta)M(c, a)R(c, -a)N(c)^{-1} = M(c, a)G(\theta)R(c, -a)N(c)^{-1} \\ &= M(c, a)R(c, -a)F(\theta)N(c)^{-1} = \overline{B}(\theta)N(c)F(\theta)N(c)^{-1} = \overline{B}(\theta)F_N(\theta). \end{aligned}$$

This proves the last assertion of the theorem.  $\square$

### 5. Stein equations

We start this section with considering two Stein equations that involve the matrices  $F_M(\theta)$  and  $G_N(\theta)$  of Eqs. (4.3) and (4.4).

**Proposition 5.1.** *Let  $e_P^\top = [e^\top P, 0]^\top \in \mathbb{R}^{2n}$  and let  $H(\theta)$  and  $Q(\theta)$  be the unique solutions to the following Stein equations:*

$$H = G_M(\theta)HG_M(\theta)^\top + e_P e_P^\top, \tag{5.1}$$

$$Q = F_N(\theta)QF_N(\theta)^\top + e_P e_P^\top. \tag{5.2}$$

Then  $Q(\theta)$  is strictly positive definite. Moreover,  $H(\theta)$  and  $Q(\theta)$  are related by

$$H(\theta) = \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix} Q(\theta) \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix}. \tag{5.3}$$

**Proof.** To show that  $Q(\theta)$  is strictly positive definite, it is sufficient to show that the pair  $(F_N(\theta), e_P)$  is controllable. Let  $T = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$ . For computational reasons it is more convenient to show controllability of the pair  $(A(\theta), b)$ , where  $A(\theta) = TF_N(\theta)T^{-1}$  and  $b = TF_N(\theta)e_P$ . Observe that  $b = e_{2n}$ , the first standard basis vector in  $\mathbb{R}^{2n}$ , whereas  $A(\theta) = \begin{pmatrix} J - ea^\top P & 0 \\ ee^\top P & J - ec^\top \end{pmatrix}$ . If one computes the controllability matrix  $(b, A(\theta)b, \dots, A^{2n-1}(\theta)b)$ , then standard calculations lead to the fact that this matrix is upper triangular with only ones on the diagonal. Hence it has full rank. By Theorem 8d.66 of [5] the matrix  $Q(\theta)$  is strictly positive definite.

Multiply Eq. (5.2) with  $Q = Q(\theta)$  on the right and on the left by the symmetric matrix  $T(\theta) = \begin{pmatrix} I & 0 \\ 0 & B(c, a) \end{pmatrix}$  and put  $H(\theta) = T(\theta)Q(\theta)T(\theta)$ . In view of relation (4.5)  $H(\theta)$  satisfies Eq. (5.1) and is thus the unique solution to this equation. This shows (5.3).  $\square$

**Corollary 5.2.** *The matrix  $H(\theta)$  is non-singular iff the polynomials  $a$  and  $c$  have no common factors.*

**Proof.** The matrix  $Q(\theta)$  is non-singular and  $B(c, a)$  is singular iff the polynomials  $a$  and  $c$  have no common factors.  $\square$

**Remark 5.3.** If the polynomials  $a$  and  $c$  have a common factor  $(1 - \phi z)$ , then the expression for  $B(c, a)$  of Eq. (3.4) can be applied to obtain a rank factorization of  $H(\theta)$ .

Along with the matrices  $H(\theta)$  and  $Q(\theta)$  of Proposition 5.1 we will also work with the matrices  $\mathcal{I}(\theta)$  and  $\mathcal{P}(\theta)$ , defined in

$$\mathcal{I}(\theta) = M(c, a)^{-1}H(\theta)M(c, a)^{-\top}, \tag{5.4}$$

$$\mathcal{P}(\theta) = N(c)^{-\top}Q(\theta)N(c)^{-1}, \tag{5.5}$$

where  $M(c, a)$  and  $N(c)$  are as in (4.1) and (4.2). In view of Eqs. (4.3) and (4.4) and Proposition 5.1, we have that  $\mathcal{J}(\theta)$  and  $\mathcal{P}(\theta)$  are the (unique) solutions to the Stein equations

$$\begin{aligned} \mathcal{J} &= F(\theta)\mathcal{J}F(\theta)^\top + BB^\top, \\ \mathcal{P} &= G(\theta)\mathcal{P}G(\theta)^\top + e_{2n}e_{2n}^\top. \end{aligned} \tag{5.6}$$

**Corollary 5.4.** *The matrix  $\mathcal{P}(\theta)$  is non-singular and the matrix  $\mathcal{J}(\theta)$  is non-singular iff the polynomials  $a$  and  $c$  have no common factors.*

**Proof.** Since the matrices  $M(c, a)$  and  $N(c)$  are non-singular, the results follow from Proposition 5.1 and Corollary 5.2.  $\square$

