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DIFFUSION LIMITS FOR A MARKOV MODULATED BINOMIAL COUNTING PROCESS

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In this paper, we study limit behavior for a *Markov-modulated* binomial counting process, also called a binomial counting process under *regime switching*. Such a process naturally appears in the context of *credit risk* when multiple obligors are present. Markov-modulation takes place when the failure/default rate of each individual obligor depends on an underlying Markov chain. The limit behavior under consideration occurs when the number of obligors increases unboundedly, and/or by accelerating the modulating Markov process, called *rapid switching*. We establish diffusion approximations, obtained by application of (semi)martingale central limit theorems. Depending on the specific circumstances, different approximations are found.

Keywords: central limit theorems, counting process, functional limit theorems, markov-modulated process

AMS subject classification: 60F17, 60F05, 60G99

1. INTRODUCTION

In this paper, we study scaling limits of a *Markov-modulated* (MM) counting process. Over the last decades Markov-modulation (as it is often referred to in the operations research literature) or *Regime switching* (common terminology in e.g. mathematical finance) has become increasingly popular. Regime switching basically explains itself with its name. The parameters of the stochastic process change with time and the behavior of the process changes. The way this is usually modeled is to make the parameters of the process a function of a *background process* (or *modulating process*), and commonly the background process is assumed to be a finite state Markov chain, say with values in a finite set of d elements. This explains the name Markov-modulation. The popularity of MM processes is due to the fact that they provide a more flexible model of reality than their non-modulated versions. It is natural to assume that a real-life phenomenon, which is modeled by a stochastic process,

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reacts to some environment which evolves autonomously. This is far more likely than the basic case in which the parameters are constant over time.

The process we consider has various applications. In (software) reliability modeling early variants are Jelinski and Moranda [33], Koch and Spreij [37], Littlewood [42]. The value of modeling (software) failures within randomly changing environments, including Markov-modulation, has been acknowledged for some time now, see for example, Özekici and Soyer [46,47], Ravishanker et al. [48]. In particular, MM variants of Jelinski and Moranda [33] have been studied, that is, in a Bayersian set-up in Landon et al. [38], with an estimation focus in Ando et al. [3], Hellmich [27] and with an added failure rate component in Subrahmaniam et al. [52]. A similar model to Jelinski and Moranda [33] has been used in epidemiology (see Andersson and Britton [2]) and a multivariate version of it in sampling design (see Berchenko et al. [9]), where the latter can also be used to model job switching behavior due to recruiters.

An early application of Markov-modulation in economic modeling is Hamilton [26]. Since then Markov-modulation has been extensively used in various branches of mathematical finance. For example, in optimal investment theory for pension funds (Chen and Delong [13]), interest rate modeling (Ang and Bekaert [4], Elliott and Mamon [19], Elliott and Siu [20]) and affine processes (van Beek et al. [53]). Other financial applications concern option and bond valuation (Buffington and Elliott [12], Elliott et al. [22], Jiang and Pistorius [35]), optimal dividend policies (Jiang [34], Jiang and Pistorius [36]), optimal portfolio and asset allocation (Elliott and Hinz [18], Elliott and Van der Hoek [21], Zhou and Yin [56]) and also most notably in the modeling of credit risk and credit derivatives (Banerjee [7], Banerjee et al. [8], Choi and Marcozzi [14], Dunbar and Edwards [16], Giampieri et al. [23], Hainaut and Colwell [25], Li and Ma [39], Liechty [40], Yin [55]). Markov-modulation has been used in insurance and risk theory as well (Asmussen and Albrecher [6]).

Outside mathematical finance, a rich area of applications of regime switching is in operations research, where there is a sizeable body of work on MM queues, see for example, Asmussen [5] and Neuts [44]. Contributions in this field with emphasis on scaling limits under rapid switching (leading to functional limit theorems which are also the subject of the present paper), are for example, Anderson et al. [1] and Blom et al. [10]. Similar scaling limits have been obtained in Huang et al. [28,29] and for example, large deviations under scaling have been treated in Huang et al. [30].

Following the considerable interest in MM financial models we consider scaling limits, also referred to as diffusion approximations, of a MM model that has a natural interpretation in Credit Risk, (see Mandjes and Spreij [43]). In the basic setting there are n obligors which have independently exponential distributed default times τ_i with intensity parameter $\lambda > 0$. In the MM case this parameter is MM, leading to an *intensity process* $\lambda_t = f(Z_t)$, say for a nonnegative function f, where Z is the Markovian background process. The process N counts the number of obligors that have defaulted. At time t, the random variable N_t is binomially distributed with parameters n and $p = 1 - \mathbb{E}\exp(-\int_0^t f(Z_s)\,\mathrm{d}s)$. Throughout the paper, we will often use the credit risk context for explanation and illustration of certain features of the model, although as was explained, applications are not limited to this branch of mathematical finance.

