A Structural Credit Risk Model with Default Contagion

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Abstract. Structural threshold models are common industry practice for modelling portfolio credit risk, but often only consider default dependence via underlying common factors. We consider a structural model extension that allows for additionally incorporating default contagion effects. A simulation study illustrates that ignoring default contagion effects may lead to significant underestimation of portfolio tail risk. As a key contribution, we propose a procedure for estimating default contagion parameters from historical default probability data.

Keywords: portfolio credit risk · default contagion · structural model

1 Introduction

The dependence between default events of obligors is a key aspect of portfolio credit risk management. A common approach in practice is to use a structural threshold model (see [5, p. 465]) in which a default event of an obligor is triggered by a latent value process reaching some threshold. The value processes of different obligors are then often assumed to be conditionally independent given underlying common factors, such as macroeconomic or industry-specific risk drivers.

Another important form of dependence may however arise due to default contagion effects, in which an increase in default risk of one obligor directly causes an increase in default risk of another obligor. In corporate parent-subsidiary relationships, for example, increased default risk can propagate from a parent company to a subsidiary. As another example, increased default risk of a sovereign issuer may propagate to entities operating in the same country.

We propose a structural threshold model that incorporates both indirect default dependence via underlying common factors, as well as direct default contagion effects. The model specifically allows for the special case where the default of one obligor guarantees the default of another, but also allows default risk to partially propagate from or to multiple different obligors. As a key contribution, we outline a procedure to estimate the contagion parameters from default

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probability data. Once calibrated, the model can be easily used for simulation of portfolio losses, similar to the structural threshold models used in practice. The combination of these desirable properties distinguishes the model from previous proposed default contagion extensions of the threshold model (see e.g. [1,3,6])[1].

Based on a simulation study, we illustrate that ignoring default contagion effects may cause significant underestimation of portfolio tail risk. This risk is relatively well captured by using estimated default contagion parameters.

2 Structural Model with Default Contagion

We consider a credit portfolio with obligors indexed by $\mathcal{I} = \{1, \dots, N\}$ that may default only at the end of some specified time horizon T, e.g. in 1 year. Following a structural approach, we consider a (latent) joint value process $\mathbf{V} = [V_i]_{i \in \mathcal{I}}$ and say that an obligor has defaulted when its value process is non-positive:

$$\{\text{Default of obligor } i \text{ at time } T\} := \{V_i(T) \leq 0\}, \ \forall i \in \mathcal{I}. \tag{1}$$

An increase in default risk, is thus represented by a decrease in value.

The joint value process $\mathbf{V}$ specifies both marginal default probabilities as well as default dependence between obligors, including possible default contagion effects. As is common in portfolio credit risk modelling, we focus on modelling the dependence between defaults. To simulate the distribution of 1-year portfolio losses, we therefore assume that the vector of marginal 1-year default probabilities $\hat{\mathbf{PD}} = [\hat{\mathrm{PD}}_i]_{i \in \mathcal{I}}$ is given. For the estimation of default contagion parameters, we assume the availability of the historical time-series $\{\hat{\mathbf{PD}}(t_m)\}_{m=0}^{M}$.

In practice, (estimated) default probabilities may for example be provided by rating agencies or internal bank models. Alternatively, default probabilities may be inferred from market data such as credit spreads. We assume that default contagion effects are already incorporated into these default probabilities.

We additionally assume that we already know the binary structure of the default contagion dependence: i.e. between which obligors there exists such direct dependence. This binary structure is indeed obvious in many practical applications[2]. The strength of the dependencies remains to be estimated.

2.1 Base Model

We first consider $\mathbf{V} = \mathbf{Y}$, where $\mathbf{Y} = [Y_i]_{i \in \mathcal{I}}$ represents the intrinsic value process of all obligors and is assumed to be a correlated N-dimensional Brownian motion:

$$\mathrm{d}\mathbf{Y}(t) = \mathbf{\Sigma}^{\frac{1}{2}} \mathrm{d}\mathbf{B}(t), \ \mathbf{Y}(0) = \mathbf{y} \in \mathbb{R}^N, \tag{2}$$

where $\mathbf{B}$ is an N-dimensional Wiener process and $\mathbf{\Sigma}$ is a correlation matrix[3].

