On Markov chains and point processes

Peter Spreij
Department of Econometrics
Free University
De Boelelaan 1105
1081 HV Amsterdam

Abstract In this paper we first review some well known results for continuous time Markov processes, that live on a finite state space. Then special attention is paid to the construction of a continuous time Markov process and a filtration in continuous time, starting from a discrete time Markov chain and a filtration in discrete time. The Markov property here holds with respect to filtrations that need not be minimal. A complete version of this paper, including proofs, will appear elsewhere.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space. Assume that the *filtration* $\mathbb{F}$ satisfies the usual conditions in the sense of Dellacherie & Meyer [3]. Let X be a $\mathbb{F}$-Markov process with a finite state space. Without loss of generality we can assume that the state space is the standard basis of the Euclidian space $\mathbb{R}^m$. Call this set $B^m = \{b_1, \ldots, b_m\}$. (Indeed, if ξ is a stochastic process with values in a set $\{z_1, \ldots, z_m\}$, where all the z_i are different, then we can define the process X with values in B^m by $X_t = b_i$ iff $\xi_t = z_i$. Hence the probabilistic strucure of ξ determines that of X and vice versa). By X is $\mathbb{F}$-Markov it is meant that for all $t \geq s$ and for all $b \in B^m$ one has $P(X_t = b|\mathcal{F}_s) = P(X_t = b|\sigma(X_s))$. Denote by $\Phi(t,s)$ the $m \times m$ matrix with elements $\Phi_{ij}(t,s) = P(X_t = b_i|X_s = b_j)$ and let (the limit is assumed to exist) $A(t) = \lim_{h \downarrow 0} \frac{1}{h}[\Phi(t+h,t) - I]$. In this paper we assume that actually $A(t)$ is independent of t, so we write A instead. We call A the generating matrix of X.

Introduce also the following notation. For $k \in \{0, 1, 2, \ldots\}$ let T_k be the time of the k-th transition of X and $S_{k+1} = T_{k+1} - T_k$. Let furthermore Λ be the diagonal matrix with elements $\Lambda_{ii} = \lambda_i = -A_{ii}$. Assume that the $\lambda_i > 0$, then Λ is invertible, and the T_k are finite a.s.

We have the following wellknown result.

Theorem 1 *(i) For all $k \geq 0$ we have that S_{k+1} has, conditionally on $\mathcal{F}_{T_k}$, an exponential distribution with mean $\frac{1}{\lambda X_{T_k}}$.*
(ii) For all $k \geq 0$ it holds that $E[X_{T_{k+1}}|\mathcal{F}_{T_k}] = [A\Lambda^{-1} + I]X_{T_k}$.
(iii) $X_{T_{k+1}}$ and $S_{T_{k+1}}$ are conditionally independent given $\mathcal{F}_{T_k}$.

Corollary 2 *The embedded process $x : \Omega \times \{0, 1, 2, \ldots\} \to B^m$, defined by $x_k = X_{T_k}$, is a Markov chain w.r.t. the dicrete time filtration $\{\mathcal{G}_n\}_{n \geq 0}$, defined by $\mathcal{G}_n = \mathcal{F}_{T_n}$, and has transition matrix $\tilde{A}$ given by $\tilde{A} = A\Lambda^{-1} + I$.*

REMARK: Observe that for an embedded Markov chain x necessarily the $\tilde{A}_{ii} = 0$, and hence $P(x_{k+1} = x_k) = 0$ for all k.

We obtained in theorem 1 and corollary 2 the distribution of the embedded chain and the distribution of the jump times of the Markov chain. We proceed with following the road in the opposite direction. That is, starting from a Markov chain in discrete time and a sequence of exponentially distributed random variables, we construct a Markov chain in continuous time. A similar construction by another approach can be found in Doob [4], section VI.1 and Gihman & Skorohod [5] sections III.1 and III.3, with the restrictions that the filtrations are generated by the processes involved. Here we allow more general filtrations. In Jacod [6], section III.2b properties of filtrations like the one that is introduced below are described, for the case where these are generated by a multivariate point process. We also mention the paper [1] by Boel, Varaiya and Wong.