In the present paper, we study diffusion approximations (functional central limit theorems, Gaussian limits) for the process N, if we scale up the transition matrix of the underlying Markov chain by a factor α and let $n, \alpha \to \infty$. We find in principle different functional limits of the scaled and centered process, depending on the order in which parameters diverge, for example, first $\alpha \to \infty$, then $n \to \infty$, or the other way around, or if both α and n jointly tend to infinity, possibly with different rates. In addition, we will also study limit behavior for the case where the intensity vanishes at a certain rate as $n \to \infty$.

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The remainder of the paper is organized as follows. In Section 2, we collect some useful results for the background Markov chain. In Section 3, we first construct the truly binomial process and prove in Section 4 the first result on diffusion approximation. Section 5, the body of the paper, is devoted to Markov modulated processes and contains the main results; we prove several limit theorems for this process in which the influence of different rates for $\alpha \to \infty$ and $n \to \infty$ is clearly visible. Some numerical examples illustrating the main results are presented in Section 6. Finally, in Section 7, we sketch some results for the case (in a credit risk context) where defaulted companies re-enter the market.

2. THE BACKGROUND PROCESS

We will always work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that the background process Z is an ergodic (also called irreducible), time homogeneous Markov chain on a finite state space. Without loss of generality, we assume that it takes values in the set of basis vectors $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$, with transition rates

$$q_{ji} = \left.\frac{d}{dt}\right|_{t=0} \mathbb{P}(Z_t = e_j | Z_0 = e_i) \geq 0 \quad i \neq j$$

and $q_{ii} := -\sum_{j \neq i} q_{ji}$. We let Q be the matrix of all q_{ij}, also called the generator (of Z). Note that $\mathbb{1}^{\mathrm{T}} Q = 0$, where $\mathbb{1}$ is the vector of all ones. Since Z is ergodic, the limits $\pi_j = \lim_{t \to \infty} \mathbb{P}(Z_t = e_j | Z_0 = e_i)$, $i, j \in \{1, \ldots, d\}$ exist and are independent of i, and we have the column vector $\pi = (\pi_1, \ldots, \pi_d)^{\mathrm{T}}$ satisfying $Q\pi = 0$. The ergodic matrix is given by $\Pi := \pi \mathbb{1}^{\mathrm{T}}$, has columns equal to π and satisfies:

$$\Pi^2 = \Pi \quad \text{and} \quad \Pi Q = Q \Pi = 0.$$

The *fundamental matrix* is given by $F := (\Pi - Q)^{-1}$. and the *deviation matrix* is defined by $D := F - \Pi$. Basic properties are:

$$QF = FQ = \Pi - I, \quad F\mathbb{1} = \mathbb{1}, \quad \text{and} \quad \mathbb{1}^{\mathrm{T}} D = 0, \quad D\pi = 0, \tag{2.1}$$

where the zeros should be read as a row or column vector. The deviation matrix can also be computed by

$$D = \int_0^\infty \exp(Qs) - \Pi \, \mathrm{d}s,$$

which follows from Glynn [24], Eq. (2.14).

The deviation matrix of an ergodic Markov process can be interpreted as a measure of total deviation of the limiting probabilities. As such it will naturally find its way into the results of our limit theorems of the stochastic processes we observe. For a survey of the main results of deviation matrices we refer to Coolen-Schrijner and Van Doorn [15].

We will use a stochastic differential equation for Z_t. Given the process Z_t on $(\Omega, \mathcal{F})$, Markovian relative to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, with initial state z_0 and with generator Q, one has by Dynkin's formula Revuz and Yor [49], Proposition 1.6 that

$$\tilde{M}_t := Z_t - z_0 - \int_0^t Q Z_s \, \mathrm{d}s$$

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is a martingale relative to $\{\mathcal{F}_t\}_{t\geq 0}$. Rewriting this into a differential notation yields

$$\mathrm{d}Z_t = QZ_t\,\mathrm{d}t + \mathrm{d}\tilde{M}_t, \quad Z_0 = z_0. \tag{2.2}$$

