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A note on the central limit theorem for the idleness process in a one-sided reflected Ornstein–Uhlenbeck model

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In this short communication, we present a (functional) central limit theorem for the idleness process of a one-sided reflected Ornstein–Uhlenbeck proces.

Keywords and Phrases: Ornstein–Uhlenbeck processes, reflection, idleness process, central limit theorems

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1 Introduction

Ornstein–Uhlenbeck (OU) processes are Markovian, mean reverting Gaussian processes and have found widespread use in a broad range of application domains, such as finance, life sciences, and operations research. In many situations, though, the stochastic process involved is not allowed to cross a certain boundary or is even supposed to remain within two boundaries. The resulting reflected OU (denoted in the sequel by ROU) processes have been studied by, for example, Ward and Glynn (2003a, 2003b, 2005), where ROU processes are used to approximate the number-in-system processes in M/M/1 and GI/GI/1 queues with reneging under a specific, reasonable scaling. Srikant and Whitt (1996) also show that the number-in-system process in a GI/M/*n* loss model can be approximated by ROU. For other applications, we refer to, for example, the introduction of Giorno, Nobile, and di Cesare (2012) and references therein.

This note is to be considered as a follow up of, and complementary to, our earlier work (Huang, Mandjes, and Spreij, 2014). That paper considered large deviation results for both one-sided and doubly reflected processes, but only central limit theo-

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rems for the 'idleness' and 'loss' processes in the doubly reflected case. The central limit theorems for the 'idleness' and 'loss' processes in one-sided ROU models are provided in the present note.

Throughout this note, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is fixed. As known (see Lions and Snitzman, 1984), the OU process is defined as the unique strong solution to the stochastic differential equation (SDE):

$$\mathrm{d}X_t = (\alpha - \gamma X_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$, $\gamma, \sigma > 0$, and W_t is a standard Brownian motion.

The OU process is *mean reverting* towards the value α/γ. To incorporate reflection at a lower boundary 0, thus constructing ROU, the following SDE is used,

$$\mathrm{d}Y_t = (\alpha - \gamma Y_t)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t, \quad Y_0 = x \geq 0. \tag{1}$$

Here, $L = \{L_t, t \geq 0\}$ could be interpreted as an 'idleness process'. More precisely, L is defined as the minimal nondecreasing process such that $Y_t \geq 0$ for $t \geq 0$; as in the deterministic Skorokhod problem, it holds that $\int_{[0,\infty]} \mathbf{1}_{\{Y_t>0\}}\mathrm{d}L_t = 0$. Hence, for any (continuous) function g, one has

$$\int_{[0,T]} g(Y_t)\mathrm{d}L_t = g(0)\int_{[0,T]} \mathbf{1}_{\{Y_t=0\}}\mathrm{d}L_t = g(0)L_T, \text{ for any} T > 0. \tag{2}$$

Existence of a strong solution to Equation (1) has been established in, for instance, Ward and Glynn (2005).

The paper mainly focuses on central limit theorems for the idleness process L. As in Huang *et al.* (2014), we use Zhang and Glynn's martingale approach, as developed in Zhang and Glynn (2011), and further elaborated on in Glynn and Wang (2015) to establish the results. In section 2, we review a previous result from Huang *et al.* (2014) for doubly reflected processes and explain why one has to modify this approach for the one-sided reflected case, whereas in section 3, we show which modifications are needed to identify the central limit theorems. We also present results for reflected processes at lower boundaries other than zero and for processes reflected at an upper bound.

