MARKOV-MODULATED ORNSTEIN-UHLENBECK PROCESSES
Author(s): G. HUANG, H. M. JANSEN, M. MANDJES, P. SPREIJ and K. DE TURCK
Source: *Advances in Applied Probability*, MARCH 2016, Vol. 48, No. 1 (MARCH 2016), pp. 235-254
Published by: Applied Probability Trust
Stable URL: https://www.jstor.org/stable/43859568

Adv. Appl. Prob. **48,** 235–254 (2016)
*doi:*10.1017/apr.2015.15

MARKOV-MODULATED ORNSTEIN–UHLENBECK PROCESSES

G. HUANG,* ** *University of Amsterdam*
H. M. JANSEN,* *** *University of Amsterdam and Ghent University*
M. MANDJES,* **** *University of Amsterdam, CWI and EURANDOM*
P. SPREIJ,* ***** *University of Amsterdam*
K. DE TURCK,****** *Ghent University*

Abstract

In this paper we consider an Ornstein–Uhlenbeck (OU) process $(M(t))_{t\geq 0}$ whose parameters are determined by an external Markov process $(X(t))_{t\geq 0}$ on a finite state space $\{1,\dots,d\}$; this process is usually referred to as *Markov-modulated Ornstein–Uhlenbeck*. We use stochastic integration theory to determine explicit expressions for the mean and variance of $M(t)$. Then we establish a system of partial differential equations (PDEs) for the Laplace transform of $M(t)$ and the state $X(t)$ of the background process, jointly for time epochs $t = t_1,\dots,t_K$. Then we use this PDE to set up a recursion that yields all moments of $M(t)$ and its stationary counterpart; we also find an expression for the covariance between $M(t)$ and $M(t+u)$. We then establish a functional central limit theorem for $M(t)$ for the situation that certain parameters of the underlying OU processes are scaled, in combination with the modulating Markov process being accelerated; interestingly, specific scalings lead to drastically different limiting processes. We conclude the paper by considering the situation of a single Markov process modulating *multiple* OU processes.

Keywords: Ornstein–Uhlenbeck process; Markov modulation; regime switching; central limit theorems; martingale techniques

2010 Mathematics Subject Classification: Primary 60K25
Secondary 60G44; 60G15

1. Introduction

The Ornstein–Uhlenbeck (OU) process is a stationary Markov–Gauss process, with the additional feature that it eventually reverts to its long-term mean; see the seminal paper [37], as well as [26] for a historic account. Having originated from physics, the process has now found widespread use in a broad range of other application domains: finance, population dynamics, climate modeling, etc. In addition, it plays an important role in queueing theory, as it can be seen as the limiting process of specific classes of infinite-server queues under a certain

Received 6 January 2015; revision received 25 February 2015.
* Postal address: Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, The Netherlands.
** Email address: g.huang@uva.nl
*** Email address: h.m.jansen@uva.nl
**** Email address: m.r.h.mandjes@uva.nl
***** Email address: p.j.c.spreij@uva.nl
***** Postal address: TELIN, Ghent University, St.-Pietersnieuwstraat 41, B9000 Gent, Belgium.
Email address: kdeturck@telin.ugent.be

scaling [34]. The OU process is characterized by three parameters (which we call α, γ, and σ^2 throughout this paper), which relate to the process mean, convergence speed towards the mean, and variance, respectively.

The probabilistic properties of the OU process have been thoroughly studied. One of the key results is that its value at a given time t has a normal distribution, with a mean and variance that can be expressed explicitly in terms of the parameters α, γ, and σ^2 of the underlying OU process; see, e.g. [26, Equation (2)]. In addition, various other quantities have been analyzed, such as the distribution of first-passage times or the maximum value attained in an interval of given length; see, e.g. [1] and the references therein.

The concept of *regime switching* (or *Markov modulation*, as it is usually referred to in the operations research literature) has become increasingly important over the past decades. In regime switching, the parameters of the underlying stochastic process are determined by an external *background process* (or *modulating process*) that is typically assumed to evolve independently of the stochastic process under consideration. Often the background process is assumed to be a Markov chain defined on a *finite* state space, say $\{1, \dots, d\}$; in the context of a Markov-modulated OU (MMOU), this means that when this Markov chain is in state i, the process locally behaves as an OU process with parameters α_i, γ_i, and σ_i^2.

Owing to its various attractive features, regime switching has become an increasingly popular concept. In a broad spectrum of application domains it offers a natural framework for modeling situations in which the stochastic process under study reacts to an autonomously evolving environment. In finance, one could identify the background process with the 'state of the economy', for instance as a two-state process (that is, alternating between a 'good' and a 'bad' state) to which, e.g. asset prices react. Likewise, in wireless networks the concept can be used to model the channel conditions that vary in time, and to which users react.

In the operations research literature there is a sizeable body of work on Markov-modulated queues; see, e.g. [4, Chapter XI] and [31], while Markov modulation has also been intensively used in insurance and risk theory [5]. In the financial economics literature the use of regime switching dates back to at least the late 1980s [22]; various specific models have been considered since then; see, e.g. [3], [15], and [16].

In this paper we present a set of new results in the context of the analysis of MMOU. Here, and in the sequel, we let $M(t)$ denote the position of the MMOU process at time t, whereas M denotes its stationary counterpart (the existence of which follows from [39, Theorem 3.1]). In the first place we derive explicit equations for the mean and variance of $M(t)$ and M, jointly with the state of the background process, relying on standard machinery from stochastic integration theory. In various special cases the resulting equations simplify drastically (for instance, when it is assumed that the background process starts off in equilibrium at time 0, or when the parameters γ_i are assumed uniform across the states $i \in \{1, \dots, d\}$).

The second contribution concerns the derivation of a system of PDEs for the Laplace transform of $M(t)$; when equating to 0 the partial derivative with respect to time we obtain a system of ordinary differential equations for the Laplace transform of M. This result is directly related to [39, Theorem 3.2], with the differences being that there the focus was on just stationary behavior, and that the system considered there had the additional feature of reflection at a lower boundary (to avoid the process attaining negative values). We set up a recursive procedure that generates all moments of $M(t)$; in each iteration a nonhomogeneous system of differential equations needs to be solved. This procedure complements the recursion for the moments of the steady-state quantity M, as presented in [39, Corollary 3.1] (in which each recursion step amounts to solving a system of linear equations). In addition, we also set up a system of PDEs for the Laplace transform associated with the joint distribution of $M(t_1), \dots, M(t_K)$, and determine the covariance $\mathrm{cov}(M(t, t+u))$.

A third contribution concerns the behavior of the MMOU process under certain parameter scalings.

- A first scaling that we consider concerns speeding up the jumps of the background process by a factor N. Using the system of PDEs that we derived earlier, it is shown that the limiting process, obtained by sending $N \to \infty$, is an *ordinary* (that is, nonmodulated) OU process, with parameters that are time averages of the individual α_i, γ_i, and σ_i^2.
- A second regime that we consider scales the transition rates of the Markovian background process by N, while the α_i and σ_i^2 are inflated by a factor N^h for some $h > 0$; the resulting process we call $M^{[N,h]}(t)$. We then center (subtract the mean, which is roughly proportional to N^h) and normalize $M^{[N,h]}(t)$, with the goal of establishing a central limit theorem (CLT). Interestingly, the appropriate normalization depends on the value of h. If $h < 1$ the variance of $M^{[N,h]}(t)$ is roughly proportional to the 'scale' at which the modulated OU process operates, namely N^h, and as a consequence the normalization looks like $N^{h/2}$; at an intuitive level, the timescale of the background process is so fast that the process essentially looks like an OU process with time-averaged parameters. If, on the contrary, $h > 1$ then the variance of $M^{[N,h]}(t)$ grows like N^{2h-1}, which is faster than N^h; as a consequence, the proper normalization looks like $N^{h-1/2}$; in this case the variance that appears in the CLT is directly related to the deviation matrix [13] associated with the background process. Importantly, we do not just prove normality for a given value of $t > 0$, but rather weak convergence (at the process level, that is) to the solution of a specific limiting stochastic differential equation.

The last contribution focuses on the situation that a single Markovian background process modulates *multiple* OU processes. This, for instance, models the situation in which different asset prices react to the same 'external circumstances' (that is, state of the economy), or the situation in which different users of a wireless network react to the same channel conditions. The probabilistic behavior of the system is captured through a system of PDEs. It is also pointed out how the corresponding moments can be found.

Importantly, there is a strong similarity between the results presented in the framework of this paper and corresponding results for Markov-modulated *infinite-server queues*. In these systems the background process modulates an M/M/∞ queue, meaning that we consider an M/M/∞ queue of which the arrival rate and service rate are determined by the state of the background process [14], [18]. For these systems, the counterparts of our MMOU results have been established: the mean and variance have been computed in, e.g. [12], [32], (partial) differential equations for the Laplace transform of $M(t)$, as well as recursions for higher moments can be found in [9], [12], [32], whereas parameter scaling results are given in [9], [12], and, for a slightly different model, in [10]. Roughly speaking, any property that can be handled explicitly for the Markov-modulated infinite-server queue can be explicitly addressed for MMOU as well, and vice versa.

This paper is organized as follows. In Section 2 we define the model and present preliminary results. Then Section 3 deals with the system's transient behavior in terms of a recursive scheme that yields all moments of $M(t)$, with explicit expressions for the mean and variance. In Section 4 we present a system of PDEs for the Laplace transform of $M(t)$. In Section 5 the parameter scalings mentioned above are applied (resulting in a process $M^{(N)}(t)$), leading to a functional CLT for an appropriately centered and normalized version of $M^{(N)}(t)$. The last section considers the setting of a single background process modulating multiple OU processes.

2. Model and preliminaries

We start by giving a detailed model description of the MMOU process. We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a random variable M_0, a standard Brownian motion $(B(t))_{t\geq 0}$, and an irreducible continuous-time Markov process $(X(t))_{t\geq 0}$ with finite state space are defined. It is assumed that M_0, X, and B are independent. The process X is the so-called background process; its state space is denoted by $\{1, \ldots, d\}$.

The idea behind MMOU is that the background process $X(\cdot)$ modulates an OU process. Intuitively, this means that while $X(\cdot)$ is in state $i \in \{1, \ldots, d\}$, the MMOU process $(M(t))_{t\geq 0}$ behaves as an Ornstein–Uhlenbeck process $U_i(\cdot)$ with parameters α_i, γ_i, and σ_i, which evolves independently of the background process $X(\cdot)$; it is assumed that for all i, $\sigma_i > 0$, and $\gamma_i \geq 0$ (with at least for one i a *strict* inequality). More formally, $M(\cdot)$ obeys the stochastic differential equation (SDE)

$$\mathrm{d}M(t) = (\alpha_{X(t)} - \gamma_{X(t)} M(t))\,\mathrm{d}t + \sigma_{X(t)}\,\mathrm{d}B(t). \tag{2.1}$$

We call a stochastic process $(M(t))_{t\geq 0}$ an MMOU process with initial condition $M(0) = M_0$ if

$$M(t) = M_0 + \int_0^t (\alpha_{X(s)} - \gamma_{X(s)} M(s))\,\mathrm{d}s + \int_0^t \sigma_{X(s)}\,\mathrm{d}B(s).$$

The following theorem provides basic facts about the existence, uniqueness, and distribution of an MMOU process. For proofs and additional details; see [25, Section A]. As mentioned in the introduction, specific aspects of MMOU have been studied earlier in the literature; see, e.g. [39].