This representation can be found in many papers, for example, in Elliott [17], where this result has a direct proof; see also Spreij [51] for a more general result. The martingale $\tilde{M}$ is square integrable, which implies that $\langle \tilde{M}\rangle_t$ exists. As a matter of fact, one has

$$\langle \tilde{M}\rangle_t = \int_0^t (\mathrm{diag}\{QZ_s\} - Q\mathrm{diag}\{Z_s\} - \mathrm{diag}\{Z_s\}Q^{\mathrm{T}})\,\mathrm{d}s, \tag{2.3}$$

see for example, Proposition 3.2 and its proof in Huang et al. [29], and

$$D\mathrm{diag}\{\pi\} + \mathrm{diag}\{\pi\}D^{\mathrm{T}} \text{ is nonnegative definite.} \tag{2.4}$$

Ergodicity of the Markov chain implies the *continuous-time ergodic theorem* (see Norris [45], Theorem 3.8.1). For $t\to\infty$, it holds that $1/t\int_0^t Z_s\,\mathrm{d}s \overset{a.s.}{\to} \pi$. Often we will use this result in the following form,

$$\frac{1}{m}\int_0^{mt} Z_s\,\mathrm{d}s \overset{a.s.}{\to} \pi t, \quad \text{when} \quad m\to\infty. \tag{2.5}$$

We close with a remark on notation. For any process X we will use the generic notation $\mathbb{F}^X$ for the filtration generated by X, that is, $\mathbb{F}^X = \{\mathcal{F}_t^X\}_{t\geq 0}$, with $\mathcal{F}_t^X = \sigma(X_s, 0\leq s\leq t)$.

3. THE MARKOV MODULATED BINOMIAL POINT PROCESS

The MM binomial point process, or counting process, (as we refer to it) is used in a variety of applications under which are software reliability and intensity-based credit risk modeling with the canonical set-up of n obligors and independent default times. Especially, the latter case provides a convenient context to explain some fundamental features of this process. Let us first introduce the non-modulated process. We assume there are n obligors with *independent* default times τ^i, $i\in\{1,\ldots,n\}$. All τ^i are exponentially distributed with parameter $\lambda>0$ which gives us that the process $Y_t^i = \mathbb{1}_{\{t\leq\tau^i\}}$ satisfies

$$\mathrm{d}Y_t^i = \lambda(1-Y_t^i)\,\mathrm{d}t + \mathrm{d}M_t^i, \quad Y_0^i = 0 \tag{3.1}$$

for a martingale M^i with respect to the filtration generated by Y^i and the τ^i. We then take $N_t := \sum_{i=1}^n Y_t^i$ as the first process we are interested in. It then follows from the independence assumption and Eq. (3.1) that we have for the process N the submartingale decomposition

$$\mathrm{d}N_t = \lambda(n-N_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad N_0 = 0 \tag{3.2}$$

where M is an $\mathbb{F}^N$-martingale. We note that this model has already been introduced in Software reliability models many years ago, see for example, Jelinski and Moranda [33], Koch and Spreij [37] for early contributions and other references in Section 1. Note that for fixed $t>0$, the random variable has a Binomial(n,p_t) distribution, with $p_t = 1-\exp(-\lambda t)$.

This model can be generalized in many ways to one in which the (default) intensity is not a constant λ, but a time-varying, random quantity λ_t. The distributional properties of N are then determined by specific choices of λ_t and equations like Eq. (3.2) and its variations further on are consequences of the general martingale characterization of counting processes, see for example, Brémaud [11], Theorem II.T8.

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Our interest is to take a MM rate $\lambda_t = \lambda^{\mathrm{T}} Z_{t-}$, where λ is now a *vector* in $\mathbb{R}^d_+$ (the meaning of the symbol λ thus depends on the context, but this should not cause any confusion), and Z is the indicator process of the Markov chain A, see Section 2. We then get the following stochastic differential equation (SDE) model for the process N,

$$\mathrm{d}N_t = \lambda^{\mathrm{T}} Z_t (n - N_t)\,\mathrm{d}t + \mathrm{d}M_t, \quad N_0 = 0 \tag{3.3}$$

where M is a martingale with respect to $\mathbb{F} = \{\mathcal{F}^Y_t \vee \mathcal{F}^Z_\infty, t \geq 0\}$, which can be justified by conditional independence of the default times, given the process Z. In this stochastic intensity case one has that N_t, given the full process Z, has a $\mathrm{Bin}(n, 1 - \exp(-\Lambda_t))$ distribution, with $\Lambda_t = \int_0^t \lambda_s\,\mathrm{d}s = \int_0^t \lambda^{\mathrm{T}} Z_s\,\mathrm{d}s$. We call N the MM binomial point process. See also Mandjes and Spreij [43] for further details on the construction of this process, and for a justification of the following reasonable assumption.