The basic assumptions for the rest of the paper are the following. Let $(\Omega, \mathcal{F}, \mathbb{G}, P)$ be a filtered probability space. $\mathbb{G}$ is a filtration in discrete time, $\mathbb{G} = \{\mathcal{G}_k\}_{k \in \{0,1,2,\ldots\}}$. Denote by $\mathcal{G}_\infty$ the

σ-algebra $\bigvee_{n\geq 0}\mathcal{G}_n$. Let $T_n : \Omega \to [0,\infty]$ for each $n \in \{0,1,2,\ldots\}$ be a random variable. Assume moreover that for all n $T_{n+1} \geq T_n$ and that strict inequality holds if $T_n < \infty$ and that the T_n are $\mathcal{G}_n$-measurable. Furthermore we assume that we have an auxiliary measurable space $(E,\mathcal{E})$ and a stochastic process $x : \Omega \times \{0,1,2,\ldots\} \to E$. Then a continuous time process $X : \Omega \times [0,\infty) \to E$ is defined by

$$X_t = \sum_{k=0}^{\infty} x_k 1_{\{T_k \leq t < T_{k+1}\}} \tag{1}$$

A filtration in continuous time is defined in

Definition 3 *Let for each $t \in [0,\infty)$ the set $\mathcal{H}_t$ be defined as follows: $\mathcal{H}_t = \{F \in \mathcal{F} : \forall k : \exists G_k \in \mathcal{G}_k$ such that $F \cap \{T_{k+1} > t\} = G_k \cap \{T_{k+1} > t\}\}$.*

Then we have the following (similar to Jacod [6], proposition (3.39))

Proposition 4 *The family $\mathbb{H} = \{\mathcal{H}_t\}_{t\in[0,\infty)}$ is a right continuous filtration on Ω, the T_n are $\mathbb{H}$-stopping times and the process X defined by equation (1) is $\mathbb{H}$-adapted.*

REMARK: One can also define filtrations by $\mathcal{H}'_t = \mathcal{H}_t \cap \mathcal{G}_\infty$. This corresponds to the filtration defined in Jacod [6] on page 84. Of course the $\mathcal{H}_t$ and $\mathcal{H}'_t$ coincide if $\mathcal{F} = \mathcal{G}_\infty$.

Proposition 5 *For all n one has $\mathcal{H}_{T_n} = \mathcal{G}_n \cap \mathcal{H}_\infty$, where $\mathcal{H}_\infty = \vee_{t\geq 0}\mathcal{H}_t$. and $\mathcal{H}_{T_{n+1}-} = \mathcal{H}_{T_n} \vee \sigma(T_{n+1})$.*

Next we discuss an application of the obtained results to multivariate point processes. Consider next to the T_n sequence a sequence of random variables Z_n^*, taking values in some auxiliary measurable space $(E,\mathcal{E})$. Define $Z_n = Z_n^* 1_{\{T_n<\infty\}}$, assuming that the product makes sense in E. Define then the $\mathcal{G}_n$ as

$$\mathcal{G}_n = \sigma(Z_0, Z_1, T_1, \ldots, Z_n, T_n). \tag{2}$$

Let now $x_n^* = (T_n, Z_n^*)$, $x_n = (T_n, Z_n)$ and for $t \in [0,\infty)$ we define X_t by $X_t = \sum_{n=0}^\infty x_n 1_{\{T_n\leq t<T_{n+1}\}} = \sum_{n=0}^\infty x_n^* 1_{\{T_n\leq t<T_{n+1}\}}$. Then X can be considered as a multivariate point process with the Z_n-sequence as marks. Following the usual convention all the events of X take place before T_∞. We claim the following:

Proposition 6 *The filtration $\mathbb{H}$, as defined with the $\mathcal{G}_n$ from equation (2), is identical with $\mathbb{F}^X$, the filtration generated by X.*

Notice that there is a little difference with for instance T30 in Brémaud [2], page 307, where (in our notation) $\mathcal{F}^X_{T_n} = \sigma\{Z_0^*, T_1, Z_1^*, \ldots, T_n, Z_n^*\}$. We cannot have this result here, since if for instance $T_1 = \infty$, then for all $n \geq 1$ one has $\mathcal{F}^X_{T_n} = \mathcal{G}_n = \sigma\{Z_0^*\} \neq \sigma\{Z_0^*, T_1, Z_1^*, \ldots, T_n, Z_n^*\}$. Of course the difference disappears if all the T_n are finite.