Theorem 2.1. *Define* $\Gamma(t) := \int_0^t \gamma_{X(s)}\,\mathrm{d}s$. *Then the stochastic process* $(M(t))_{t\geq 0}$ *given by*

$$M(t) = M_0 \mathrm{e}^{-\Gamma(t)} + \int_0^t \mathrm{e}^{-(\Gamma(t)-\Gamma(s))} \alpha_{X(s)}\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(\Gamma(t)-\Gamma(s))} \sigma_{X(s)}\,\mathrm{d}B(s)$$

is the unique MMOU process with initial condition M_0. *Conditional on the process* X, *the random variable* $M(t)$ *has a normal distribution with random mean*

$$\mu(t) = M_0 \exp(-\Gamma(t)) + \int_0^t \exp(-(\Gamma(t) - \Gamma(s)))\alpha_{X(s)}\,\mathrm{d}s$$

and random variance

$$v(t) = \int_0^t \exp(-2(\Gamma(t) - \Gamma(s)))\sigma^2_{X(s)}\,\mathrm{d}s.$$

This result is analogous with the corresponding result for the Markov-modulated infinite-server queue in [12] and [14]: there it was shown that the number of jobs in the system has a Poisson distribution with random parameter.

For later use, we now recall some concepts pertaining to the theory of deviation matrices of Markov processes. For an introduction to this topic, we refer the reader to standard texts such as [29], [30], and [36]. For a compact survey, see [13]. Let the transition rates corresponding to the continuous-time Markov chain $(X(t))_{t\geq 0}$ be given by $q_{ij} \geq 0$ for $i \neq j$ and $q_i := -q_{ii} := \sum_{j\neq i} q_{ij}$; they define the *intensity matrix* or *generator* Q. The (unique) invariant distribution corresponding to Q is denoted by (the column vector) $\boldsymbol{\pi}$, that is, it obeys $\boldsymbol{\pi}^\top Q = \mathbf{0}$ and $\boldsymbol{e}^\top \boldsymbol{\pi} = 1$, where $\boldsymbol{e}$ is a d-dimensional all-1s vector.

Let $\Pi := \boldsymbol{e}\boldsymbol{\pi}^\top$ denote the *ergodic matrix*. Then the *fundamental matrix* is given by $F := (\Pi - Q)^{-1}$, whereas the *deviation matrix* is defined by $D := F - \Pi$. Standard identities are $QF = FQ = \Pi - I$, as well as $\Pi D = D\Pi = 0$ (here 0 is to be read as an all-0s $d \times d$ matrix) and $F\boldsymbol{e} = \boldsymbol{e}$. The (i, j)th entry of the deviation matrix, with $i, j \in \{1, \dots, d\}$, can be alternatively computed as $D_{ij} = \int_0^\infty (p_{ij}(t) - \pi_j)\,\mathrm{d}t$, where $p_{ij}(t) := \mathbb{P}(X(t) = j \mid X(0) = i)$, which in matrix form can be expressed as

$$D = \int_0^\infty (\exp(Qt) - \boldsymbol{e}\boldsymbol{\pi}^\top)\,\mathrm{d}t.$$

3. Transient behavior: moments

In this section we analyze the moments of $M(t)$ using stochastic integration theory. First considering the mean and variance in the general situation, we then concentrate on more specific cases in which the expressions simplify greatly (that is, equal γ_is with $X(t)$ starting off in equilibrium at time 0, and in the steady-state regime). The section is completed by deriving an expression for the covariance between $M(t)$ and $M(t + u)$ (for $t, u \geq 0$), and an Itô-based recursive procedure to determine all moments.

3.1. Mean and variance: general case

Let $\boldsymbol{Z}(t) \in \{0, 1\}^d$ be the vector of indicator functions associated with the Markov chain $(X(t))_{t \geq 0}$, that is, we let $Z_i(t) = 1$ if $X(t) = i$ and 0 otherwise. Let $\boldsymbol{p}_t$ denote the vector of transient probabilities of the background process, that is, $(\mathbb{P}(X(t) = 1), \dots, \mathbb{P}(X(t) = d))^\top$ (where we have not specified the distribution of the initial state $X(0)$ yet). We subsequently find expressions for the mean $\mu_t := \mathbb{E}M(t)$ and variance $v_t := \operatorname{var} M(t)$.

- The mean can be computed as follows. We consider the mean of $M(t)$ jointly with the state of the background process at time t. To this end, we define $\boldsymbol{Y}(t) := \boldsymbol{Z}(t)M(t)$, and $\boldsymbol{\nu}_t := \mathbb{E}\boldsymbol{Y}(t)$. It is clear that

$$\mathrm{d}\boldsymbol{Z}(t) = Q^\top \boldsymbol{Z}(t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{K}(t) \tag{3.1}$$

for a d-dimensional martingale $\boldsymbol{K}(t)$. With Itô's rule we obtain, with the strictly diagonally dominant, and hence invertible [23, Theorem 6.2.27]), matrix $\bar{Q}_{\boldsymbol{\gamma}}$ defined by $Q^\top - \operatorname{diag}\{\boldsymbol{\gamma}\}$,

$$\begin{aligned}\mathrm{d}\boldsymbol{Y}(t) &= M(t)(Q^\top \boldsymbol{Z}(t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{K}(t)) + \boldsymbol{Z}(t)((\boldsymbol{\alpha}^\top \boldsymbol{Z}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Y}(t))\,\mathrm{d}t + \boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t)) \\ &= (\bar{Q}_{\boldsymbol{\gamma}}\boldsymbol{Y}(t) + \operatorname{diag}(\boldsymbol{\alpha})\boldsymbol{Z}(t))\,\mathrm{d}t + \operatorname{diag}\{\boldsymbol{\sigma}\}\boldsymbol{Z}(t)\,\mathrm{d}B(t) + M(t)\,\mathrm{d}\boldsymbol{K}(t); \end{aligned} \tag{3.2}$$

here we use identities such as $\boldsymbol{Z}(t)\boldsymbol{\gamma}^\top \boldsymbol{Y}(t) = \operatorname{diag}\{\boldsymbol{\gamma}\}\boldsymbol{Y}(t)$ and $\boldsymbol{Z}(t)\boldsymbol{\sigma}^\top \boldsymbol{Z}(t) = \operatorname{diag}\{\boldsymbol{\sigma}\}\boldsymbol{Z}(t)$, which follow due to $Z_i(t) = (Z_i(t))^2$ and $Z_i(t)Z_j(t) = 0$ for $i \neq j$. Taking expectations of both sides in (3.2), we obtain the system $\boldsymbol{\nu}_t' = \bar{Q}_{\boldsymbol{\gamma}}\boldsymbol{\nu}_t + \operatorname{diag}\{\boldsymbol{\alpha}\}\boldsymbol{p}_t$. This is a nonhomogeneous linear system of differential equations that is solved by

$$\boldsymbol{\nu}_t = \mathrm{e}^{\bar{Q}_{\boldsymbol{\gamma}}t}\boldsymbol{\nu}_0 + \int_0^t \mathrm{e}^{\bar{Q}_{\boldsymbol{\gamma}}(t-s)}\operatorname{diag}\{\boldsymbol{\alpha}\}\boldsymbol{p}_s\,\mathrm{d}s;$$

then $\mu_t = \boldsymbol{e}^\top \boldsymbol{\nu}_t$. Realize that $\boldsymbol{\nu}_0 = m_0\boldsymbol{p}_0$, as we assumed that $M(0)$ equals m_0.
The equations simplify drastically if the background process starts off in equilibrium at time 0; then evidently $\boldsymbol{p}_t = \boldsymbol{\pi}$ for all $t \geq 0$. As a result, we find that $\boldsymbol{\nu}_t = \mathrm{e}^{\bar{Q}_{\boldsymbol{\gamma}}t}\boldsymbol{\nu}_0 - \bar{Q}_{\boldsymbol{\gamma}}^{-1}(I - \mathrm{e}^{\bar{Q}_{\boldsymbol{\gamma}}t})\operatorname{diag}\{\boldsymbol{\alpha}\}\boldsymbol{\pi}$.

We now consider the steady-state regime (that is, $t \to \infty$). From the above expressions, it immediately follows that

$$\boldsymbol{v}_\infty = -\bar{Q}_{\boldsymbol{\gamma}}^{-1}\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{\pi}, \qquad \mu_\infty = \boldsymbol{e}^\top \boldsymbol{v}_\infty = -\boldsymbol{e}^\top \bar{Q}_{\boldsymbol{\gamma}}^{-1}\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{\pi}.$$

We further note that $\boldsymbol{\gamma} = -(Q - \mathrm{diag}\{\boldsymbol{\gamma}\})\boldsymbol{e}$, and, hence, $\boldsymbol{\gamma}^\top \bar{Q}_{\boldsymbol{\gamma}}^{-1} = -\boldsymbol{e}^\top$, so that $\boldsymbol{\gamma}^\top \boldsymbol{v}_\infty = \boldsymbol{\pi}^\top \boldsymbol{\alpha}$.

- The variance can be found in a similar way. Define $\bar{\boldsymbol{Y}}(t) := \boldsymbol{Z}(t)M^2(t)$, and $\boldsymbol{w}_t := \mathbb{E}\bar{\boldsymbol{Y}}(t)$. Now our starting point is the relation

$$\mathrm{d}(M(t) - \mu_t) = (\boldsymbol{\alpha}^\top(\boldsymbol{Z}(t) - \boldsymbol{p}_t) - \boldsymbol{\gamma}^\top(\boldsymbol{Y}(t) - \boldsymbol{v}_t))\,\mathrm{d}t + \boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t),$$

so that

$$\begin{aligned}\mathrm{d}(M(t) - \mu_t)^2 &= 2(M(t) - \mu_t)(\boldsymbol{\alpha}^\top(\boldsymbol{Z}(t) - \boldsymbol{p}_t) - \boldsymbol{\gamma}^\top(\boldsymbol{Y}(t) - \boldsymbol{v}_t))\,\mathrm{d}t\\ &\qquad + 2(M(t) - \mu_t)\boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t) + \boldsymbol{\sigma}^\top \mathrm{diag}\{\boldsymbol{Z}(t)\}\boldsymbol{\sigma}\,\mathrm{d}t.\end{aligned}$$

Taking expectations of both sides, we obtain

$$v_t' = 2\boldsymbol{\alpha}^\top \boldsymbol{v}_t - 2\mu_t \boldsymbol{\alpha}^\top \boldsymbol{p}_t - 2\boldsymbol{\gamma}^\top \boldsymbol{w}_t + 2\mu_t \boldsymbol{\gamma}^\top \boldsymbol{v}_t + \boldsymbol{\sigma}^\top \mathrm{diag}\{\boldsymbol{p}_t\}\boldsymbol{\sigma}.$$

Clearly, to evaluate this expression, we first need to identify $\boldsymbol{w}_t$. To this end, we set up an equation for $\mathrm{d}\bar{\boldsymbol{Y}}(t)$ as before, take expectations so as to obtain

$$\boldsymbol{w}_t' = \bar{Q}_{2\boldsymbol{\gamma}}\boldsymbol{w}_t + 2\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{v}_t + \mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{p}_t;$$

here $\boldsymbol{\sigma}^2$ is the vector $(\sigma_1^2, \ldots, \sigma_d^2)^\top$. This leads to

$$\boldsymbol{w}_t = \mathrm{e}^{\bar{Q}_{2\boldsymbol{\gamma}}t}\boldsymbol{w}_0 + \int_0^t \mathrm{e}^{\bar{Q}_{2\boldsymbol{\gamma}}(t-s)}(2\,\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{v}_s + \mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{p}_s)\,\mathrm{d}s, \tag{3.3}$$

so that $v_t = \boldsymbol{e}^\top \boldsymbol{w}_t - \mu_t^2$. Observe that $\boldsymbol{w}_0 = m_0^2 \boldsymbol{p}_0$.
Again, simplifications can be made if $\boldsymbol{p}_0 = \boldsymbol{\pi}$ (and hence $\boldsymbol{p}_t = \boldsymbol{\pi}$ for all $t \geq 0$). In that case, we had already found an expression for $\boldsymbol{v}_s$ above, and as a result (3.3) can be explicitly evaluated.
For the stationary situation ($t \to \infty$, that is), we obtain

$$\boldsymbol{w}_\infty = -\bar{Q}_{2\boldsymbol{\gamma}}^{-1}(2\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{v}_\infty + \mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{\pi}), \qquad v_\infty = \boldsymbol{e}^\top \boldsymbol{w}_\infty - \mu_\infty^2.$$