Assumption 3.1: *The processes N and Z never jump at the same time, that is, the optional quadratic covariation process $[N, Z]$ is identically zero (with probability one).*

There are also situations known where this assumption is violated by construction, see Spreij [50] for an example.

4. LIMIT THEOREMS FOR THE NON-MODULATED BINOMIAL PROCESS

Let us first, as a warming up and for future reference, consider the truly binomial non-modulated process. Recall Eq. (3.2), where we have that $\lambda > 0$ is a constant. Since the process N is distributed $\mathrm{Bin}(n, 1 - \exp(-\lambda t))$ we have $\mathbb{E}N_t = n(1 - \exp(-\lambda t))$. Below we will use $\varrho_t := 1 - \exp(-\lambda t))$, which satisfies the ODE

$$\dot{\varrho}_t = \lambda(1 - \varrho_t), \quad \varrho_0 = 0 \tag{4.1}$$

This will function as the centering process for N, as $\mathbb{E}N_t = n\varrho_t$, in the following proposition.

Proposition 4.1: *Let $\lambda > 0$ be constant and let N be given by Eq. (3.2). Then the scaled and centered process*

$$\hat{N}^n_t := n^{-1/2}(N_t - n\varrho_t)$$

converges weakly to the solution of the following SDE,

$$\mathrm{d}\hat{N}_t = -\lambda \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t, \quad \hat{N}_0 = 0$$

as $n \to \infty$. Here B is a continuous Gaussian martingale with $\langle B \rangle_t = 1 - e^{-\lambda t}$.

Proof: First we will determine the limit of the martingale $M^n = M/\sqrt{n}$ in Eq. (3.2). Note that $|\Delta M^n_t| = |\Delta M_t|/\sqrt{n} \leq 1/\sqrt{n} \to 0$, and that $\mathbb{E}M_t^2 < \infty$ for all t. We want to prove that $\langle M^n \rangle_t \xrightarrow{\mathbb{P}} C_t$ for some deterministic C_t, so that can apply the martingale central limit theorem Jacod and Shiryaev [31], Theorem VIII.3.11. By standard results for the compensator of a counting process, $\langle M \rangle_t = \int_0^t \lambda(n - N_s)\,\mathrm{d}s$.

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Using this expression for $\langle M\rangle_t$ we have that (if $n\to\infty$)

$$\begin{aligned}\langle M^n\rangle_t &= \int_0^t \lambda - \frac{\lambda N_s}{n}\,\mathrm{d}s\\ &\overset{a.s.}{\to} \int_0^t \lambda - \lambda \mathbb{E}Y_t^1\,\mathrm{d}s\\ &= \int_0^t \lambda e^{-\lambda s}\,\mathrm{d}s = 1-e^{-\lambda t},\end{aligned}$$

where we applied the dominated convergence theorem to establish almost sure convergence of N_t/n (dominated by 1) to $\mathbb{E}Y_t^1$ by the strong law of large numbers. Hence we can apply the citetd martingale central limit theorem to find that M converges weakly to a Gaussian martingale B with $\langle B\rangle_t = 1-e^{-\lambda t}$.

Now we consider the process $\hat{N}_t^n = n^{-1/2}(N_t - n\varrho_t)$. Taking the differentials and rewriting gives us

$$\mathrm{d}\hat{N}_t^n = -\lambda \hat{N}_t^n\,\mathrm{d}t + \mathrm{d}M_t^n.$$

Now we define $\hat{X}_t^n := \exp(\lambda t)\hat{N}_t^n$, to get $\mathrm{d}\hat{X}_t^n = e^{\lambda t}\,\mathrm{d}M_t^n$. By similar reasoning as in proofs of the next section where we spell out the details, we find that $\hat{X}^n$ converges in distribution to $\hat{X} = \int_0^{\cdot} \exp(\lambda t)\,\mathrm{d}B_t$ and we find that $\hat{N}_t^n \xrightarrow{d} \hat{N}$, where $\hat{N}$ satisfies the SDE

$$\mathrm{d}\hat{N}_t = -\lambda\hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t, \quad \hat{N}_0 = 0. \qquad \blacksquare$$

Remark 4.2: The binomial distribution of N_t for fixed t, can be exploited in an application of the ordinary central limit theorem, which tells us that $\hat{N}_t^n$ has a limiting normal distribution with variance $e^{-\lambda t}(1-e^{-\lambda t})$. This is, of course, in full agreement with the functional limit result of Proposition 4.1, as can quickly be seen by computing the variance of $\hat{N}_t$.