[1] In particular, earlier proposed calibration of default contagion parameters often relies on expert input or on strong ad hoc assumptions.

[2] For example in corporate parent-subsidiary relationships. Alternatively, the binary dependence structure may be identified using a network-based approach, see e.g. [2].

[3] That is, the process is scaled such that $\Sigma_{ii} = 1$ for all $i \in \mathcal{I}$.

In the base model, the marginal default probability for $i \in \mathcal{I}$ can be computed as

$$\mathrm{PD}_i = \mathbb{P}\left[Y_i(T) \leq 0 \;\middle|\; Y_i(0) = y_i\right] = \Phi\left(-\frac{y_i}{\sqrt{T}}\right) \,, \tag{3}$$

where Φ is the standard Gaussian CDF. Conversely, given a default probability $\hat{\mathrm{PD}}_i \in (0, 1)$, we can solve for the corresponding starting point $\hat{y}_i$ to obtain

$$\hat{y}_i = -\sqrt{T} \cdot \Phi^{-1}\left(\hat{\mathrm{PD}}_i\right). \tag{4}$$

We note that the base model effectively corresponds to a Gaussian threshold model, similar to multi-factor extensions of the Merton model that are popular in industry [5, p. 430]. We assume here that the correlation matrix $\mathbf{\Sigma}$ is known or has already been calibrated by using such a model. The distribution of losses can then be simulated when the default probability vector $\hat{\mathbf{PD}}$ is also given.

2.2 Default Contagion Extension

We propose a structural extension of the base model in which default risk can propagate via a parent structure represented by a weights matrix $\mathbf{W} = \left[W_{ij}\right]_{i,j\in\mathcal{I}}$ with non-negative entries and row-sums equal to 1. We say $j \in \mathcal{I}$ is a parent of child $i \in \mathcal{I}$ when $W_{Ij} > 0$ and $i \neq j$. We say that obligor i has no parents when $W_{ii} = 1$. We assume the existence of a partition $\mathcal{I} = \{\mathcal{I}_P, \mathcal{I}_C\}$, where all obligors without parents are in $\mathcal{I}_P$ and the child obligors in $\mathcal{I}_C$ only have parents in $\mathcal{I}_P$[4].

In the proposed extended model, the value $\mathbf{V}$ is the (element-wise) minimum of the intrinsic value $\mathbf{Y}$ and the propagated value $\mathbf{X} = [X_i]_{i\in\mathcal{I}}$. The intrinsic value $\mathbf{Y}$ is modelled as in (2) and the propagated value $\mathbf{X}$ is modelled as a convex combination of the intrinsic value and the value of parents:

$$\mathbf{V}(t) = \min\left[\mathbf{Y}(t), \mathbf{X}(t)\right], \;\; \mathbf{X}(t) = \mathbf{W}\mathbf{Y}(t). \tag{5}$$

A decrease in value V_i of an obligor i may be caused by either a decrease in intrinsic value Y_i or by a decrease in value of its parents propagated via X_i. Similarly, the default event as defined in (1) can be triggered by a decrease in either intrinsic or propagated value:

$$\{\text{Default of obligor } i \text{ at time } T\} = \{Y_i(T) \leq 0\} \cup \{X_i(T) \leq 0\}\,, \;\; \forall i \in \mathcal{I}. \tag{6}$$

We highlight two special cases that can be captured by this model:

1. In the special case that no obligor has parents, i.e. $\mathcal{I} = \mathcal{I}_P$ and $\mathbf{W} = \mathbf{I}$, the model reduces to the base model. This is because each obligor $i \in \mathcal{I}_P$ without parents has $W_{Ii} = 1$ and therefore value $V_i = \min\left[Y_i, Y_i\right] = Y_i$.
2. If obligor i has one parent j with weight $W_{ij} = 1$, then $V_i \leq X_i = Y_j = V_j$, so that a default event of parent j implies a default event of obligor i.