2 A previous result

Let us briefly summarize the result in Huang *et al.* (2014). In that paper, the object of study was a doubly reflected (at the lower bound zero and an upper bound d) OU process Z, satisfying the SDE

$$\mathrm{d}Z_t = (\alpha - \gamma Z_t)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}U_t.$$

For a twice continuously differentiable function h on $\mathbb{R}$, by Itô's formula, we have

$$\mathrm{d}h(Z_t) = \left((\alpha - \gamma Z_t)h'(Z_t) + \frac{\sigma^2}{2}h''(Z_t)\right)\mathrm{d}t + \sigma h'(Z_t)\mathrm{d}W_t + h'(Z_t)\mathrm{d}L_t - h'(Z_t)\mathrm{d}U_t.$$

On the basis of the key properties of L (e.g., Equation (2)) and U (which takes care of the reflection at the upper level d), this reduces to

$$\begin{aligned}\mathrm{d}h(Z_t) &= \big((\alpha-\gamma Z_t)h'(Z_t)+\frac{\sigma^2}{2}h''(Z_t)\big)\mathrm{d}t+\sigma h'(Z_t)\mathrm{d}W_t+h'(0)\mathrm{d}L_t-h'(d)\mathrm{d}U_t\\ &= (\mathcal{L}h)(Z_t)\mathrm{d}t+\sigma h'(Z_t)\mathrm{d}W_t+h'(0)\mathrm{d}L_t-h'(d)\mathrm{d}U_t,\end{aligned}$$

where the operator $\mathcal{L}$ is defined through

$$\mathcal{L} := (\alpha-\gamma x)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

The following lemma taken from Huang *et al.* (2014) presented a judicious choice of the function h that was instrumental for the proof of the central limit theorem for the process U.

Lemma 2.1. Consider the ordinary differential equation (ODE) with real variable right-hand side $q \in \mathbb{R}$

$$(\mathcal{L}h) = q, \quad 0 \leqslant x \leqslant d,$$

under the mixed initial/boundary conditions $h(0) = 0$, $h'(0) = 0$, and $h'(d) = 1$. It has the unique solution $(h, q) \in C^2(\mathbb{R}) \times \mathbb{R}$ given by

$$q = q_U := \frac{\sigma^2}{2}\frac{W(d)}{\int_0^d W(v)\mathrm{d}v}, \qquad h(x) = \frac{2q_U}{\sigma^2}\int_0^x\int_0^u \frac{W(v)}{W(u)}\mathrm{d}v\,\mathrm{d}u,$$

where

$$W(v) := \exp\left(\frac{2\alpha v}{\sigma^2}-\frac{\gamma v^2}{\sigma^2}\right).$$

Indeed, with this choice of h, we have

$$\mathrm{d}h(Z_t) = \sigma h'(Z_t)\mathrm{d}W_t - (\mathrm{d}U_t - q_U\,\mathrm{d}t).$$

Boundedness of $h(Z_t)$ (Z is a bounded process) combined with a central limit theorem for the martingale $\int_0^{\cdot}\sigma h'(Z_t)\mathrm{d}W_t$ was central in the proof of the central limit theorem for U_t; see Huang *et al.* (2014) for further details.

In the present paper, we deal with the one-sided reflected process Y and with a twice differential function h one obtains

$$\mathrm{d}h(Y_t) = (\mathcal{L}h)(Y_t)\mathrm{d}t+\sigma h'(Y_t)\mathrm{d}W_t+h'(0)\mathrm{d}L_t. \tag{3}$$

Two facts obstruct a direct application of the method earlier: (i) the process $h(Y_t)$ is not bounded, and (ii) we cannot immediately apply Lemma 2.1 to obtain a proper choice of h. Indeed, we needed three initial/boundary conditions to also determine the constant q, whereas now, we can only specify $h(0)$ and $h'(0)$. In the next section, we will see how to overcome these difficulties.

3 The central limit theorems

The main objective of this section is to derive a central limit theorem for L_t, for $t \to \infty$, and a functional version of it. We do so relying on the martingale techniques initiated in ZHANG and GLYNN (2011). We also consider other reflected processes.

3.1 Main results

Dealing with only a one-sided reflected process, we argue that the procedure as outlined in section 2 breaks down. In order to remedy this difficulty, we modify the procedure as follows. We need a separate argument that establishes the value of q_L that appears in Theorems 3.7 and 3.8 and the following variant of Lemma 2.1.