We consider now an even more special case: $\gamma_i \equiv \gamma$ for all i (in addition to $\boldsymbol{p}_t = \boldsymbol{\pi}$; we let $t \geq 0$). It is directly seen that $\mu_\infty = \boldsymbol{\pi}^\top \boldsymbol{\alpha}/\gamma$. Since $\boldsymbol{\gamma}^\top \bar{Q}_{\boldsymbol{\gamma}}^{-1} = -\boldsymbol{e}^\top$ implies that $\boldsymbol{e}^\top \bar{Q}_{\delta \boldsymbol{e}}^{-1} = -\delta^{-1}\boldsymbol{e}^\top$ for any $\delta > 0$, it follows that

$$\begin{aligned}v_\infty &= \boldsymbol{e}^\top \boldsymbol{w}_\infty - \mu_\infty^2\\ &= \frac{\boldsymbol{e}^\top \mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{v}_\infty}{\gamma} + \frac{\boldsymbol{\pi}^\top \boldsymbol{\sigma}^2}{2\gamma} - \left(\frac{\boldsymbol{\pi}^\top \boldsymbol{\alpha}}{\gamma}\right)^2\\ &= -\frac{\boldsymbol{e}^\top \mathrm{diag}\{\boldsymbol{\alpha}\}\bar{Q}_{\gamma \boldsymbol{e}}^{-1}\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{\pi}}{\gamma} + \frac{\boldsymbol{\pi}^\top \boldsymbol{\sigma}^2}{2\gamma} - \left(\frac{\boldsymbol{\pi}^\top \boldsymbol{\alpha}}{\gamma}\right)^2.\end{aligned}$$

Now observe that, with $\check{D}_{ij}(\gamma) := \int_0^\infty p_{ij}(v)\mathrm{e}^{-\gamma v}\,\mathrm{d}v$ for $\gamma > 0$, integration by parts yields

$$\begin{aligned} Q\check{D}(\gamma) &= \int_0^\infty QP(v)\mathrm{e}^{-\gamma v}\,\mathrm{d}v \\ &= \int_0^\infty P'(v)\mathrm{e}^{-\gamma v}\,\mathrm{d}v \\ &= -I + \int_0^\infty \gamma P(v)\mathrm{e}^{-\gamma v}\,\mathrm{d}v \\ &= -I + \gamma\check{D}(\gamma). \end{aligned}$$

As a consequence, $-(Q-\gamma I)\check{D}(\gamma) = I$, so that

$$v_\infty = \frac{\boldsymbol{\pi}^\top\boldsymbol{\sigma}^2}{2\gamma} + \frac{1}{\gamma}\boldsymbol{\alpha}^\top\mathrm{diag}\{\boldsymbol{\pi}\}\check{D}(\gamma)\boldsymbol{\alpha} = \left(\frac{\boldsymbol{\pi}^\top\boldsymbol{\alpha}}{\gamma}\right)^2\frac{\boldsymbol{\pi}^\top\boldsymbol{\sigma}^2}{2\gamma} + \frac{1}{\gamma}\boldsymbol{\alpha}^\top\mathrm{diag}\{\boldsymbol{\pi}\}D(\gamma)\boldsymbol{\alpha}, \tag{3.4}$$

using $D_{ij}(\gamma) := \int_0^\infty (p_{ij}(v) - \pi_j)\mathrm{e}^{-\gamma v}\,\mathrm{d}v = \check{D}_{ij}(\gamma) - \pi_j/\gamma$. In the next section we further study the case in which the γ_is are equal, that is, $\gamma_i \equiv \gamma$, and the background process is in a steady state at time 0, that is, and $\boldsymbol{p}_t = \boldsymbol{\pi}$. As it turns out, under these conditions the mean and variance can also be found by an alternative elementary, insightful argument.

3.2. Mean and variance: special case of equal γ, starting in equilibrium

We now consider the special case $\gamma_i \equiv \gamma$ for all i, while $X(t)$ is assumed to start off in equilibrium at time 0 (that is, $\mathbb{P}(X(t) = i) = \mathbb{P}(X(0) = i) = \pi_i$ for all $t \geq 0$), allowing an explicit evaluation of μ_t and v_t, particularly when specific scalings are imposed (one of which plays a key role later in the paper).

We first concentrate on computing the transient mean μ_t. We denote by X the path $(X(s), s \in [0,t])$. Now using the representation of Theorem 2.1 and recalling the standard fact that μ_t can be written as $\mathbb{E}(\mathbb{E}(M(t) \mid X))$, it is immediately seen that μ_t can be written as a convex mixture of m_0 and $\boldsymbol{\pi}^\top\boldsymbol{\alpha}/\gamma$:

$$\mu_t = m_0\mathrm{e}^{-\gamma t} + \mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}\,\mathrm{d}s\left(\sum_{i=1}^d \pi_i\alpha_i\right) = m_0\mathrm{e}^{-\gamma t} + \frac{\boldsymbol{\pi}^\top\boldsymbol{\alpha}}{\gamma}(1-\mathrm{e}^{-\gamma t});$$

using the fact that $(X(t))_{t\geq 0}$ started off in equilibrium at time 0. This expression converges as $t \to \infty$, to the stationary mean $\boldsymbol{\pi}^\top\boldsymbol{\alpha}/\gamma$, as expected.

The variance v_t can be computed similarly, relying on the so-called *law of total variance*, which says that $\mathrm{var}\, M(t) = \mathbb{E}(\mathrm{var}(M(t) \mid X)) + \mathrm{var}(\mathbb{E}(M(t) \mid X))$. Regarding the first term, it is seen that Theorem 2.1 directly yields that $\mathbb{E}(\mathrm{var}(M(t) \mid X))$ can be expressed as

$$\mathbb{E}\left(\int_0^t \mathrm{e}^{-2\gamma(t-s)}\sigma^2_{X(s)}\,\mathrm{d}s\right) = \int_0^t \mathrm{e}^{-2\gamma(t-s)}\mathbb{E}(\sigma^2_{X(s)})\,\mathrm{d}s = \sum_{i=1}^d \pi_i\sigma_i^2\left(\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\right).$$

Along similar lines, $\mathrm{var}(\mathbb{E}(M(t) \mid X))$ can be expressed as

$$\begin{aligned} \mathrm{var}\left(\int_0^t \mathrm{e}^{-\gamma(t-s)}\alpha_{X(s)}\,\mathrm{d}s\right) &= \int_0^t\int_0^t \mathrm{cov}(\mathrm{e}^{-\gamma(t-s)}\alpha_{X(s)}, \mathrm{e}^{-\gamma(t-u)}\alpha_{X(u)})\,\mathrm{d}u\,\mathrm{d}s \\ &= \mathrm{e}^{-2\gamma t}\int_0^t\int_0^t \mathrm{e}^{\gamma(s+u)}\,\mathrm{cov}(\alpha_{X(s)}, \alpha_{X(u)})\,\mathrm{d}u\,\mathrm{d}s. \end{aligned}$$

The latter integral expression can be made more explicit. Using the fact that $X(0)$ has distribution $\boldsymbol{\pi}$, we have

$$
\begin{aligned}
&2\mathrm{e}^{-2\gamma t}\int_0^t\int_0^s \mathrm{e}^{\gamma(s+u)}\operatorname{cov}(\alpha_{X(s)},\alpha_{X(u)})\,\mathrm{d}u\,\mathrm{d}s\\
&\quad= 2\mathrm{e}^{-2\gamma t}\int_0^t\int_0^s \mathrm{e}^{\gamma(s+u)}\sum_{i=1}^d\sum_{j=1}^d \alpha_i\alpha_j\pi_i(p_{ij}(s-u)-\pi_j)\,\mathrm{d}u\,\mathrm{d}s\\
&\quad= \frac{1}{\gamma}\sum_{i=1}^d\sum_{j=1}^d \alpha_i\alpha_j\int_0^t (\mathrm{e}^{-\gamma v}-\mathrm{e}^{-\gamma(2t-v)})\pi_i(p_{ij}(v)-\pi_j)\,\mathrm{d}v
\end{aligned}
$$

(where the last equation follows after changing the order of integration and some elementary calculus). We have thus found that

$$
v_t = \sum_{i=1}^d \pi_i\sigma_i^2\bigg(\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\bigg) + \sum_{i=1}^d\sum_{j=1}^d \alpha_i\alpha_j\int_0^t\bigg(\frac{\mathrm{e}^{-\gamma v}-\mathrm{e}^{-\gamma(2t-v)}}{\gamma}\bigg)\pi_i(p_{ij}(v)-\pi_j)\,\mathrm{d}v.
$$

We conclude this section by considering two specific limiting regimes, to which we return in Section 5, where we will derive limit distributions under parameter scalings.

- Specializing to the situation that $t\to\infty$, we recover, after some algebra, (3.4).
- Scale $\boldsymbol{\alpha}\mapsto N^h\boldsymbol{\alpha}$, $\boldsymbol{\sigma}^2\mapsto N^h\boldsymbol{\sigma}^2$, and $Q\mapsto NQ$ for some $h\geq 0$. We find that $\operatorname{var} M(t)$ can be expressed as

$$
\begin{aligned}
&N^h\sum_{i=1}^d \pi_i\sigma_i^2\bigg(\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\bigg)\\
&\quad+ N^{2h}\sum_{i=1}^d\sum_{j=1}^d \alpha_i\alpha_j\int_0^t\bigg(\frac{\mathrm{e}^{-\gamma v}-\mathrm{e}^{-\gamma(2t-v)}}{\gamma}\bigg)\pi_i(p_{ij}(vN)-\pi_j)\,\mathrm{d}v,
\end{aligned}
$$

which for large N behaves as, with $D:=D(0)$, the deviation matrix introduced in Section 2,

$$
\begin{aligned}
&\bigg(\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\bigg)\bigg(N^h\sum_{i=1}^d \pi_i\sigma_i^2 + 2N^{2h-1}\sum_{i=1}^d\sum_{j=1}^d \alpha_i\alpha_j\pi_i D_{ij}\bigg)\\
&\qquad= \bigg(\frac{1-\mathrm{e}^{-2\gamma t}}{2\gamma}\bigg)(N^h\boldsymbol{\pi}^\top\boldsymbol{\sigma}^2 + 2N^{2h-1}\boldsymbol{\alpha}^\top\operatorname{diag}\{\boldsymbol{\pi}\}D\boldsymbol{\alpha}).
\end{aligned}
\tag{3.5}
$$

We observe an interesting dichotomy: for $h<1$ the variance is essentially linear in the 'scale' of the OU processes N^h, while for $h>1$ it behaves superlinearly in N^h (more specifically, proportionally to N^{2h-1}). It is this dichotomy that also featured in an earlier work on Markov-modulated infinite-server queues [2], [9], and that we further explore in Section 5.

The intuition behind the dichotomy is the following. If $h<1$ then the timescale of the background process systematically exceeds that of the d underlying OU processes (that is, the background process is 'faster'). As a result, the system essentially behaves as an *ordinary* (that is, nonmodulated) OU process with 'time-averaged' parameters $\alpha_\infty := \boldsymbol{\pi}^\top\boldsymbol{\alpha}$, γ, and $\sigma_\infty^2 := \boldsymbol{\pi}^\top\boldsymbol{\sigma}^2$. If $h>1$, on the contrary the background process jumps at a slow rate, relative to the typical timescale of the OU processes; as a result, the process $(M(t))_{t\geq 0}$ moves between multiple local limits (where the individual 'variance coefficients' σ_i^2 do not play a role).

Note that it follows from (3.5) that $\mathrm{diag}\{\boldsymbol{\pi}\}D$ is a nonnegative definite matrix, although singular and nonsymmetric in general; more precisely, it is a consequence of the fact that (3.5) is a variance and hence nonnegative, in conjunction with the fact the we can pick $\boldsymbol{\sigma}^2 = \mathbf{0}$. Below we state and prove the nonnegativity by independent arguments; cf. [2, Proposition 3.2].