[4] This imposes a restriction where it is not allowed for parents to have also parents themselves and is similar to the primary-secondary structure assumed in [4].

In the extended model, we have

$$\begin{bmatrix}\mathbf{Y}(T)\\ \mathbf{X}(T)\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mathbf{y}\\ \mathbf{W}\mathbf{y}\end{bmatrix},\ T\cdot\begin{bmatrix}\mathbf{\Sigma} & \mathbf{\Sigma}\mathbf{W}^T\\ \mathbf{W}\mathbf{\Sigma} & \mathbf{W}\mathbf{\Sigma}\mathbf{W}^T\end{bmatrix}\right), \tag{7}$$

so that the marginal default probability of an obligor $i \in \mathcal{I}$ can be computed as

$$\begin{aligned}\mathrm{PD}_i\big(\mathbf{y};\mathbf{W}\big) &= \mathbb{P}\left[Y_i(T) \le 0 \ \text{ or } X_i(T) \le 0 \ \middle|\ \mathbf{Y}(0) = \mathbf{y}\right]\\ &= \Phi\left(-\frac{y_i}{\sqrt{T}}\right) + \Phi\left(-\frac{x_i}{\sigma_{X_i}\sqrt{T}}\right) - \Phi_2^{\rho_i}\left(-\frac{y_i}{\sqrt{T}},\ -\frac{x_i}{\sigma_{X_i}\sqrt{T}}\right),\end{aligned} \tag{8}$$

where $x_i = [\mathbf{W}\mathbf{y}]_i$, $\sigma_{X_i} = \sqrt{[\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T]_{ii}}$ and $\Phi_2^{\rho_i}(\cdot,\cdot)$ is the bi-variate Gaussian CDF with correlation $\rho_i = [\mathbf{W}\mathbf{\Sigma}]_{ii}/\sigma_{X_i}$. For obligor $i \in \mathcal{I}_P$ without parents, Eq. (8) indeed simplifies to as in the base model in Eq. (3).

Given a weights matrix $\mathbf{W}$, correlation matrix $\mathbf{\Sigma}$ and default probability vector $\hat{\mathbf{PD}}$, we want to find the corresponding starting point $\hat{\mathbf{y}}$ that satisfies $\mathrm{PD}_i\big(\hat{\mathbf{y}};\mathbf{W}\big) = \hat{\mathrm{PD}}_i$ for all $i \in \mathcal{I}$. We can first find $\hat{\mathbf{y}}_P := [\hat{y}_i]_{i\in\mathcal{I}_P}$ by using the result in (4). Then given $\hat{\mathbf{y}}_P$, we can numerically solve for the remaining $\hat{\mathbf{y}}_C := [\hat{y}_i]_{i\in\mathcal{I}_C}$ by using the result in (8). The latter is possible because in (8) PD_i is strictly decreasing in y_i and has range equal to $(0, 1)$ when $W_{ii} > 0$.[5]

The distribution of portfolio losses can be simulated when the weights matrix $\mathbf{W}$, correlation matrix $\mathbf{\Sigma}$ and default probability vector $\hat{\mathbf{PD}}$ are estimated or given. This is analogous to the base model, where $\mathbf{W} = \mathbf{I}$.

3 Estimation of the Weights Matrix W

Intuitively, we propose to estimate the weights matrix $\mathbf{W}$ by using that the value processes of the obligors should have correlation $\mathbf{\Sigma}$ after filtering out default contagion effects. We therefore consider a moment condition on the normalized discrete-time increments of the intrinsic value process $\mathbf{Y}$ as defined in (2):

$$\mathbb{E}\left[\mathbf{Z}(t_m)\mathbf{Z}(t_m)^T\right] = \mathbf{\Sigma}, \quad \text{for } \ \mathbf{Z}(t_m) := \frac{\mathbf{Y}(t_m) - \mathbf{Y}(t_{m-1})}{\sqrt{t_m - t_{m-1}}}. \tag{9}$$