LEMMA 3.1. Let q be a given constant. Consider the ODE

$$(\mathcal{L}h)(x) = q,\ x \geq 0,$$

under the initial conditions $h(0) = 0$, $h'(0) = 1$. It has the unique solution $h \in C^2(\mathbb{R})$ given by

$$h(x) = \int_0^x \frac{1}{W(u)} \left(1 + \frac{2q}{\sigma^2} \int_0^u W(v)\mathrm{d}v\right) \mathrm{d}u,$$

where

$$W(v) := \exp\left(\frac{2\alpha v}{\sigma^2} - \frac{\gamma v^2}{\sigma^2}\right).$$

PROOF. One easily verifies (like in the proof of the corresponding result in HUANG *et al.*, 2014) that the general solution is

$$h(x) = C_2 + C_1 \int_0^x \frac{1}{W(u)}\mathrm{d}u + \frac{2q}{\sigma^2} \int_0^x \int_0^u \frac{W(v)}{W(u)}\mathrm{d}v\,\mathrm{d}u.$$

Then, the initial conditions $h(0) = 0, h'(0) = 1$ uniquely determine the values of C_1, C_2. Indeed, we obtain $C_2 = 0$ and $C_1 = 1$, and so

$$h(x) = \int_0^x \frac{1}{W(u)}\mathrm{d}u + \frac{2q}{\sigma^2} \int_0^x \int_0^u \frac{W(v)}{W(u)}\mathrm{d}v\,\mathrm{d}u.$$

□

Next, we give the solution as presented in Lemma 3.1 a different appearance. First, we need the fact that the invariant distribution of Y is truncated normal; see Ward and Glynn, 2003b; Proposition 1). If X is the solution to a nonreflected equation, an ordinary OU process, its invariant distribution is $N(\frac{\alpha}{\gamma}, \frac{\sigma^2}{2\gamma})$. Let ξ be a random variable having this distribution and denote its density by $p_{\mathrm{OU}}(x)$. By a simple computation, one obtains

$$W(x) = p_{\mathrm{OU}}(x) \exp\left(\frac{\alpha^2}{\gamma\sigma^2}\right) \sqrt{\pi\sigma^2/\gamma}.$$

Furthermore, $\mathbb{P}(\xi > 0) = \int_0^\infty p_{\mathrm{OU}}(x)\,\mathrm{d}x = \Phi(\frac{\alpha}{\sqrt{\sigma^2\gamma/2}})$ with Φ the cumulative distribution function of the standard normal distribution. The invariant density p_Y of Y is given by (here and further below $y \geq 0$)

$$p_Y(y) = \frac{p_{\mathrm{OU}}(y)}{\int_0^\infty p_{\mathrm{OU}}(u)\,\mathrm{d}u} = \frac{W(y)}{\int_0^\infty W(u)\,\mathrm{d}u}$$

or, in explicit terms,

$$p_Y(y) = \frac{1}{\mathbb{P}(\xi > 0)\sqrt{\pi\sigma^2/\gamma}} \exp\left(-\frac{\gamma}{\sigma^2}(y - \frac{\alpha}{\gamma})^2\right).$$

Note further that

$$p_Y(0) = \frac{p_{\mathrm{OU}}(0)}{\mathbb{P}(\xi > 0)} = \frac{\exp(-\frac{\alpha^2}{\gamma\sigma^2})}{\mathbb{P}(\xi > 0)\sqrt{\pi\sigma^2/\gamma}},$$

from which it follows that

$$W(y) = \frac{p_Y(y)}{p_Y(0)}. \tag{4}$$

Let η be a random variable with density p_Y. We proceed by computing $\mathbb{E}\eta$.