Proposition 3.1. *The matrix* $D^\top\mathrm{diag}\{\boldsymbol{\pi}\} + \mathrm{diag}\{\boldsymbol{\pi}\}D$ *is symmetric and nonnegative definite.*

Proof. First we prove the claim that the matrix $Q^\top\mathrm{diag}\{\boldsymbol{\pi}\} + \mathrm{diag}\{\boldsymbol{\pi}\}Q$ is (symmetric and) nonpositive definite. To that end we start from the semimartingale decomposition (3.1) for $\boldsymbol{Z}$. By the product rule we obtain, collecting all the martingale terms in $\mathrm{d}M(t)$,

$$\mathrm{d}(\boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top) = Q^\top\boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top\,\mathrm{d}t + \boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top Q\,\mathrm{d}t + \mathrm{d}\langle\boldsymbol{Z}\rangle_t + \mathrm{d}M(t).$$

As the predictable quadratic variation of $\boldsymbol{Z}$ is absolutely continuous and increasing, we can write $\mathrm{d}\langle\boldsymbol{Z}\rangle_t = P_t\,\mathrm{d}t$, where P_t is a nonnegative definite matrix. Next we make the obvious observation that $\boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top = \mathrm{diag}\{\boldsymbol{Z}(t)\}$. Hence, by combining (3.1) and the above display, we have

$$\mathrm{diag}\{Q^\top\boldsymbol{Z}(t)\} = Q^\top\mathrm{diag}\{\boldsymbol{Z}(t)\} + \mathrm{diag}\{\boldsymbol{Z}(t)\}Q + P_t.$$

Taking expectations with respect to the stationary distribution of $\boldsymbol{Z}_t$ and using $Q^\top\pi = 0$, we obtain $0 = Q^\top\mathrm{diag}\{\boldsymbol{\pi}\} + \mathrm{diag}\{\boldsymbol{\pi}\}Q + \mathbb{E}P_t$, from which it follows that $Q^\top\mathrm{diag}\{\boldsymbol{\pi}\} + \mathrm{diag}\{\boldsymbol{\pi}\}Q$ is (symmetric and) nonpositive definite.

This in turn implies that $-D^\top(Q^\top\mathrm{diag}\{\boldsymbol{\pi}\} + \mathrm{diag}\{\boldsymbol{\pi}\}Q)D$ is symmetric and nonnegative definite. Recall now that $FQ = \Pi - I$ and, hence, $DQ = \Pi - I$. Then $D^\top Q^\top\mathrm{diag}\{\boldsymbol{\pi}\}D = -(\mathrm{diag}\{\boldsymbol{\pi}\} - \boldsymbol{\pi}\boldsymbol{\pi}^\top)D$. But $\boldsymbol{\pi}^\top D = 0$, so $D^\top Q^\top\mathrm{diag}\{\boldsymbol{\pi}\}D = -\mathrm{diag}\{\boldsymbol{\pi}\}D$. The result now follows. □

3.3. Covariances

In this section we point out how to compute the covariance $c(t, u) := \mathrm{cov}\,(M(t), M(t+u))$ for $t, u \geq 0$. To this end, we observe that by applying a time shift, we first assume in the computations to follow that $t = 0$, and we consider $c(t) := \mathrm{cov}(M(t), M(0))$. Below we make frequent use of the additional quantities $\boldsymbol{C}(t) = \mathrm{cov}(\boldsymbol{Y}(t), M(0))$ and $\boldsymbol{B}(t) = \mathrm{cov}(\boldsymbol{Z}(t), M(0))$. Note that $c(t) = \boldsymbol{e}^\top\boldsymbol{C}(t)$. Multiplying (3.1) and (3.2) by $M(0)$, we obtain upon taking expectation the following system of ODEs, with initial conditions $\boldsymbol{B}(0) = \mathrm{cov}(\boldsymbol{Z}(0), M(0))$ and $\boldsymbol{C}(0) = \mathrm{cov}(\boldsymbol{Y}(0), M(0))$:

$$\begin{pmatrix}\boldsymbol{B}'(t)\\ \boldsymbol{C}'(t)\end{pmatrix} = R\begin{pmatrix}B(t)\\ C(t)\end{pmatrix}, \quad \text{where } R := \begin{pmatrix}Q^\top & 0\\ \mathrm{diag}\{\boldsymbol{\alpha}\} & \bar{Q}_\gamma\end{pmatrix}\begin{pmatrix}\boldsymbol{B}(t)\\ \boldsymbol{C}(t)\end{pmatrix}.$$

In a more compact and obvious notation, we write $\boldsymbol{A}'(t) = R\boldsymbol{A}(t)$, and hence $\boldsymbol{A}(t) = \exp(Rt)\boldsymbol{A}(0)$. Likewise, we can compute

$$A(t, u) := \begin{pmatrix}\mathrm{cov}(\boldsymbol{Z}(t+u), M(t))\\ \mathrm{cov}(\boldsymbol{Y}(t+u), M(t))\end{pmatrix} = \exp(Ru)\begin{pmatrix}\mathrm{cov}(\boldsymbol{Z}(t), M(t))\\ \mathrm{cov}(\boldsymbol{Y}(t), M(t))\end{pmatrix}.$$

It remains to derive an expression for the last covariances. For $\mathrm{cov}(\boldsymbol{Z}(t), M(t))$ we need $\mathbb{E}M(t)\boldsymbol{Z}(t) = \mathbb{E}\boldsymbol{Y}(t)$, $\mathbb{E}M(t)$, and $\mathbb{E}\boldsymbol{Z}(t)$. For $\mathrm{cov}(\boldsymbol{Y}(t), M(t))$ we need $\mathbb{E}M(t)\boldsymbol{Y}(t) = \mathbb{E}M(t)^2\boldsymbol{Z}(t)$, $\mathbb{E}\boldsymbol{Y}(t)$, and $\mathbb{E}M(t)$. All these quantities were obtained in Section 3.1.

3.4. Recursive scheme for higher-order moments

The objective of this section is to set up a recursive scheme to generate all transient moments, that is, the expected value of $M(t)^k$ for any $k \in \{1, 2, \dots\}$, jointly with the indicator function $\mathbf{1}_{\{X(t)=i\}}$. To that end we consider the expectation of $(M(t))^k \boldsymbol{Z}(t)$. First we recast (2.1) in the form $\mathrm{d}M(t) = (\boldsymbol{\alpha}^\top \boldsymbol{Z}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Z}(t)X(t))\,\mathrm{d}t + \boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t)$. Applying Itô's lemma to this SDE, we obtain

$$\begin{aligned}\mathrm{d}(M(t))^k &= k(M(t))^{k-1}(\boldsymbol{\alpha}^\top \boldsymbol{Z}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Z}(t)X(t))\,\mathrm{d}t + k(M(t))^{k-1}\boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t)\\ &\quad + \tfrac{1}{2}k(k-1)(M(t))^{k-2}\boldsymbol{\sigma}^\top \mathrm{diag}\{\boldsymbol{Z}(t)\}\boldsymbol{\sigma}\,\mathrm{d}t.\end{aligned}$$

Then we apply the product rule to $M(t)^k \boldsymbol{Z}(t)$, together with the just obtained equation and (3.1) to obtain

$$\begin{aligned}\mathrm{d}((M(t))^k \boldsymbol{Z}(t)) &= k(M(t))^{k-1}(\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{Z}(t) - \mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{Z}(t)M(t))\,\mathrm{d}t\\ &\quad + k(M(t))^{k-1}\mathrm{diag}\{\boldsymbol{\sigma}\}\boldsymbol{Z}(t)\,\mathrm{d}B(t) + \tfrac{1}{2}k(k-1)(M(t))^{k-2}\mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{Z}(t)\,\mathrm{d}t\\ &\quad + (M(t))^k(Q^\top \boldsymbol{Z}(t)\,\mathrm{d}t + \mathrm{d}K(t)).\end{aligned}$$

All martingale terms on the right-hand side are genuine martingales and thus have expectation 0. Putting $\boldsymbol{H}_k(t) := \mathbb{E}M(t)^k \boldsymbol{Z}(t)$, we obtain the following recursion in ODE form:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_k(t) &= k\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{H}_{k-1}(t) - k\mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{H}_k(t) + \frac{1}{2}k(k-1)\mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{H}_{k-2}(t) + Q^\top \boldsymbol{H}_k(t)\\ &= \bar{Q}_{k\gamma}\boldsymbol{H}_k(t) + k\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{H}_{k-1}(t) + \tfrac{1}{2}k(k-1)\mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{H}_{k-2}(t).\end{aligned}$$

Stacking $\boldsymbol{H}_0(t), \dots, \boldsymbol{H}_n(t)$ into a single vector $\bar{\boldsymbol{H}}_n(t)$, we obtain $\mathrm{d}\bar{\boldsymbol{H}}_n(t)/\,\mathrm{d}t = A_n\bar{\boldsymbol{H}}_n(t)$, with $A_n \in \mathbb{R}^{(n+1)d\times(n+1)d}$ denoting a lower block triangular matrix whose solution is $\boldsymbol{H}_n(t) = \exp(A_n t)\boldsymbol{H}_n(0)$. Then $h_k(t) := \mathbb{E}M(t)^k = \boldsymbol{e}^\top \boldsymbol{H}_k(t)$. For $k = 1, 2$ the results of Section 3.1 can be recovered.

For the stationary version M, the above procedure becomes a normal recursion; existence of all moments can be established as in [39].

4. Transient behavior: PDEs

The goal of this section is to characterize, for a given vector $\boldsymbol{t} \in \mathbb{R}^K$ (with $K \in \mathbb{N}$) such that $0 \le t_1 \le \cdots \le t_K$, the Laplace transform of $(M(t+t_1), \dots, M(t+t_K))$ (together with the state of the background process at these time instances). More specifically, we set up a system of PDEs for

$$g_{\boldsymbol{i}}(\boldsymbol{\vartheta}, t) := \mathbb{E}\mathrm{e}^{-(\vartheta_1 M(t_1+t)+\cdots+\vartheta_K M(t_K+t))}\,\mathbf{1}_{\{X(t_1+t)=i_1,\dots,X(t_K+t)=i_K\}};$$

here $t \ge 0$, $\boldsymbol{i} \in \{1, \dots, d\}^K$, and $\boldsymbol{\vartheta} \in \mathbb{R}^K$. The system of PDEs is with respect to t and ϑ_1 up to ϑ_K. We first point out the line of reasoning for the $K = 1$ case, and then present the PDE for $K = 2$. The $K \in \{3, 4, \dots\}$ cases can be dealt with fully analogously, but lead to notational inconveniences and are therefore left out.

It is noted that the stationary version of the result below (that is, $t \to \infty$) for the special case $K = 1$ has appeared in [39] (where we remark that in [39] the additional issue of reflection at 0 was incorporated). For $K = 1$, the object of interest is $g_i(\vartheta, t) := \mathbb{E}\mathrm{e}^{-\vartheta M(t)}Z_i(t)$ for $i = 1, \dots, d$; realize that, without loss of generality, we have taken $t_1 = 0$. For a more compact notation we stack the g_i in a single vector $\boldsymbol{g}$, so $\boldsymbol{g}(\vartheta, t) = \mathbb{E}\mathrm{e}^{-\vartheta M(t)}\boldsymbol{Z}(t)$. Replacing in this expression ϑ by $-\imath u$ for $u \in \mathbb{R}$ gives the characteristic function of $M(t)$ jointly with $\boldsymbol{Z}(t)$.

Theorem 4.1. *Consider the case $K = 1$ and $t_1 = 0$. The Laplace transforms $\boldsymbol{g}(\vartheta, t)$ satisfy the following system of PDEs:*

$$\frac{\partial}{\partial t}\boldsymbol{g}(\vartheta, t) = \left(Q^\top - \vartheta \mathrm{diag}\{\boldsymbol{\alpha}\} + \frac{1}{2}\vartheta^2 \mathrm{diag}\{\boldsymbol{\sigma}^2\}\right)\boldsymbol{g}(\vartheta, t) - \vartheta \mathrm{diag}\{\boldsymbol{\gamma}\}\frac{\partial}{\partial \vartheta}\boldsymbol{g}(\vartheta, t). \quad (4.1)$$

The corresponding initial conditions are $\boldsymbol{g}(0, t) = \boldsymbol{p}_t$ and $\boldsymbol{g}(\vartheta, 0) = \mathrm{e}^{-\vartheta m_0}\boldsymbol{p}_0$.