### 6. Fisher’s information matrix

We turn back to the ARMA process  $y$  defined by (2.1) and the Fisher information matrix. Recall Eq. (2.4) for the computation of the matrix  $\mathcal{J}_n(\theta)$ . Remember that for  $\dot{\varepsilon}_t(\theta)$  we have the realizations of Propositions 2.2 and 2.3. For  $i_t(\theta) := \mathbb{E}_\theta \dot{\varepsilon}_t(\theta)^\top \dot{\varepsilon}_t(\theta)$  we get from Eq. (2.11) and the independence of  $\varepsilon_t(\theta)$  and  $\dot{\varepsilon}_t(\theta)$  the recursion

$$i_{t+1}(\theta) = F(\theta)i_t(\theta)F(\theta)^\top + \sigma^2 BB^\top. \tag{6.1}$$

Because of stationarity we have  $i_{t+1}(\theta) = i_t(\theta)$  for all  $t$ . Hence  $\mathcal{J}_n(\theta) = \frac{n}{\sigma^2} \mathbb{E}_\theta i_1(\theta)$ . We call

$$\frac{1}{\sigma^2} \mathbb{E}_\theta i_1(\theta) \tag{6.2}$$

the *asymptotic Fisher information matrix*. Summing up intermediate results, we obtain the following theorem.

**Theorem 6.1.** *The asymptotic Fisher information matrix of the ARMA process defined by (2.1) is the same as the matrix  $\mathcal{J}(\theta)$  defined in Eq. (5.4). Hence it is the unique solution to the Stein equation (5.6) and thus independent of  $\sigma^2$ . Moreover this matrix is non-singular iff the polynomials  $a$  and  $c$  have no common factors.*

**Proof.** From Eqs. (6.1) and (6.2) and the stationarity assumption, one immediately sees that the asymptotic Fisher information matrix satisfies (5.6). Hence it is equal to  $\mathcal{J}(\theta)$  and the characterization of non-singularity is nothing else but Corollary 5.4.  $\square$

The conclusion of this theorem has been proved in [10] by different means, involving representations of the Fisher information matrix as an integral in the complex plane and the following lemma of which we give an alternative proof.

**Lemma 6.2.** *Let  $\mathcal{J}(\theta)$  be the Fisher information matrix and  $\mathcal{P}(\theta)$  as in (5.5). Then the following factorization holds:*

$$\mathcal{J}(\theta) = R(c, -a)\mathcal{P}(\theta)R(c, -a)^\top. \tag{6.3}$$

**Proof.** This follows from Proposition 4.1, combined with Eqs. (5.3), (5.4) and (5.5).  $\square$

Since the matrix  $\mathcal{P}(\theta)$  is non-singular, also Lemma 6.2 illustrates the fact that  $\mathcal{I}(\theta)$  is non-singular iff  $a$  and  $c$  have no common factors. Moreover, looking at Eq. (5.6), we see that  $\mathcal{I}(\theta)$  is non-singular iff the pair  $(F(\theta), B)$  is controllable. But the controllability matrices  $\mathcal{R}(F(\theta), B)$  and  $\mathcal{R}(G(\theta), e)$  satisfy the easily verified relation  $\mathcal{R}(F(\theta), B) = R(c, -a)\mathcal{R}(G(\theta), e)$ . Since the matrix  $\mathcal{R}(G(\theta), e)$  has full rank, we see that  $(F(\theta), e)$  is controllable iff  $R(c, -a)$  is non-singular, which leads to another way of showing the conclusion of Theorem 6.1.

**Appendix A. Proofs of Section 3**

**Proof of Proposition 3.1.** Eq. (3.1) follows from elementary computations. To prove (3.2) we premultiply both sides of the equation by  $u_n(z)^\top$  and postmultiply them by  $u_n(w)$ . The obtained left hand side then is obviously equal to the left hand side of (3.1). To show that the right hand sides coincide one uses that  $u_n(z)^\top U_{\alpha_1} = (1 - \alpha_1 z)(u_{n-1}(z)^\top, 0) + (0, \dots, 0, z^{n-1})$ . Then the assertion easily follows from the definition of  $B(a_{-1}, b_{-1})$ . To prove the other assertions, we proceed as follows. First we show how the right hand sides of Eqs. (3.3) and (3.1) are related. We pre-multiply the right hand side of (3.1) by  $u_n(z)^\top$ . The important key relation is

$$u_n(z)^\top U_{\alpha_1} \cdots U_{\alpha_{k-1}} U_{\beta_{k+1}} \cdots U_{\beta_n} e = \prod_{j=1}^{k-1} (1 - \alpha_j z) \prod_{j=k+1}^n (1 - \beta_j z),$$

which is easily shown to be true. Of course the right hand side of this equation is nothing else but

$$\frac{a(z)}{a_{-(k-1)}(z)} b_{-k}(z).$$

Then post-multiplication of the obtained expression by  $u_n(w)$  obviously results in the right hand side of (3.3).