Lemma 3.2. It holds that

$$\mathbb{E}\eta = \int_0^\infty y p_Y(y)\,\mathrm{d}y = \frac{\sigma^2}{2\gamma} p_Y(0) + \frac{\alpha}{\gamma}.$$

Let Y be in its stationary regime. Then, $q_L := \frac{\mathrm{d}\mathbb{E}L_t}{\mathrm{d}t} = \frac{\sigma^2}{2} p_Y(0)$.

Proof. Note the identity

$$\frac{\mathrm{d}p_Y(y)}{\mathrm{d}y} = -\frac{2\gamma}{\sigma^2}\left(y - \frac{\alpha}{\gamma}\right) p_Y(y).$$

Hence,

$$\begin{aligned}
\mathbb{E}\eta &= \int_0^\infty \left(y - \frac{\alpha}{\gamma}\right) p_Y(y)\,\mathrm{d}y + \int_0^\infty \frac{\alpha}{\gamma} p_Y(y)\,\mathrm{d}y \\
&= -\frac{\sigma^2}{2\gamma}\int_0^\infty \frac{\mathrm{d}p_Y(y)}{\mathrm{d}y}\,\mathrm{d}y + \frac{\alpha}{\gamma} \\
&= \frac{\sigma^2}{2\gamma} p_Y(0) + \frac{\alpha}{\gamma}.
\end{aligned}$$

The next step is to determine q_L. Let Y be in its stationary regime. From the SDE for Y, we acquire

$$0 = (\alpha - \gamma\,\mathbb{E}\eta) + q_L.$$

Using the aforementioned expression for $\mathbb{E}\eta$, we obtain $q_L = \gamma\mathbb{E}\eta - \alpha = \frac{\sigma^2}{2}p_Y(0)$. □

REMARK 3.3. In GLYNN and WANG (2015), a paper on more general reflecting diffusion processes, with a broader scope also covering results on large deviations, the value of q_L for a one-sided reflected process is postulated in Eq. (23), but the exact derivation is not completely given. The result can be derived from results in that paper for two-sided reflected processes with an upper boundary b, if one imposes an extra boundary condition, $\lim_{b\to\infty} h'(b) = 0$ with sufficiently fast convergence, but some additional motivation is needed to justify this. To offset the effect of the nonconverging integral for $x \to \infty$ in Eq. (20) of the cited paper, when applied to OU case, one needs that the first factor in Eq. (19) tends to zero, which implies Eq. (23). See Proposition 3.5 and Remark 3.6 for a follow-up discussion and a more precise argument. Not imposing this additional boundary condition motivated our alternative approach, as already alluded to at the end of section 2.

LEMMA 3.4. Let $q = -q_L$ and let h be the function as in Lemma 3.1. Then, for $x > 0$, $h'(x) = \frac{p_Y(0)}{p_Y(x)}\bar{F}_Y(x)$, where $\bar{F}_Y(x) = 1 - F_Y(x)$, with F_Y the distribution function associated with the invariant density p_Y. Moreover,

$$\int_0^\infty h'(x)^2 p_Y(x)\,\mathrm{d}x < \infty. \tag{5}$$

PROOF. First, we note that

$$h'(x) = \frac{1}{W(x)}\left(1 - \frac{2q_L}{\sigma^2}\int_0^x W(v)\mathrm{d}v\right). \tag{6}$$

Use now $q_L = \frac{\sigma^2}{2}p_Y(0)$ to acquire $1 - \frac{2q_L}{\sigma^2}\int_0^x W(v)\mathrm{d}v = 1 - p_Y(0)\int_0^x W(v)\,\mathrm{d}v$, and recall Equation (4), $p_Y(0)W(v) = p_Y(v)$, to arrive at $1 - \frac{2q_L}{\sigma^2}\int_0^x W(v)\mathrm{d}v = 1 - \int_0^x p_Y(v)\,\mathrm{d}v = \bar{F}_Y(x)$. The first result follows.