Proof. The proof mimics the procedure used in Section 3.4 to determine the moments of $M(t)$. Letting $f(\vartheta, t) := \mathrm{e}^{-\vartheta M(t)}$, applying Itô's formula to

$$\mathrm{d}M(t) = (\boldsymbol{\alpha}^\top \boldsymbol{Z}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Z}(t) X(t))\,\mathrm{d}t + \boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t)$$

yields

$$\begin{aligned}\mathrm{d}f(\vartheta, t) = &-\vartheta f(\vartheta, t)((\boldsymbol{\alpha}^\top \boldsymbol{Z}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Z}(t) M(t))\,\mathrm{d}t + \boldsymbol{\sigma}^\top \boldsymbol{Z}(t)\,\mathrm{d}B(t)) \\ &+ \tfrac{1}{2}\vartheta^2 f(\vartheta, t)\mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{Z}(t)\,\mathrm{d}t.\end{aligned}$$

We then apply the product rule to $f(\vartheta, t)\boldsymbol{Z}(t)$, using the just obtained equation in combination with (3.1). This leads to

$$\begin{aligned}\mathrm{d}(f(\vartheta, t)\boldsymbol{Z}(t)) = &-\vartheta f(\vartheta, t)((\mathrm{diag}\{\boldsymbol{\alpha}\}\boldsymbol{Z}(t) - \mathrm{diag}\{\boldsymbol{\gamma}\}\boldsymbol{Z}(t)M(t))\,\mathrm{d}t + \mathrm{diag}\{\boldsymbol{\sigma}\}\boldsymbol{Z}(t)\,\mathrm{d}B(t)) \\ &+ \tfrac{1}{2}\vartheta^2 f(\vartheta, t)\mathrm{diag}\{\boldsymbol{\sigma}^2\}\boldsymbol{Z}(t)\,\mathrm{d}t + f(\vartheta, t)(Q^\top \boldsymbol{Z}(t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{K}(t)).\end{aligned}$$

Taking expectations, recalling that $\boldsymbol{g}(\vartheta, t) = \mathbb{E} f(\vartheta, t)\boldsymbol{Z}(t)$ and that the martingale terms have expectation 0, we obtain (4.1); realize that $\partial \boldsymbol{g}/\partial\vartheta = -\mathbb{E}(f(\vartheta, t)M(t)\boldsymbol{Z}(t))$. □

In [25, Section 4.2] we have included explicit expressions relating to the case $K = 1$ and $d = 2$.

It is remarked that the above system (4.1) of PDEs coincides for $t \to \infty$, with the stationary result of [39] (where it is mentioned that in [39] the feature of reflection at 0 was incorporated). In addition, it is noted that this system can be converted into a system of *ordinary* differential equations, as follows. Let T be exponentially distributed with mean τ^{-1}, independent of all other random features involved in the model. Define $\boldsymbol{g}(\vartheta) := \mathbb{E}\mathrm{e}^{-\vartheta M(T)}\boldsymbol{Z}(t)$. Now multiply the PDE featuring in Theorem 4.1 by $\tau \mathrm{e}^{-\tau t}$ and integrate over $t \in [0, \infty)$ to obtain (use integration by parts for the left-hand side)

$$\lambda(\boldsymbol{g}(\vartheta) - \mathrm{e}^{-\vartheta m_0}\boldsymbol{p}_0) = Q^\top \boldsymbol{g}(\vartheta) - \left(\vartheta \mathrm{diag}\{\boldsymbol{\alpha}\} - \tfrac{1}{2}\vartheta^2 \mathrm{diag}\{\boldsymbol{\sigma}^2\}\right)\boldsymbol{g}(\vartheta) - \vartheta \mathrm{diag}\{\boldsymbol{\gamma}\}\frac{\partial}{\partial\vartheta}\boldsymbol{g}(\vartheta).$$

All the above results are related to the $K = 1$ case. For higher values of K the same procedure can be followed; as announced we now present the result for $K = 2$. Let i, k be elements of $\{1, \ldots, d\}$, and $\boldsymbol{\vartheta} \equiv (\vartheta_1, \vartheta_2) \in \mathbb{R}^2$. We obtain the following system of PDEs:

$$\begin{aligned}\frac{\partial}{\partial t} g_{i,k}(\boldsymbol{\vartheta}, t) = &\sum_{j=1}^{d} q_{ji}\, g_{j,k}(\boldsymbol{\vartheta}, t) + \sum_{\ell=1}^{d} q_{\ell k} g_{i,\ell}(\boldsymbol{\vartheta}, t) \\ &- \left(\vartheta_1 \alpha_i + \vartheta_2 \alpha_k - \frac{1}{2}\vartheta_1^2 \sigma_i^2 - \frac{1}{2}\vartheta_2^2 \sigma_k^2\right) g_{i,k}(\boldsymbol{\vartheta}, t) - \vartheta_1 \gamma_i \frac{\partial}{\partial \vartheta_1} g_{i,k}(\boldsymbol{\vartheta}, t) \\ &- \vartheta_2 \gamma_k \frac{\partial}{\partial \vartheta_2} g_{i,k}(\boldsymbol{\vartheta}, t),\end{aligned}$$

or in self-evident matrix notation, suppressing the arguments $\boldsymbol{\vartheta}$ and t,

$$\frac{\partial G}{\partial t} = Q^\top G + GQ - \vartheta_1 \mathrm{diag}\{\boldsymbol{\alpha}\}G - \vartheta_2 G\mathrm{diag}\{\boldsymbol{\alpha}\} + \frac{1}{2}\vartheta_1^2 \mathrm{diag}\{\boldsymbol{\sigma}^2\}G + \frac{1}{2}\vartheta_2^2 G\mathrm{diag}\{\boldsymbol{\sigma}^2\} - \vartheta_1 \mathrm{diag}\{\boldsymbol{\gamma}\}\frac{\partial G}{\partial \vartheta_1} - \vartheta_2 \frac{\partial G}{\partial \vartheta_2}\mathrm{diag}\{\boldsymbol{\gamma}\}.$$

The initial conditions follow directly from the $K = 1$ case; taking without loss of generality $t_1 = 0$, we have that the (i, k)th entry of $G(\mathbf{0}, t)$ equals $\mathbb{P}(X(t) = i, X(t + t_2) = k)$, whereas the (i, k)th entry of $G(\boldsymbol{\vartheta}, 0)$ equals $\mathrm{e}^{-\vartheta_1 m_0}\mathbb{E}\mathrm{e}^{-\vartheta_2 M(t_2)}\, \mathbf{1}_{\{X(0)=i, X(t_2)=k\}}$.

This matrix-valued system of PDEs can be converted into its vector-valued counterpart. Define the d^2-dimensional vector $\check{\boldsymbol{g}}(\boldsymbol{\vartheta}, t) := \mathrm{vec}(G(\boldsymbol{\vartheta}, t))$. Recall the definitions of the Kronecker sum (denoted by '$\oplus$') and the Kronecker product (denoted by '$\otimes$'). Using the relations $\mathrm{vec}(ABC) = (C^\top \otimes A)\mathrm{vec}(B)$ and $A \oplus B = A \otimes I + I \otimes B$, for matrices A, B, and C of appropriate dimensions, we obtain the vector-valued PDE, again suppressing the arguments $\boldsymbol{\vartheta}$ and t,

$$\frac{\partial \check{\boldsymbol{g}}}{\partial t} = (Q^\top \oplus Q^\top)\check{\boldsymbol{g}} - \vartheta_1(I \otimes \mathrm{diag}\{\boldsymbol{\alpha}\})\check{\boldsymbol{g}} - \vartheta_2(\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I)\check{\boldsymbol{g}} + \frac{\vartheta_1^2}{2}(I \otimes \mathrm{diag}\{\boldsymbol{\sigma}^2\})\check{\boldsymbol{g}} + \frac{\vartheta_2^2}{2}(\mathrm{diag}\{\boldsymbol{\sigma}^2\} \otimes I)\check{\boldsymbol{g}} - \vartheta_1(I \otimes \mathrm{diag}\{\boldsymbol{\gamma}\})\frac{\partial \check{\boldsymbol{g}}}{\partial \vartheta_1} - \vartheta_2(\mathrm{diag}\{\boldsymbol{\gamma}\} \otimes I)\frac{\partial \check{\boldsymbol{g}}}{\partial \vartheta_2}.$$

It is clear how this procedure should be extended to $K \in \{3, 4, \ldots\}$, but, as indicated, we do not include this because of the cumbersome notation needed; the initial conditions follow from the $K = 1$ case.

5. Parameter scaling

So far we have characterized the distribution of $M(t)$ in terms of an algorithm to determine moments and a PDE for the Fourier–Laplace transform. In other words, so far we have not presented any explicit results on the distribution of $M(t)$ itself. In this section we consider asymptotic regimes in which this is possible; these regimes can be interpreted as parameter scalings.

More specifically, in this section we consider the following two scaled versions of the MMOU model.

- In the first we (linearly) speed up the background process (that is, we replace $Q \mapsto NQ$ or, equivalently, $X(t) \mapsto X(Nt)$). Our main result is that, as $N \to \infty$, the MMOU essentially experiences the time-averaged parameters, that is, $\alpha_\infty := \boldsymbol{\pi}^\top\boldsymbol{\alpha}$, $\gamma_\infty := \boldsymbol{\pi}^\top\boldsymbol{\gamma}$, and $\sigma_\infty^2 := \boldsymbol{\pi}^\top\boldsymbol{\sigma}^2$. As a consequence, it behaves as an OU process with these parameters.
- The second regime considered concerns a *simultaneous* scaling of the background process and the OU processes. This is done as in Section 3.2: Q on the one hand, and $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}^2$ on the other hand are scaled at *different* rates: we replace $\boldsymbol{\alpha} \mapsto N^h\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}^2 \mapsto N^h\boldsymbol{\sigma}^2$, but $Q \mapsto NQ$ for some $h \geq 0$). We obtain essentially two regimes, in line with the observations in Section 3.2.

As mentioned above, we are particularly interested in the limiting behavior in the regime that N grows large. It is shown that the process $M(t)$, which we now denote as $M^{[N]}(t)$ to stress the dependence on N, converges to the solution of a specific SDE. Importantly, we establish *weak* *convergence*, that is, in the sense of convergence at the process level; our result can be seen as the counterpart of the result for Markov-modulated infinite-server queues in [2].

We consider sequences of MMOU processes, indexed by N, subject to the following scaling: $Q \mapsto NQ$; $\boldsymbol{\alpha} \mapsto N^h\boldsymbol{\alpha}$; $\boldsymbol{\sigma} \mapsto N^{h/2}\boldsymbol{\sigma}$, where $h \geq 0$. Note that by appropriately choosing h we enter the two regimes described above as we let N grow large (see Corollaries 5.1 and 5.2). The definitions of $M(t)$, $\boldsymbol{Z}(t)$, and $\boldsymbol{K}(t)$ (the latter two having been defined in Section 3) then take the following form (where superscripts are being used to make the dependence on N and h explicit):

$$\mathrm{d}M^{[N,h]}(t) = (N^h\boldsymbol{\alpha} - \boldsymbol{\gamma} M^{[N,h]}(t))^\top \boldsymbol{Z}^{[N]}(t)\,\mathrm{d}t + N^{h/2}\boldsymbol{\sigma}^\top \boldsymbol{Z}^{[N]}(t)\,\mathrm{d}B(t),$$
$$\mathrm{d}\boldsymbol{Z}^{[N]}(t) = NQ^\top \boldsymbol{Z}^{[N]}(t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{K}^{[N]}(t).$$

We keep the initial condition $M^{[N,h]}(0)$ at a fixed level $M(0)$. Let, with the definitions of $\alpha_\infty, \gamma_\infty$, and σ_∞^2 given above, the 'average path' $\varrho(t)$ be defined by $\mathrm{d}\varrho(t) = (\alpha_\infty - \gamma_\infty\varrho(t))\,\mathrm{d}t$ with $\varrho(0) = \mathbf{1}_{\{h=0\}}\,M(0)$, such that $\varrho(t) = \mathrm{e}^{-\gamma_\infty t}\varrho(0) + (\alpha_\infty/\gamma_\infty)\,(1 - \mathrm{e}^{-\gamma_\infty t})$. It is possible to show that $\varrho(t)$ coincides with $\lim_{N\to\infty} N^{-h}\mathbb{E}M^{[N,h]}(t)$; in particular $\varrho(0) = \lim_{N\to\infty} N^{-h}\mathbb{E}M(0) = \mathbf{1}_{\{h=0\}}\,M(0)$.