Equation (9) implies the following moment condition for the subcomponents $\mathbf{Z}_C := [Z_i]_{i\in\mathcal{I}_C}, \mathbf{Z}_P := [Z_i]_{I\in\mathcal{I}_P}$ of obligors with and without parents respectively:

$$\mathbb{E}\left[\mathbf{Z}_C(t_m)\mathbf{Z}_P(t_m)^T\right] = \mathbf{\Sigma}_{CP}, \ \text{ where } \ \mathbf{\Sigma}_{CP} := [\Sigma_{ij}]_{i\in\mathcal{I}_C, j\in\mathcal{I}_P}. \tag{10}$$

Although the increments $\{\mathbf{Z}(t_m)\}_{m=1}^M$ are not directly observed, we can use $\{\hat{\mathbf{PD}}(t_m)\}_{m=0}^M$ to compute the inferred increments $\{\hat{\mathbf{Z}}(t_m)\}_{m=1}^M$ for a given $\mathbf{W}$:

$$\hat{\mathbf{Z}}(t_m;\mathbf{W}) := \frac{\hat{\mathbf{y}}(t_m) - \hat{\mathbf{y}}(t_{m-1})}{\sqrt{t_m - t_{m-1}}}, \text{where } \ \mathbf{PD}\big(\hat{\mathbf{y}}(t_m), \mathbf{W}\big) \equiv \hat{\mathbf{PD}}(t_m). \tag{11}$$

[5] Note that PD_i has lower bound $\lim\limits_{y_i\to\infty} \mathrm{PD}_i(\mathbf{y};\mathbf{W}) = \Phi\big(-\frac{x_i}{\sqrt{T}\sigma_{X_i}}\big)$ when $W_{ii} = 0$, since the probability of default triggered by propagated value X_i is then independent of intrinsic value y_i. For example, we have $\mathrm{PD}_i > \mathrm{PD}_j$ when $W_{ij} = 1$.

We can thus estimate $\mathbf{W}$ by (numerically) minimizing the loss function $L(\mathbf{W})$, which is based on the Frobenius norm of the difference between $\mathbf{\Sigma}_{CP}$ and the sample correlation of the inferred value increments:

$$L(\mathbf{W}) := \left\| \mathrm{Corr}\left(\hat{\mathbf{Z}}_C(t_m; \mathbf{W}), \hat{\mathbf{Z}}_P(t_m) \right) - \mathbf{\Sigma}_{CP} \right\|_F, \text{ for } \begin{array}{l} \hat{\mathbf{Z}}_C := [\hat{Z}_i]_{i \in \mathcal{I}_C} \\ \hat{\mathbf{Z}}_P := [\hat{Z}_i]_{i \in \mathcal{I}_P} \end{array}. \quad (12)$$

We note two advantages of the loss function in (12) for numerical optimization. First, each row $\mathbf{W}_i := [W_{ij}]_{j \in \mathcal{I}}$ can be estimated separately for each $i \in \mathcal{I}_C$, as each row affects a different element of $\hat{\mathbf{Z}}_C$. So, if each child only has a few parents, only a few weights have to be estimated per row. Second, we again note that the weights corresponding to obligors without parents $i \in \mathcal{I}_P$ are equal to $W_{ii} = 1$. This means that the increments $\hat{\mathbf{Z}}_P$ have to be computed only once.

4 Numerical Example and Possible Extensions

To illustrate the possible impact of default contagion effects on aggregate portfolio risk, we consider a numerical example. We first simulate portfolio losses using the extended model in (5) for a randomly drawn weights matrix $\mathbf{W}_{\text{true}}$. We then also simulate losses with estimated weights $\mathbf{W}_{\text{est}}$ and naive weights $\mathbf{W}_{\text{naive}} = \mathbf{I}$, to assess the impact on the expected losses and Value-at-Risk (VaR).