To prove the second result, we note that $p_Y(x)/\bar{F}_Y(x) = p_{\mathrm{OU}}(x)/\bar{F}_{\mathrm{OU}}(x)$. Recall that for $x \to \infty$, it holds that $\frac{\bar{\Phi}(x)}{\phi(x)} \sim \frac{1}{x}$ (ϕ is the density of $N(0,1)$). Hence, we have $\frac{\bar{F}_Y(x)}{p_Y(x)} \sim \frac{\sigma^2/2\gamma}{x}$, and for large x, it follows that $\int_x^\infty h'(y)^2 p_Y(y)\,\mathrm{d}y < \infty$. □

PROPOSITION 3.5. Consider the ODE of Lemma 3.1 with the given initial conditions and $q > 0$. To have that the solution h satisfies Equation (5), it is necessary and sufficient that $q = -q_L$ as given in Lemma 3.2.

Proof. Consider the analogue of expression 6 for arbitrary q, that is,

$$h'(x) = \frac{1}{W(x)}\left(1 + \frac{2q}{\sigma^2}\int_0^x W(v)\mathrm{d}v\right).$$

Assume that Equation (5) holds. From Equation (4), this amounts to

$$\int_0^\infty \frac{p_Y(0)}{W(x)}\left(1 + \frac{2q}{\sigma^2}\int_0^x W(v)\,\mathrm{d}v\right)^2 \mathrm{d}x < \infty.$$

As $\frac{1}{W(x)}$ is not integrable over $[0,\infty)$, the aforementioned integral can only be finite if the square in the integrand tends to zero for $x \to \infty$, so one must have $q = -\frac{\sigma^2}{2}\int_0^\infty W(v)\,\mathrm{d}v$. But $\int_0^\infty W(v)\,\mathrm{d}v = \frac{1}{p_Y(0)}\int_0^\infty p_Y(v)\,\mathrm{d}v = \frac{1}{p_Y(0)}$. It follows that $q = -q_L$ is a necessary condition. Sufficiency has been proved in Lemma 3.4. □

Remark 3.6. Finiteness of the integral in Equation (5) is essential for the central limit theorems further down to hold. If we consider the situation of Lemma 3.1, but with variable q, then the integrability condition leads to the unique solution $q = -q_L$ along with the given solution h to the ODE for $q = -q_L$. As such, the initial conditions in the lemma together with the integrability condition are the three conditions needed to uniquely solve the ODE with variable q. The integrability condition can thus be viewed as an alternative to determine the value of q_L, without invoking the stationary distribution of Y as in Lemma 3.2.

Here is the first central limit theorem, the counterpart of Proposition 6 in Huang *et al.* (2014).

Theorem 3.7. Let h be as in Lemma 3.1 for $q = -q_L$. For $t \to \infty$, the law of $\frac{L_t - q_L t}{\sqrt{t}}$ weakly converges to $N(0,\tau^2)$, where $\tau^2 = \sigma^2\int_0^\infty h'(x)^2 p_Y(x)\,\mathrm{d}x < \infty$.

Proof. Let h be as in Lemma 3.1 for $q = -q_L$, and consider $h(Y_t)$. Itô's rule gives

$$h(Y_t) - h(Y_0) = \int_0^t \mathcal{L}h(Y_s)\,\mathrm{d}s + \sigma\int_0^t h'(Y_s)\,\mathrm{d}W_s + \int_0^t h'(Y_s)\,\mathrm{d}L_s.$$

Property 2 of L together with $h'(0) = 1$ give $\int_0^t h'(Y_s)\,\mathrm{d}L_s = L_t$. Combine this with $\mathcal{L}h(Y_s) = -q_L$ to arrive at

$$h(Y_t) - h(Y_0) = -q_L t + \sigma\int_0^t h'(Y_s)\,\mathrm{d}W_s + L_t.$$

Hence,

$$\frac{L_t - q_L t}{\sqrt{t}} = \frac{h(Y_t) - h(Y_0)}{\sqrt{t}} - \frac{1}{\sqrt{t}}\sigma\int_0^t h'(Y_s)\,\mathrm{d}W_s. \tag{7}$$

As $Y_t \to \eta$ in distribution, where η is distributed according to the invariant distribution of Y, we also have by the continuous mapping theorem $h(Y_t) \to h(\eta)$ in distribution, and one then also has the convergence $\frac{h(Y_t)-h(Y_0)}{\sqrt{t}} \to 0$ in probability, denoted $\frac{h(Y_t)-h(Y_0)}{\sqrt{t}} \stackrel{\mathbb{P}}{\to} 0$. Hence, the distributional limit of $\frac{L_t - q_L t}{\sqrt{t}}$ is completely determined by that of the second term on the right in Equation (7).