We can now state the main theorem of this section.

Theorem 5.1. *Under the scaling $Q \mapsto NQ$; $\boldsymbol{\alpha} \mapsto N^h\boldsymbol{\alpha}$; $\boldsymbol{\sigma} \mapsto N^{h/2}\boldsymbol{\sigma}$, it follows that the scaled and centered process $\hat{M}^{[N,h]}(t)$, as defined through*

$$\hat{M}^{[N,h]}(t) := N^{-\beta}(M^{[N,h]}(t) - N^h\varrho(t)),$$

converges weakly to the solution of the following SDE:

$$\mathrm{d}\hat{M}(t) = -\gamma_\infty\hat{M}(t)\,\mathrm{d}t + \sqrt{\sigma_\infty^2\,\mathbf{1}_{\{h\leq 1\}} + V'(t)\,\mathbf{1}_{\{h\geq 1\}}}\,\mathrm{d}B(t), \qquad \hat{M}(0) = 0,$$

where $\beta := \max\{h/2, h - \frac{1}{2}\}$, B a Brownian motion, and

$$V(t) := \int_0^t (\boldsymbol{\alpha} - \boldsymbol{\gamma}\varrho(s))^\top(\mathrm{diag}\{\boldsymbol{\pi}\}D + D^\top\mathrm{diag}\{\boldsymbol{\pi}\})(\boldsymbol{\alpha} - \boldsymbol{\gamma}\varrho(s))\,\mathrm{d}s. \tag{5.1}$$

Before proving this result, we observe that the above theorem provides us with the limiting behavior in the two regimes described at the beginning of this section. In the first corollary we simply take $h = 0$.

Corollary 5.1. *Under the scaling $Q \mapsto NQ$, with $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ kept at their original values, it follows that $M^{[N,0]}(t)$ converges weakly to a process $\mathcal{M}_1(t)$, which is an (ordinary, that is, nonmodulated) OU process with parameters $(\alpha_\infty, \gamma_\infty, \sigma_\infty)$, defined through the SDE*

$$\mathrm{d}\mathcal{M}_1(t) = (\alpha_\infty - \gamma_\infty\mathcal{M}_1(t))\,\mathrm{d}t + \sigma_\infty\,\mathrm{d}B(t).$$

The second corollary describes the situation in which both the background process and the OU process are scaled, but at different rates. As it turns out, there are three regimes.

Corollary 5.2. *Under the scaling $Q \mapsto NQ$; $\boldsymbol{\alpha} \mapsto N^h\boldsymbol{\alpha}$; $\boldsymbol{\sigma} \mapsto N^{h/2}\boldsymbol{\sigma}$, it follows that $\hat{M}^{[N,h]}(t)$ converges weakly to a process $\mathcal{M}_2(t)$, defined through one of the following SDEs: if $0 < h < 1$ then*

$$\mathrm{d}\mathcal{M}_2(t) = -\gamma_\infty\mathcal{M}_2(t)\,\mathrm{d}t + \sigma_\infty\,\mathrm{d}B(t);$$

if $h = 1$ *then*

$$\mathrm{d}\mathcal{M}_2(t) = -\gamma_\infty \mathcal{M}_2(t)\,\mathrm{d}t + \sqrt{\sigma_\infty^2 + V'(t)}\,\mathrm{d}B(t);$$

if $h > 1$ *then*

$$\mathrm{d}\mathcal{M}_2(t) = -\gamma_\infty \mathcal{M}_2(t)\,\mathrm{d}t + \sqrt{V'(t)}\,\mathrm{d}B(t).$$

These corollaries are trivial consequences of Theorem 5.1, and therefore we direct our attention to the proof of this main theorem itself. We remark that Corollary 5.2 confirms an observation we made in Section 3. For $h < 1$ the system essentially behaves as a nonmodulated OU process, while for $h > 1$ the background process plays a role through its deviation matrix D.

In the proof of Theorem 5.1 we need an auxiliary result, which we present first.

Lemma 5.1. *Let the d-dimensional row vectors* $\boldsymbol{\Psi}^{[N]}$ *be a sequence of predictable processes such that* $\boldsymbol{\Psi}^{[N]}(t) \to \boldsymbol{\Psi}(t)$ *in probability uniformly on compact sets, that is, as* $N \to \infty$,

$$\sup_{t \le T} |\boldsymbol{\Psi}^{[N]}(t) - \boldsymbol{\Psi}(t)| \to 0$$

in probability for every $T > 0$*; here* $\boldsymbol{\Psi}$ *is deterministic, satisfying* $\int_0^t \boldsymbol{\Psi}(s)\boldsymbol{\Psi}(s)^\top\,\mathrm{d}s < \infty$ *for every* $t > 0$. *Furthermore, let* $\boldsymbol{X}^{[N]}$ *be continuous semimartingales that converge weakly to a d-dimensional scaled Brownian motion* $\boldsymbol{B}$ *with quadratic variation* $\langle \boldsymbol{B} \rangle_t = Ct$ *(where* $C \in \mathbb{R}^{d \times d}$*). Then, as* $N \to \infty$, *the stochastic integrals*

$$\int_0^\cdot \boldsymbol{\Psi}^{[N]}(s)\,\mathrm{d}\boldsymbol{X}^{[N]}(s)$$

converge weakly to the time-inhomogeneous Brownian motion $B^{\boldsymbol{\Psi}} := \int_0^\cdot \boldsymbol{\Psi}(s)\,\mathrm{d}\boldsymbol{B}(s)$ *with quadratic variation*

$$\langle B^{\boldsymbol{\Psi}} \rangle_t = \int_0^t \boldsymbol{\Psi}(s) C \boldsymbol{\Psi}(s)^\top\,\mathrm{d}s.$$

The claim of Lemma 5.1 essentially follows from [27, Theorem VI.6.22]. To check the condition of the cited theorem, one needs weak convergence of the pair $(\boldsymbol{\Psi}^{[N]}, \boldsymbol{X}^{[N]})$, but this is guaranteed by the uniform convergence in probability of the $\boldsymbol{\Psi}^{[N]}(t)$.

We now proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1. The proof consists of four steps.

Step 1. We describe the dynamics of the process $\hat{M}^{[N,h]}(t)$ through

$$\begin{aligned}\mathrm{d}\hat{M}^{[N,h]}(t) &= N^{h-\beta}(\boldsymbol{\alpha} - \rho(t)\boldsymbol{\gamma})^\top(\boldsymbol{Z}^{[N]}(t) - \boldsymbol{\pi})\,\mathrm{d}t + N^{h/2-\beta}\boldsymbol{\sigma}^\top \boldsymbol{Z}_t^{[N]}\,\mathrm{d}B(t) \\ &\quad - \boldsymbol{\gamma}^\top \boldsymbol{Z}_t^{[N]} \hat{M}^{[N,h]}(t)\,\mathrm{d}t \\ &=: N^{h-\beta-1/2}\,\mathrm{d}G^{[N]}(t) + N^{h/2-\beta}\,\mathrm{d}\hat{B}^{[N]}(t) - \boldsymbol{\gamma}^\top \boldsymbol{Z}_t^{[N]} \hat{M}^{[N,h]}(t)\,\mathrm{d}t.\end{aligned}$$

Defining $\boldsymbol{\zeta}^{[N]}(t) := \int_0^t \boldsymbol{Z}^{[N]}(s)\,\mathrm{d}s$ and $Y^{[N,h]}(t) := \mathrm{e}^{\boldsymbol{\gamma}^\top \boldsymbol{\zeta}^{[N]}(t)} \hat{M}^{[N,h]}(t)$, we obtain

$$\mathrm{d}Y^{[N,h]}(t) = N^{h-\beta-1/2}\mathrm{e}^{\boldsymbol{\gamma}^\top \boldsymbol{\zeta}_t^{[N]}}\,\mathrm{d}G^{[N]}(t) + N^{h/2-\beta}\mathrm{e}^{\boldsymbol{\gamma}^\top \boldsymbol{\zeta}_t^{[N]}}\,\mathrm{d}\hat{B}^{[N]}(t). \tag{5.2}$$

In the next two steps we analyze the two terms in the right-hand side of (5.2).

Step 2. We first consider the first term on the right-hand side of (5.2). To analyze it, we need the functional CLT for the martingale $\boldsymbol{K}_\circ^{[N]} := \boldsymbol{K}^{[N]}/\sqrt{N}$. From the proof of Proposition 3.1, we know that

$$\frac{1}{N}\langle \boldsymbol{K}_\circ^{[N]}\rangle_t = \int_0^t (\operatorname{diag}\{Q^\top \boldsymbol{Z}^{[N]}(s)\} - Q^\top \operatorname{diag}\{\boldsymbol{Z}^{[N]}(s)\} - \operatorname{diag}\{\boldsymbol{Z}^{[N]}(s)\}Q)\,\mathrm{d}s,$$

which by the ergodic theorem [4, Section VI.3] converges to $-(Q^\top \operatorname{diag}\{\boldsymbol{\pi}\} + \operatorname{diag}\{\boldsymbol{\pi}\}Q)t$. As the jumps of $\boldsymbol{K}_\circ^{[N]}$ are of order $O(1/\sqrt{N})$, the martingale CLT (see, e.g. [27, Theorem VIII.3.11] or [17, Theorem 7.1.4]) gives the weak convergence of $\boldsymbol{K}_\circ^{[N]}$ to a d-dimensional scaled Brownian motion $\boldsymbol{B}_\circ$ with

$$\langle \boldsymbol{B}_\circ\rangle_t = -(Q^\top \operatorname{diag}\{\boldsymbol{\pi}\} + \operatorname{diag}\{\boldsymbol{\pi}\}Q)t.$$

Moreover, we then also deduce the weak convergence of the process

$$\boldsymbol{Z}^{[N,Q]} := \sqrt{N}\int_0^\cdot Q^\top \boldsymbol{Z}^{[N]}(s)\,\mathrm{d}s$$

to $-\boldsymbol{B}_\circ$, and, hence, to $\boldsymbol{B}_\circ$ as well.

- We first apply Lemma 5.1, with $\boldsymbol{\Psi}^{[N]}(t) := -(\boldsymbol{\alpha} - \rho(t)\boldsymbol{\gamma})^\top D^\top$ and $X^{[N]} := \boldsymbol{Z}^{[N,Q]}$, to

$$\begin{aligned} G^{[N]} &= \sqrt{N}\int_0^\cdot (\boldsymbol{\alpha} - \rho(s)\boldsymbol{\gamma})^\top (\boldsymbol{Z}^{[N]}(s) - \boldsymbol{\pi})\,\mathrm{d}s \\ &= -\sqrt{N}\int_0^\cdot (\boldsymbol{\alpha} - \rho(s)\boldsymbol{\gamma})^\top (QD)^\top \boldsymbol{Z}^{[N]}(s)\,\mathrm{d}s, \end{aligned}$$

  where the last equality follows from $QD = \boldsymbol{e}\boldsymbol{\pi}^\top - I$ (see the proof of Proposition 3.1). Note that $\boldsymbol{\Psi}^{[N]}(t) = \boldsymbol{\Psi}(t)$ for all N, and therefore it is immediate that the weak limit can be identified as a continuous Gaussian martingale G, where it turns out that $\langle G\rangle_t = V(t)$ with $V(t)$ defined in (5.1), which again follows from the proof of Proposition 3.1.