We now show by induction that this is equal to  $u_n(z)^\top B(a, b)u_n(w)$ . Let  $A(z) = (1 - \alpha_0 z)a(z)$ ,  $B(z) = (1 - \beta_0 z)b(z)$ . Define

$$A_{-k}(z) = \frac{A(z)}{\prod_{j=0}^k (1 - \alpha_j z)}$$

and define  $B_{-k}(z)$  likewise ( $k = 0, \dots, n$ ). We also let  $A_1(z) = A(z)$  and  $B_1(z) = B(z)$ . We will use the following trivial identities. For  $k = 1, \dots, n$  we have  $A_{-k}(z) = a_{-k}(z)$  and  $B_{-k}(z) = b_{-k}(z)$ .

$$\begin{aligned} A(z)B(w) \sum_{k=0}^n (\beta_k - \alpha_k) \frac{A_{-k}(w)B_{-k}(z)}{A_{-(k-1)}(z)B_{-(k-1)}(w)} \\ = (\beta_0 - \alpha_0)a(w)b(z) + (1 - \alpha_0 z)(1 - \beta_0 w)a(z)b(w) \sum_{k=1}^n (\beta_k - \alpha_k) \frac{a_{-k}(w)b_{-k}(z)}{a_{-(k-1)}(z)b_{-(k-1)}(w)}. \end{aligned}$$

In view of (3.1) and the induction assumption, this equals  $\frac{A(z)B(w) - A(w)B(z)}{z - w}$ . The proposition has been proved.  $\square$

**Proof of Corollary 3.2.** This is a straightforward consequence of Proposition 3.1.  $\square$

**Proof of Corollary 3.3.** First we have to show that the vectors  $v_k^j$  are independent. Explicit computation of these vectors show that, after multiplication with  $P$ , they are columns of the confluent

Vandermonde matrix associated with all zeros of  $\hat{a}$ , from which independence then follows. For  $j = 1$ , it follows immediately from Corollary 3.2 that the  $v_k^1$  belong to the kernel of the Bezout matrix. When  $\phi_k$  is common zero with multiplicity  $j > 1$  we can factor the matrix  $B(a_{-1}, b_{-1})$  in (3.4) like  $B(a, b)$ , but with one dimension less. However, one can then show that also

$$B(a, b) = U_{\phi_k}^2 \begin{pmatrix} B(a_{-2}, b_{-2}) & 0 \\ 0 & 0 \end{pmatrix} (U_{\phi_k}^\top)^2,$$

where for instance the 0-matrix in the lower right corner now has size  $2 \times 2$ . Continuation of this procedure leads to

$$B(a, b) = U_{\phi_k}^j \begin{pmatrix} B(a_{-j}, b_{-j}) & 0 \\ 0 & 0 \end{pmatrix} (U_{\phi_k}^\top)^j$$

for  $j = 1, \dots, n_k$ . Since the last  $j$  columns of  $B(a, b)$  are thus zero vectors, one immediately sees that  $B(a, b)v_k^j = 0$ . The proof is complete upon noticing that the kernel of the Bezout matrix has dimension equal to  $n_1 + \dots + n_m$  (cf [7, Theorem 8.4.3]).  $\square$

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## References

- [1] K.J. Åström, T. Söderström, Uniqueness of the maximum likelihood estimates of the parameters of an ARMA model, *IEEE Trans. Automat. Control* 19 (1974) 769–773.
- [2] S. Barnett, A new formulation of the theorems of Hurwitz, Routh and Sturm, *J. Inst. Math. Appl.* 8 (1971) 240–250.
- [3] S. Barnett, Matrices, polynomials and linear time invariant systems, *IEEE Trans. Automat. Control* 18 (1973) 1–10.
- [4] P.J. Brockwell, R.A. Davis, *Time Series: Theory and Methods*, Springer, 1991.
- [5] F.M. Callier, C.A. Desoer, *Linear System Theory*, Springer, 1991.
- [6] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1951.
- [7] P.A. Fuhrmann, *A Polynomial Approach to Linear Algebra*, Springer, 1996.
- [8] R.E. Kalman, Mathematical description of linear dynamical systems, *SIAM J. Contr.* 1 (1963) 152–192.
- [9] A. Klein, P.J.C. Spreij, On Fisher's information matrix of an ARMAX process and Sylvester's resultant matrices, *Linear Algebra Appl.* 237/238 (1996) 579–590.
- [10] A. Klein, P.J.C. Spreij, On Fisher's information matrix of an ARMA process, in: I. Csiszar, Gy. Michaletzky (Eds.), *Stochastic Differential and Difference Equations*, Birkhäuser, 1997, pp. 273–284.
- [11] A. Klein, P.J.C. Spreij, Some results on Vandermonde matrices with an application to time series analysis, *SIAM J. Matrix Anal.* 25 (1) (2003) 213–223.
- [12] P. Lancaster, M. Tismenetsky, *The Theory of Matrices*, Academic Press, 1985.
- [13] A.I. McLeod, A note on ARMA model redundancy, *J. Time Series Anal.* 14 (2) (1993) 207–208.
- [14] B.M. Pötscher, The behaviour of the Lagrangian multiplier test in testing the orders of an ARMA model, *Metrika* 32 (1985) 129–150.
- [15] C.R. Rao, *Linear Statistical Inference and its Applications*, Wiley, 1965.
- [16] T.J. Rothenberg, Identification in parametric models, *Econometrica* 39 (1971) 577–591.
- [17] T. Söderström, P. Stoica, *System Identification*, Prentice-Hall, 1989.
- [18] B.L. van der Waerden, *Algebra I*, Springer, 1966.