Next, we apply the large time central limit theorem for continuous local martingales. It says the following. Let M be a continuous local martingale with quadratic variation process $\langle M \rangle$. If $\frac{\langle M \rangle_t}{t} \stackrel{\mathbb{P}}{\to} \tau^2$ as $t \to \infty$, where τ^2 is a positive constant, then $\frac{M_t}{\sqrt{t}}$ converges in distribution, as $t \to \infty$, to a random variable with a $N(0, \tau^2)$ distribution. We apply this to the local martingale $M_t = -\sigma \int_0^t h'(Y_s)\,\mathrm{d}W_s$ for which we have $\langle M \rangle_t = \sigma^2 \int_0^t h'(Y_s)^2\,\mathrm{d}s$.

By the ergodic theorem (Gīhman and Skorohod, 1972; p. 134),

$$\frac{1}{t}\langle M \rangle_t \stackrel{\mathbb{P}}{\to} \sigma^2 \int_0^\infty h'(x)^2 p_Y(x)\,\mathrm{d}x = \tau^2, \tag{8}$$

where the right-hand side is finite according to Lemma 3.4. Hence, $-\frac{1}{\sqrt{t}}\sigma \int_0^t h'(Y_s)\,\mathrm{d}W_s \to N(0, \tau^2)$ in distribution. □

With a bit more effort, we obtain a functional version of the aforementioned theorem.

Theorem 3.8. The centered and scaled loss process L^n defined by $L_t^n := \frac{L_{nt} - q_L nt}{\tau\sqrt{n}}$ converges weakly in $C[0,\infty)$ with the locally uniform metric to a standard Brownian motion as $n \to \infty$.

Proof. We have, as in the proof of Theorem 3.7,

$$\frac{L_{nt} - q_L nt}{\sqrt{n}} = \frac{h(Y_{nt}) - h(Y_0)}{\sqrt{n}} - \frac{\sigma}{\sqrt{n}} \int_0^{nt} h'(Y_s)\mathrm{d}W_s.$$

Let M^n be the local martingale given by $M_t^n = -\frac{\sigma}{\sqrt{n}} \int_0^{nt} h'(Y_s)\mathrm{d}W_s$, $n \geq 1$. Then, the quadratic variation processes are given by

$$\langle M^n \rangle_t = \frac{\sigma^2}{n} \int_0^{nt} h'(Y_s)^2 \mathrm{d}s.$$

By the ergodic theorem (Gīhman and Skorohod, 1972; p. 134) again, and similar to Equation (8), for arbitrary $t \in [0, \infty)$,

$$\langle M^n \rangle_t = \sigma^2 t \frac{1}{nt} \int_0^{nt} h'(Y_s)^2 \mathrm{d}s \stackrel{\mathbb{P}}{\to} \tau^2 t, \text{ as } n \to \infty,$$

and hence, by the functional martingale central limit theorem, we have weak convergence of the local martingales $\frac{1}{\tau}M^n$ to a standard Brownian motion. The claim will be proved by applying the functional limit theorem for semimartingales (Shiryayev, 1981; Thm. 3) to L^n, for which it is now sufficient to show that for every $T > 0$,

$$\sup_{t\le T} \frac{h(Y_0) - h(Y_{nt})}{\sqrt{n}} \to 0 \text{ in probability for } n \to \infty.$$