- In the next step we consider the processes $\int_0^\cdot \Psi^{[N]}(s)\,\mathrm{d}G^{[N]}(s)$ with $\Psi^{[N]}(s) := \exp(\boldsymbol{\gamma}^\top \boldsymbol{\zeta}^{[N]}(s))$. As these processes are increasing, we have the almost sure convergence of

$$\sup_{s\le T} |\exp(\boldsymbol{\gamma}^\top \boldsymbol{\zeta}^{[N]}(s)) - \exp(\boldsymbol{\gamma}^\top \boldsymbol{\pi} s)| \to 0 \quad \text{as } N \to \infty,$$

  by combining the ergodic theorem with [27, Theorem VI.2.15(c)] (which states that pointwise convergence of increasing functions to a continuous limit implies uniform convergence on compacts). As an immediate consequence of the above and Lemma 5.1, we obtain the weak convergence of $\int_0^\cdot \exp(\boldsymbol{\gamma}^\top \boldsymbol{\zeta}^{[N]}(s))\,\mathrm{d}G^{[N]}(s)$ to $\int_0^\cdot \exp(\boldsymbol{\gamma}^\top \boldsymbol{\pi}\, s)\,\mathrm{d}G(s) = \int_0^\cdot \exp(\gamma_\infty s)\,\mathrm{d}G(s)$.

Step 3. We now consider the second term on the right-hand side of (5.2). For the Brownian term $\hat{B}^{[N]}$, we have by the martingale CLT weak convergence to the Gaussian martingale $\hat{B}$ with quadratic variation $\langle \hat{B}\rangle_t = \sigma_\infty^2 t$. The convergence of $\int_0^\cdot \exp(\boldsymbol{\gamma}^\top \boldsymbol{\zeta}^{[N]}(s))\,\mathrm{d}\hat{B}^{[N]}(s)$ can be handled as above to obtain weak convergence to the Gaussian martingale $\int_0^\cdot \exp(\boldsymbol{\gamma}^\top \boldsymbol{\pi} s)\,\mathrm{d}\hat{B}(s) = \int_0^\cdot \exp(\gamma_\infty s)\,\mathrm{d}\hat{B}(s)$.

Step 4. In order to finally obtain the weak limit of $Y^{[N,h]}$, we use

$$h - \beta - \frac{1}{2} = \frac{1}{2}\min\{h-1, 0\}, \qquad \frac{h}{2} - \beta = \frac{1}{2}\min\{1-h, 0\}.$$

Clearly, for $h < 1$ we have convergence of $Y^{[N,h]}$ to $\int_0^{\cdot} \exp(\boldsymbol{\gamma}^{\top}\boldsymbol{\pi}s)\,\mathrm{d}\hat{B}(s)$, whereas for $h > 1$ we have convergence to $\int_0^{\cdot} \exp(\boldsymbol{\gamma}^{\top}\boldsymbol{\pi}s)\,\mathrm{d}G(s)$. For $h = 1$, we obtain weak convergence to the sum of these. To see this, recall that the weak convergence of the $G^{[N]}$ was based on properties of the Markov chain, whereas the convergence of the $B^{[N]}$ resulted from considerations involving the Brownian motion B, and these basic processes are independent. Furthermore, note that $Y^{[N,h]}(0) = N^{-\beta}M(0) - N^{h-\beta}\mathbf{1}_{\{h=0\}}\,M(0) \to 0$. Combining these results, we find that $Y^{[N,h]}$ converges to a Gaussian martingale Y given by

$$Y(t) = \int_0^t \mathrm{e}^{\gamma_\infty s}(\mathbf{1}_{\{h\le 1\}}\,\mathrm{d}\hat{B}(s) + \mathbf{1}_{\{h\ge 1\}}\,\mathrm{d}G(s)),$$

and hence the $\hat{M}^{[N,h]}$ converge weakly to the limit $\hat{M}$ given by $\hat{M}(t) = \mathrm{e}^{-\gamma_\infty t}Y(t)$, and this process satisfies the SDE

$$\mathrm{d}\hat{M}(t) = -\gamma_\infty \hat{M}(t)\,\mathrm{d}t + (\mathbf{1}_{\{h\le 1\}}\,\mathrm{d}\hat{B}(t) + \mathbf{1}_{\{h\ge 1\}}\,\mathrm{d}G(t)).$$

In this equation the (continuous, Gaussian) martingale has quadratic variation $\mathbf{1}_{\{h\le 1\}}\,\sigma_\infty^2 t + \mathbf{1}_{\{h\ge 1\}}\,V(t)$. Hence, we can identify its distribution with that of

$$\int_0^{\cdot} \sqrt{\mathbf{1}_{\{h\le 1\}}\,\sigma_\infty^2 + \mathbf{1}_{\{h\ge 1\}}\,V'(s)}\,\mathrm{d}B(s),$$

where B is a standard Brownian motion. This completes the proof. □

6. Multiple MMOU processes driven by the same background process

In this section we consider a single background process X, taking as before values in $\{1, \dots, d\}$, modulating *multiple* OU processes. Suppose that there are $J \in \mathbb{N}$ such processes, with parameters $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\gamma}^{(1)}, \boldsymbol{\sigma}^{(1)})$ up to $(\boldsymbol{\alpha}^{(J)}, \boldsymbol{\gamma}^{(J)}, \boldsymbol{\sigma}^{(J)})$. It is further assumed that the OU processes are driven by *independent* Brownian motions $B_1(\cdot), \dots, B_J(\cdot)$. Combining the above, this leads to the J coupled SDEs

$$\mathrm{d}M_j(t) = (\alpha^{(j)}_{X(t)} - \gamma^{(j)}_{X(t)}M_j(t))\,\mathrm{d}t + \sigma^{(j)}_{X(t)}\,\mathrm{d}B_j(t) \quad \text{for } j = 1, \dots, J.$$

We call the process a J-MMOU process.

Interestingly, this construction yields J components that have common features, as they react to the same background process, as well as component-specific features, as a consequence of the fact that the driving Brownian motions are independent. This model is particularly useful in settings with multidimensional stochastic processes whose components are affected by the same external factors.

An example of a situation where this idea can be exploited is that of multiple asset prices reacting to the (same) state of the economy, which could be represented by a background process (for instance with two states, that is, alternating between a ‘good’ and a ‘bad’ state). In this way the dependence between the individual components can be naturally modeled. In mathematical finance, one of the key challenges is to develop models that incorporate the correlation between the individual components in a sound way. Some proposals were too simplistic, ignoring too many relevant details, while others correspond with models with overly many parameters, with its repercussions in terms of the calibration that needs to be performed. Another setting in which such a coupling may offer a natural modeling framework is that of a wireless network. Channel conditions may be modeled as alternating between various levels, and users' transmission rates may react in a similar way to these fluctuations.

Many of the results derived in the previous sections covering the $J = 1$ case, can be generalized to the situation of J-MMOU processes described above. To avoid unnecessary repetition, we restrict ourselves to a few of these extensions. In particular, we present

(i) the counterpart of Theorem 2.1, stating that $\boldsymbol{M}(t)$ is, conditionally on the path of the background process, multivariate normally distributed;

(ii) some explicit calculations for the means and (co-)variances for certain special cases;

(iii) the generalization of the PDE of Theorem 4.1;

(iv) explicit expressions for the steady-state (mixed) moments.

Neither procedures for transient moments nor scaling results (such as a J-dimensional CLT) are included in this paper, but can be developed as in the single-dimensional case.

Conditional normality. Evidently, conditioning on $(X(s), s \in [0, t])$ the individual components of $\boldsymbol{M}(t)$ are independent. The following result describes this setting in greater detail.

Proposition 6.1. *Define* $\Gamma^{(j)}(t) := \int_0^t \gamma_{X(s)}^{(j)}\,\mathrm{d}s$ *for* $j = 1, \ldots, J$. *Then the* J*-dimensional stochastic process* $(\boldsymbol{M}(t))_{t\geq 0}$ *given by*

$$M^{(j)}(t) = M_0^{(j)}\mathrm{e}^{-\Gamma^{(j)}(t)} + \int_0^t \mathrm{e}^{-(\Gamma^{(j)}(t)-\Gamma^{(j)}(s))}\alpha_{X(s)}^{(j)}\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(\Gamma^{(j)}(t)-\Gamma^{(j)}(s))}\sigma_{X(s)}^{(j)}\,\mathrm{d}B(s)$$

is the unique J*-MMOU process with initial condition* $\boldsymbol{M}_0$. *Conditional on the process* X, *the random vector* $\boldsymbol{M}(t)$ *has a multivariate normal distribution with, for* $j = 1, \ldots, J$, *random mean*

$$\mu^{(j)}(t) = M_0^{(j)}\exp(-\Gamma^{(j)}(t)) + \int_0^t \exp(-(\Gamma^{(j)}(t) - \Gamma^{(j)}(s)))\alpha_{X(s)}^{(j)}\,\mathrm{d}s$$

and random covariance $v^{(j,k)}(t) = 0$ *if* $j \neq k$, *and*

$$v^{(j,j)}(t) = \int_0^t \exp(-2(\Gamma^{(j)}(t) - \Gamma^{(j)}(s)))(\sigma_{X(s)}^{(j)})^2\,\mathrm{d}s.$$

Mean and (co-)variance. The mean and (co-)variance of $\boldsymbol{M}(t)$ for a J-MMOU can be computed relying on stochastic integration theory, with a procedure similar to the one relied on in Section 3; we do not include the resulting expressions.

We consider in greater detail the special case that $\gamma_i^{(j)} \equiv \gamma^{(j)}$ for all $i \in \{1, \ldots, d\}$ (as in Section 3.2) because in this situation expressions simplify. The means and variances can be found as in Proposition 2.1; we now point out how to compute the covariance $v_t^{(j,k)} := \operatorname{cov}(M^{(j)}(t), M^{(k)}(t))$ (with $j \neq k$), relying on the *law of total covariance.* We write, in self-evident notation,

$$v_t^{(j,k)} = \mathbb{E}(\operatorname{cov}(M^{(j)}(t), M^{(k)}(t) \mid X)) + \operatorname{cov}(\mathbb{E}(M^{(j)}(t) \mid X), \mathbb{E}(M^{(k)}(t) \mid X)).$$

The first term obviously cancels (cf. Proposition 6.1), while the second reads

$$\frac{1}{\gamma^{(j)}+\gamma^{(k)}}\left(\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\alpha_{i_1}^{(j)}\alpha_{i_2}^{(k)}\int_0^t(\mathrm{e}^{-\gamma^{(k)}v}-\mathrm{e}^{-(\gamma^{(j)}+\gamma^{(k)})t+\gamma^{(j)}v})\pi_{i_1}(p_{i_1i_2}(v)-\pi_{i_2})\,\mathrm{d}v\right.$$
$$\left.\times\sum_{i_1=1}^{d}\sum_{i_2=1}^{d}\alpha_{i_1}^{(k)}\alpha_{i_2}^{(j)}\int_0^t(\mathrm{e}^{-\gamma^{(j)}v}-\mathrm{e}^{-(\gamma^{(k)}+\gamma^{(j)})t+\gamma^{(k)}v})\pi_{i_1}(p_{i_1i_2}(v)-\pi_{i_2})\,\mathrm{d}v\right).$$

As before, this expression further simplifies in particular asymptotic regimes, as pointed out in [25, Section 6.2]; in [25, Example 6.2] the case of $d=2$, $J=2$ is explicitly analyzed for $t\to\infty$.