We consider a portfolio with $N = 800$ obligors, partitioned into a set of obligors without parents $\mathcal{I}_P = \{1, \ldots, 400\}$ and a set of obligors $\mathcal{I}_C = \{401, \ldots, 800\}$ that all have the same two parents $\{1, 2\}$. We choose an exposure of 100 for all obligors and default probabilities of 0.005, 0.008 for the obligors in $\mathcal{I}_P$ and $\mathcal{I}_C$ respectively. We also choose a correlation matrix $\mathbf{\Sigma}$ with off-diagonal entries equal to 0.3, corresponding to a 1-factor model with equal factor loadings for all obligors. Finally, for each child $i \in \mathcal{I}_C$, we fix randomly drawn weights to both parents in $\{1, 2\}$, independent of all other obligors. The weights are drawn from uniform distributions, such that $W_{i1} \sim \mathrm{Unif}(0, 1)$ and $W_{i2} \sim \mathrm{Unif}(0, 1 - W_{i1})$.

For estimating $\mathbf{W}$, we simulate $M = 250$ daily observations of the default probability vector, corresponding to roughly 1 year of trading days. We then minimize the loss function in (12) by an initial grid search, which is followed by a numerical gradient-based optimization. This combination mitigates the issue of possible local minima and is not too computationally intensive given that only two parameters have to be estimated per row of $\mathbf{W}$.

Figure 1 shows the simulated (relative) expected losses (EL) and VaR for different quantiles for the three different weights matrices, based on 10^7 simulations. For easier visual comparison, all figures are relative to the corresponding figure for the true weights $\mathbf{W}_{\text{true}}$. The results show that the simulated expected losses are very close for all three weights matrices. This is as expected, because the expected losses depend only on the marginal default probabilities to which the model is calibrated by construction. The results also show that the VaR at higher quantiles is significantly underestimated when ignoring default contagion effects (i.e. using when $\mathbf{W}_{\text{naive}}$). In contrast, the results indicate that using the estimated $\mathbf{W}_{\text{est}}$ allows for estimating the VaR reasonably well.

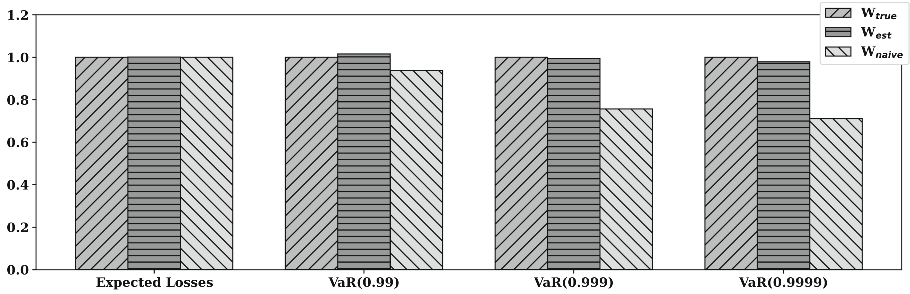

Fig. 1. Simulated expected losses and VaR(q) for $q = 0.99, 0.999, 0.9999$

We note that although the aggregate portfolio risk is estimated reasonable well, the estimation of the weights is relatively volatile on an obligor level. Also, although daily historical default probability data may be inferred from market data such as credit spreads, data provided by rating agencies or internal bank models will likely be less frequent. It may therefore be worthwhile to further improve the efficiency of the proposed estimation procedure: for example by explicitly incorporating data from the underlying factor model. Nevertheless, we have illustrated that it is possible to estimate default contagion parameters from historical default probability data, overcoming the reliance on expert input.

An interesting extension of the proposed model would be to allow parents to also have parents themselves. In that case, chain-like propagation of default contagion effects can be incorporated. In such an extension, the propagated value in (5) could for example become $\mathbf{X}(t) = \mathbf{WV}(t)$. However, the results derived in e.g. (7) and (8) would then take a less simple form. More generally, other functional forms can be considered for the default contagion model extension. For example, using a maximum instead of a minimum in (5) could represent the case where one obligor acts as a guarantor for another obligor.

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