We have seen in the proof of Lemma 3.4 that $h'(x)$ behaves as a constant times $\frac{1}{x}$ for large values of x. Hence ,$h(x)$ for large x behaves as $\log x$, and therefore, it is sufficient to show that

$$\sup_{t\le T} \frac{\log(Y_{nt} + 1)}{\sqrt{n}} \to 0 \text{ in probability for } n \to \infty. \tag{9}$$

We use the following trivial estimate. Because $Y_t \ge 0$, we have $Y_t \le Y_0 + \alpha t + \sigma W_t + L_t$. Hence, denoting $W_t^* = \sup_{s\le t} |W_s|$ and recalling Doob's inequality, $\mathbb{E}(W_t^*)^2 \le 4t$, we have $\sup_{t\le T} Y_{nt} \le \alpha nT + \sigma W_{nT}^* + L_{nT}$ and $\mathbb{E}\sup_{t\le T} Y_{nt} \le \alpha nT + \sqrt{\mathbb{E}(\sigma W_{nT}^*)^2} + \mathbb{E}L_{nT} = \alpha nT + \sqrt{\sigma^2 4nT} + \mathbb{E}L_{nT}$.

But then, by Jensen's inequality,

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\log(Y_{nt}+1) &= \mathbb{E}\log(\sup_{t\le T} Y_{nt}+1) \le \log\mathbb{E}(\sup_{t\le T} Y_{nt}+1)\\ &\le \log(\alpha nT + \sqrt{\sigma^2 4nT} + \mathbb{E}L_{nT})),\end{aligned}$$

from which it follows that ($\mathbb{E}L_{nT}/n$ has a finite limit $q_L T$ for $n \to \infty$)

$$\limsup_{n\to\infty} \mathbb{E}\sup_{t\le T} \frac{\log(Y_{nt}+1)}{\sqrt{n}} \le \limsup_{n\to\infty} \frac{1}{\sqrt{n}}(\log n + \log(\alpha T + \sqrt{\sigma^2 4T/n} + \mathbb{E}L_{nT}/n)) = 0.$$

As the supremum is trivially nonnegative, we conclude that $\sup_{t\le T} \frac{\log(Y_{nt}+1)}{\sqrt{n}} \to 0$ in L^1, as $n \to \infty$, which is sufficient for Equation (9) to hold. □

3.2 Other reflecting boundaries

In this section, we study processes that are (lower or upper) reflected at other boundaries. As the results immediately follow from Theorem 3.8 or can be proven in a similar fashion, we state them without proofs.

First, we consider a process reflected at another lower boundary than zero, which we reduce by translation to the previous case. Let ℓ be this boundary and consider the one-sided lower reflected process Y^ℓ that is such that $Y_t^\ell \ge \ell$ and given by the SDE

$$\mathrm{d}Y_t^\ell = (\alpha - \gamma Y_t^\ell)\,\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t^\ell,$$

where L^ℓ is the minimal increasing process that renders $Y_t^\ell \geq \ell$ for all $t \geq 0$. Put $Y_t := Y_t^\ell - \ell$ to find

$$\mathrm{d}Y_t = (\alpha^\ell - \gamma Y_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}L_t^\ell,$$

with $\alpha^\ell = \alpha - \gamma\ell$. Note that $\int_0^\infty \mathbf{1}_{\{Y_t>0\}}\,\mathrm{d}L_t^\ell = 0$. It follows that one can obtain a central limit theorem for $Ł^\ell$ from the result in the previous section. We need that the stationary density of Y^ℓ is (at ℓ) truncated normal. For $y > \ell$, one has

$$p_{Y^\ell}(y) = \frac{1}{\Phi(\frac{\alpha-\gamma\ell}{\sqrt{\sigma^2\gamma/2}})}\frac{1}{\sqrt{\pi\sigma^2/\gamma}}\exp\left(-\frac{\gamma}{\sigma^2}(y-\frac{\alpha}{\gamma})^2\right).$$