Transient behavior: PDEs. In order to uniquely characterize the joint distribution of $\boldsymbol{M}(t)$, we now set up a system of PDEs for the objects $\boldsymbol{g}(\boldsymbol{\vartheta},t):=\mathbb{E}\exp(\sum_{j=1}^{J}\vartheta_j M^{(j)}(t))\boldsymbol{Z}(t)$ with $i\in\{1,\dots,d\}$. Relying on the machinery used when establishing the system of PDEs featuring in Theorem 4.1, we find that $\partial\boldsymbol{g}(\boldsymbol{\vartheta},t)/\partial t$ can be expressed as

$$\left(Q^\top-\sum_{j=1}^{J}\left(\vartheta_j\mathrm{diag}\{\boldsymbol{\alpha}^{(j)}\}-\frac{1}{2}\vartheta_j^2\mathrm{diag}\{(\boldsymbol{\sigma}^{(j)})^2\}\right)\right)\boldsymbol{g}(\boldsymbol{\vartheta},t)-\sum_{j=1}^{J}\vartheta_j\mathrm{diag}\{\boldsymbol{\gamma}^{(j)}\}\frac{\partial}{\partial\vartheta_j}\boldsymbol{g}(\boldsymbol{\vartheta},t).$$

Recursive scheme for higher-order moments. The above system of PDEs can be used to determine all (transient and stationary) moments related to the J-MMOU. We restrict ourselves to the stationary moments here. Define $\boldsymbol{h_k}=(h_{1,\boldsymbol{k}},\dots,h_{d,\boldsymbol{k}})^\top$, where

$$h_{i,\boldsymbol{k}}:=\mathbb{E}((-1)^{\sum_{j=1}^{J}k_j}(M^{(1)})^{k_1}\cdots(M^{(J)})^{k_J}\,\mathbf{1}_{\{X=i\}}).$$

Observe that $\boldsymbol{h_0}=\boldsymbol{\pi}$. With techniques similar to those applied earlier, $\boldsymbol{u}_j\in\mathbb{R}^J$ denoting the jth unit vector, we obtain the recursion

$$\boldsymbol{h_k}=\left(Q^\top-\sum_{j=1}^{J}k_j\mathrm{diag}\{\boldsymbol{\gamma}^{(j)}\}\right)^{-1}\left(\sum_{j=1}^{J}k_j\left(\mathrm{diag}\{\boldsymbol{\alpha}^{(j)}\}\boldsymbol{h}_{\boldsymbol{k}-\boldsymbol{u}_j}-\frac{k_j-1}{2}\mathrm{diag}\{(\boldsymbol{\sigma}^{(j)})^2\}\boldsymbol{h}_{\boldsymbol{k}-2\boldsymbol{u}_j}\right)\right).$$

This procedure allows us to compute all mixed moments, thus facilitating the calculation of covariances as well. In the situation of $J=2$, for instance, we find that, with $\boldsymbol{h}_{0,1}$ and $\boldsymbol{h}_{1,0}$ as in Section 3.4,

$$\mathbb{E}M^{(1)}M^{(2)}=\boldsymbol{e}(Q^\top-\mathrm{diag}\{\boldsymbol{\gamma}^{(1)}\}-\mathrm{diag}\{\boldsymbol{\gamma}^{(2)}\})^{-1}(\mathrm{diag}\{\boldsymbol{\alpha}^{(1)}\}\boldsymbol{h}_{0,1}+\mathrm{diag}\{\boldsymbol{\alpha}^{(2)}\}\boldsymbol{h}_{1,0}).$$

Remark 6.1. The model proposed in this section describes a J-dimensional stochastic process with dependent components. In many situations, the dimension d can be chosen relatively small (see, e.g. [6], [19]), whereas J tends to be large (e.g. in the context of asset prices). Importantly, the $\frac{1}{2}J(J+1)=O(J^2)$ entries of the covariance matrix of $\boldsymbol{M}(t)$ (or its stationary counterpart $\boldsymbol{M}$) are endogenously determined by the model and need not be estimated from data. Instead, this approach requires the calibration of just the $d(d-1)$ entries of the Q-matrix, as well as the $3dJ$ parameters of the underlying OU processes, totaling $O(J)$ parameters. We conclude that, as a consequence, this framework offers substantial potential advantages.

7. Discussion and concluding remarks

In this paper we have presented a set of results on MMOU processes, ranging from procedures to compute moments and a PDE for the Fourier–Laplace transform, to weak convergence results under specific scalings and a multivariate extension in which multiple MMOUs are modulated by the same background process. Although a relatively large number of aspects is covered, there are many issues that still need to be studied. One such area concerns the large-deviations behavior under specific scalings, so as to obtain the counterparts of the results obtained in, e.g. [7], [8], and [11], for the Markov-modulated infinite-server queue.

In this paper we have assumed that $\gamma_i \geq 0$, with strict inequality for at least one $i \in \{1, \ldots, d\}$, so as to make sure the stationary version M exists, but one can actually do with less. It is sufficient [20] to require $\gamma_\infty < 0$ (also if $X(t)$ corresponds to a real-valued process); see also [40, Example 5.1].

It is further remarked that in this paper we looked at an regime-switching version of the OU process, but of course we could have considered various other processes. One option is the Markov-modulated version of the so-called Cox–Ingersoll–Ross process:

$$\mathrm{d}M(t) = (\alpha_{X(t)} - \gamma_{X(t)} M(t))\,\mathrm{d}t + \sigma_{X(t)} \sqrt{M(t)}\,\mathrm{d}B(t).$$

Some results we have established for MMOU processes have their immediate MMCIR counterpart, while for others there are crucial differences. It is relatively straightforward to adapt the procedure used in [25, Section A] to set up a system of PDEs for the Fourier–Laplace transforms (essentially based on Itô's rule). Interestingly, the recursions needed to generate all moments are now one-step (rather than two-step) recursions. A further objective would be to see to what extent the results of our paper generalize to more general classes of diffusions; see, e.g. [24].

Acknowledgements

M. Mandjes was partly supported by the NWO Gravitation project NETWORKS (grant no. 024.002.003). K. De Turck was supported by Fonds Wetenschappelijk Onderzoek – Vlaanderen (FWO) for Flanders, and is currently supported by a postdoctoral fellowship.

References

[1] Alili, L., Patie, P. and Pedersen, J. L. (2005). Representations of the first hitting time density of an Ornstein–Uhlenbeck process. *Stoch. Models* **21,** 967–980.

[2] Anderson, D., Blom, J., Mandjes, M., Thorsdottir, H. and De Turck, K. (2014). A functional central limit theorem for a Markov-modulated infinite-server queue. To appear in *Methodology Comput. Appl. Prob.*.

[3] Ang, A. and Bekaert, G. (2002). Regime switches in interest rates. *J. Business Econom. Statist.* **20,** 163–182.

[4] Asmussen, S. (2003). *Applied Probability and Queues*, 2nd edn. Springer, New York.

[5] Asmussen, S. and Albrecher, H. (2010). *Ruin Probabilities*, 2nd edn. World Scientific, Hackensack, NJ.

[6] Banachewicz, K., Lucas, A. and van der Vaart, A. (2008). Modelling portfolio defaults using hidden Markov models with covariates. *Econometrics J.* **11,** 155–171.

[7] Blom, J. and Mandjes, M. (2013). A large-deviations analysis of Markov-modulated inifinite-server queues. *Operat. Res. Lett.* **41,** 220–225.

[8] Blom, J., De Turck, K. and Mandjes, M. (2013). Rare event analysis of Markov-modulated infinite-server queues: a Poisson limit. *Stoch. Models* **29,** 463–474.

[9] Blom, J., De Turck, K. and Mandjes, M. (2015). Analysis of Markov-modulated infinite-server queues in the central-limit regime. *Prob. Eng. Inf. Sci.* **29,** 433–459.

[10] Blom, J., Mandjes, M. and Thorsdottir, H. (2013). Time-scaling limits for Markov-modulated infinite-server queues. *Stoch. Models* **29,** 112–127.

[11] Blom, J., De Turck, K., Kella, O. and Mandjes, M. (2014). Tail asymptotics of a Markov-modulated infinite-server queue. *Queueing Systems* **78,** 337–357.

[12] BLOM, J., KELLA, O., MANDJES, M. AND THORSDOTTIR, H. (2014). Markov-modulated infinite-server queues with general service times. *Queueing Systems* **76,** 403–424.

[13] COOLEN-SCHRIJNER, P. AND VAN DOORN, E. A. (2002). The deviation matrix of a continuous-time Markov chain. *Prob. Eng. Inf. Sci.* **16,** 351–366.

[14] D'AURIA, B. (2008). M/M/∞ queues in semi-Markovian random environment. *Queueing Systems* **58,** 221–237.

[15] ELLIOTT, R. J. AND MAMON, R. S. (2002). An interest rate model with a Markovian mean reverting level. *Quant. Finance* **2,** 454–458.

[16] ELLIOTT, R. J. AND SIU, T. K. (2009). On Markov-modulated exponential-affine bond price formulae. *Appl. Math. Finance* **16,** 1–15.

[17] ETHIER, S. N. AND KURTZ, T. G. (1986). *Markov Processes. Characterization and Convergence.* John Wiley, New York.

[18] FRALIX, B. H. AND ADAN, I. J. B. F. (2009). An infinite-server queue influenced by a semi-Markovian environment. *Queueing Systems* **61,** 65–84.

[19] GIAMPIERI, G., DAVIS, M. AND CROWDER, M. (2005). Analysis of default data using hidden Markov models. *Quant. Finance* **5,** 27–34.

[20] GUYON, X., IOVLEFF, S. AND YAO, J.-F. (2004). Linear diffusion with stationary switching regime. *ESAIM Prob. Statist.* **8,** 25–35.

[21] HALE, J. K. (1980). *Ordinary Differential Equations*, 2nd edn. Krieger, Huntington, New York.

[22] HAMILTON, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica* **57,** 357–384.

[23] HORN, R. A. AND JOHNSON, C. R. (1985). *Matrix Analysis*. Cambridge University Press.

[24] HUANG, G., MANDJES, M. AND SPREIJ, P. (2014). Weak convergence of Markov-modulated diffusion processes with rapid switching. *Statist. Prob. Lett.* **86,** 74–79.

[25] HUANG, G. *et al.* (2014). Markov-modulated Ornstein–Uhlenbeck processes. Preprint. Available at http://arxiv.org/abs/1412.7952v1.

[26] JACOBSEN, M. (1996). Laplace and the origin of the Ornstein–Uhlenbeck process. *Bernoulli* **2,** 271–286.

[27] JACOD, J. AND SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edn. Springer, Berlin.

[28] KARATZAS, I. AND SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd edn. Springer, New York.

[29] KEILSON, J. (1979). *Markov Chain Models—Rarity and Exponentiality*. Springer, New York.

[30] KEMENY, J. G. AND SNELL, J. L. (1960). *Finite Markov Chains.* Van Nostrand, Princeton, NJ.

[31] NEUTS, M. F. (1981). *Matrix–Geometric Solutions in Stochastic Models: An Algorithmic Approach.* John Hopkins University Press, Baltimore, MD.

[32] O'CINNEIDE, C. A. AND PURDUE, P. (1986). The M/M/∞ queue in a random environment. *J. Appl. Prob.* **23,** 175–184.

[33] REVUZ, D. AND YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin.

[34] ROBERT, P. (2003). *Stochastic Networks and Queues.* Springer, Berlin.

[35] ROGERS, L. C. G. AND WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales,* Vol. 2, *Itô Calculus*, 2nd edn. Cambridge University Press.

[36] SYSKI, R. (1978). Ergodic potential. *Stoch. Process. Appl.* **7,** 311–336.

[37] UHLENBECK, G. E. AND ORNSTEIN, L. S. (1930). On the theory of Brownian motion. *Phys. Rev.* **36,** 823–841.

[38] WILLIAMS, D. (1991). *Probability with Martingales.* Cambridge University Press.

[39] XING, X., ZHANG, W. AND WANG, Y. (2009). The stationary distributions of two classes of reflected Ornstein–Uhlenbeck processes. *J. Appl. Prob.* **46,** 709–720.

[40] YUAN, C. AND MAO, X. (2003). Asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Stoch. Process. Appl.* **103,** 277–291.