As a side remark we notice that the constructions in this paper allow for a generalization of the notion of a multivariate or marked point process as a sequence of pairs of random times and σ-algebras $\{(T_n, \mathcal{G}_n)\}$, where the T_n are $\mathcal{G}_n$-measurable.

In addition to the assumptions made before we impose the following conditions on the random variables x_n and T_n. Each x_n assumes its values in the set B^m and the sequence $\{x_n\}$ is Markov w.r.t. the filtration $\mathbb{G}$. Denote by A' the matrix of transition probalities of x, so $P(x_{n+1} = b_i | x_n = b_j) = A'_{ij}$. We do not assume that the A'_{ii} are zero, which is a necessary property of an embedded Markov chain (see the remark following corollary 2).
Define for all $n \in \mathbb{N}$ S_{n+1} as $T_{n+1} - T_n$ and assume that S_{n+1} has, conditionally on $\mathcal{G}_n$, an exponential distribution with density on $(0,\infty)$

$$\mathbf{1}^T \Lambda \exp(-\Lambda s) x_n,$$

where Λ is a diagonal matrix with entries $\Lambda_{ii} = \lambda_i$ and all $\lambda_i \geq 0$ and moreover that S_{n+1} and x_{n+1} are conditionally independent given $\mathcal{G}_n$. Here $\mathbf{1}^T$ is the vector $[1,\ldots,1] \in \mathbb{R}^{1\times m}$ and superscript T denotes transposition.

The main result of this paper is that the process X is a continuous time Markov process w.r.t. the filtration $\mathbb{H}$. The proof of this uses a key result that is contained in the following lemma, which tells us how to compute certain conditional expectations given $\mathcal{H}_t$ in terms of conditional expectations given $\mathcal{G}_n$.

Lemma 7 *Let Z be an integrable random variable. Then, with the convention $\frac{0}{0} = 0$,*

$$E[Z 1_{\{T_n\leq t<T_{n+1}\}} | \mathcal{H}_t] =$$

$$1_{\{T_n\leq t<T_{n+1}\}} \frac{E[Z 1_{\{T_n\leq t<T_{n+1}\}} | \mathcal{G}_n]}{E[1_{\{T_n\leq t<T_{n+1}\}} | \mathcal{G}_n]},$$

with

$$E[1_{\{T_n\leq t<T_{n+1}\}} | \mathcal{G}_n] =$$

$$1_{\{T_n \le t\}} x_n^T \exp(-\Lambda(t - T_n))\mathbf{1}.$$

Theorem 8 *For $t \ge s \ge 0$ it holds that*

$$E[X_t|\mathcal{H}_s] = e^{A(t-s)}X_s, \qquad (3)$$

with $A = (A' - I)\Lambda$. So X is a Markov process with respect to the filtration $\mathbb{H}$ with transition intensities given by A.

Notice that the Markov process X constructed this way has jump times that in general differ from the T_k, since it is not explicitely assumed that the diagonal elements of A' are zero. Denote by $\tilde{T}_k$ the jump times of X.
In order to avoid uninteresting complications, we assume that x has no absorbing states, so all the $A'_{ii} \neq 1$. If we denote in this case the embedded Markov chain by $\tilde{x}$, then it follows from 1, that $\tilde{x}$ has the transition matrix $\tilde{A}$ and the interarrival times $\tilde{S}_{k+1} = \tilde{T}_{k+1} - \tilde{T}_k$ are exponentially distributed given $\mathcal{H}_{\tilde{T}_k}$ with mean $\mathbf{1}^T\tilde{\Lambda}^{-1}\tilde{x}_{\tilde{T}_k}$, where the matrix $\tilde{A}$ has as its entries zeros on the diagonal and outside it

$$\tilde{A}_{ij} = \frac{A'_{ij}}{1 - A'_{jj}},$$

and the diagonal matrix $\tilde{\Lambda}$ has entries $\tilde{\Lambda}_{ii} = \lambda_i(1 - A'_{ii})$. (Notice that $\tilde{\Lambda} = \Lambda$ and $\tilde{A} = A'$ if all the A'_{ii} are zero).

References

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