We also need $q_\ell = \frac{\sigma^2}{2}p_{Y^\ell}(\ell)$ and the function h_ℓ, which is for $y > \ell$ given by $h_\ell(y) = \int_\ell^y h_\ell'(x)\,\mathrm{d}x$, with $h_\ell'(x) = \frac{p_{Y^\ell}(\ell)}{p_{Y^\ell}(x)}\bar{F}_{Y^\ell}(x)$ and $\tau_\ell^2 = \sigma^2\int_\ell^\infty (h_\ell'(x))^2 p_{Y^\ell}(x)\,\mathrm{d}x$. Note that $\mathcal{L}h(x) = -q_\ell$. The precise result is as follows.

Theorem 3.9. The centered and scaled loss process $L^{\ell,n}$ defined by $L_t^{\ell,n} := \frac{L_{nt}^\ell - q_\ell nt}{\tau_\ell\sqrt{n}}$ converges weakly in $C[0,\infty)$ with the locally uniform metric to a Brownian motion as $n \to \infty$.

Next, we turn to upper reflected processes. Let $d \in \mathbb{R}$ and consider the one-sided upper reflected process Z that is such that $Z_t \leq d$ and given by the SDE

$$\mathrm{d}Z_t = (\alpha - \gamma Z_t)\,\mathrm{d}t + \sigma\mathrm{d}W_t - \mathrm{d}U_t,$$

where U is the minimal increasing process that renders $Z_t \leq d$ for all $t \geq 0$. Note that $\int_0^\infty \mathbf{1}_{\{Z_t<d\}}\,\mathrm{d}U_t = 0$. By 'flipping', we can reduce this case to the one with reflection at a lower boundary. Let $\tilde{Y}_t := d - Z_t$, then we find

$$\mathrm{d}\tilde{Y}_t = (\tilde{\alpha} - \gamma\tilde{Y}_t)\,\mathrm{d}t - \sigma\mathrm{d}W_t + \mathrm{d}U_t,$$

with $\tilde{\alpha} = -\alpha + \gamma d$. It follows that one can obtain a central limit theorem for U from the results in the previous section. Almost all that is needed is to express all quantities needed in terms of $\tilde{\alpha} = -\alpha + \gamma d$ instead of in α. For instance, the invariant density p_Z of Z can be derived from the invariant density of $\tilde{Y}$. It is at zero truncated $N(\frac{\tilde{\alpha}}{\gamma}, \frac{\sigma^2}{2\gamma})$, and one has (for $z < d$) $p_Z(z) = p_{\tilde{Y}}(d-z)$, explicitly,

$$p_Z(z) = \frac{1}{\Phi(\frac{\gamma d-\alpha}{\sigma^2\gamma/2})}\frac{1}{\sqrt{\frac{\pi\sigma^2}{2\gamma}}}\exp\left(-\frac{\gamma}{\sigma^2}(z-\frac{\alpha}{\gamma})^2\right).$$

We also need (for $z \leq d$) $h_Z(z) = -\int_z^d h_Z'(u)\,\mathrm{d}u$, where $h_Z'(z) := \frac{p_Z(d)}{p_Z(z)}F_Z(z)$, with $F_Z(z) = \int_{-\infty}^z p_Z(u)\,\mathrm{d}u$. Note that $h_Z(d) = 0$, $h_Z'(d) = 1$ and $\mathcal{L}h_Z(z) = q_U$, where $q_U = \frac{\sigma^2}{2}p_Z(d)$.
The precise result is as follows.

PROPOSITION 3.10. Let $q_U = \frac{\sigma^2}{2} p_Z(d)$ and $\tau_U^2 = \sigma^2 \int_{-\infty}^{d} h'_Z(z)^2 p_Z(z)\,dz$. For $n \to \infty$, we have weak convergence of the scaled and centered process U^n defined by $U_t^n = (n\tau_U^2)^{-1/2}(U_{nt} - q_U nt)$ to a standard Brownian motion.

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