5. LIMIT RESULTS FOR THE MM BINOMIAL PROCESS

In this section, we will prove limit results for the MM binomial point process with a Markov modulated rate. In principle, one can prove various types of limit theorems. We focus on results in central limit form, that is, on diffusion approximations. These are obtained for $n\to\infty$ in Eq. (3.3), whereas we also investigate limit behaviors by scaling the generator of the background process Markov chain via $Q\mapsto\alpha Q$, and letting $\alpha\to\infty$. As we are interested in the limit behavior for both $n\to\infty$ and $\alpha\to\infty$, various possibilities occur. We will investigate iterated limits (first $n\to\infty$, then $\alpha\to\infty$ or vice versa), or joint limit behavior when certain specified relationships between n and α come into play. We shall also investigate the impact of different choices for the centering processes.

As a side remark, we mention that alternative scalings may lead to completely different limit results. For instance, if one would scale the vector λ to λ/n, keeping Q fixed, one would get a MM Poisson process, with intensity process $\lambda^{\mathrm{T}}Z_t$, see for example, Jacod and Shiryaev [31], Theorem VIII.4.10, or Liptser and Shiryaev [41], Theorem 1, p. 588. Another case, where the intensity is scaled as λ/n^γ with $\gamma\in(0,1)$, leading again to a diffusion limit, is treated at the end of this section.

Contrary to the non-modulated case, in the MM case the consequences of a scaling $Q\mapsto\alpha Q$ for some $\alpha\to\infty$, will have a major impact. To make the dependence of the corresponding processes on n and α explicit, we denote the resulting processes by $N^{n,\alpha}$, $M^{n,\alpha}$

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and Z^α, giving the following SDE which is an analogy to Eq. (3.3)

$$\mathrm{d}N_t^{n,\alpha} = \lambda^{\mathrm{T}} Z_t^\alpha (n - N_t^{n,\alpha})\,\mathrm{d}t + \mathrm{d}M_t^{n,\alpha}, \quad N_0^{n,\alpha} = 0. \tag{5.1}$$

We will prove functional limit theorems of central limit type. However the centering process ϱ will, in the case $n \to \infty$, not always be the asymptotic limit of the expectation. It may depend on α and we will make this explicit in the notation. We first present a theorem for $n \to \infty$ and then $\alpha \to \infty$. Second comes the theorem in which the limits are interchanged. We write λ_t^α for $\lambda^{\mathrm{T}} Z_t^\alpha$ and $\Lambda_t^\alpha = \int_0^t \lambda_s^\alpha\,\mathrm{d}s$.

THEOREM 5.1: *Let $N^{n,\alpha}$ be given by Eq. (5.1) for $\lambda \in \mathbb{R}_+^d$ and let ϱ^α be given by*

$$\dot{\varrho}_t^\alpha = \lambda^T Z_t^\alpha (1 - \varrho_t^\alpha), \quad \varrho_0^\alpha = 0.$$

Then the scaled and compensated process

$$\hat{N}_t^{n,\alpha} = n^{-1/2}(N_t^{n,\alpha} - n\varrho_t^\alpha),$$

converges, as $n \to \infty$, weakly to the solution of the following SDE

$$\mathrm{d}\hat{N}_t^\alpha = -\lambda^T Z_t^\alpha \hat{N}_t^\alpha\,\mathrm{d}t + \mathrm{d}B_t^\alpha, \quad \hat{N}_0^\alpha = 0 \tag{5.2}$$

where B^α is a continuous martingale with $\langle B^\alpha \rangle_t = 1 - \exp(-\Lambda_t^\alpha)$.

Moreover, for $\alpha \to \infty$, the process $\hat{N}^\alpha$ converges weakly to the solution of

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t, \quad \hat{N}_0 = 0 \tag{5.3}$$

where B is a Gaussian martingale with $\langle B \rangle_t = 1 - \exp(-\lambda_\infty t)$ where $\lambda_\infty = \lambda^T \pi$.

PROOF: We modify the proof of Proposition 4.1. We first view the scaled martingale $M^{n,\alpha}/\sqrt{n}$, with $M^{n,\alpha}$ defined in Eq. (5.1). As in the proof of Proposition 4.1 we see that $\Delta M^{n,\alpha}/\sqrt{n} \to 0$. Following the same arguments, we find that for the quadratic variation we have the expression $\langle M^{n,\alpha} \rangle_t = \int_0^t \lambda_s^\alpha (n - N_s^{n,\alpha})\,\mathrm{d}s$. Hence for the scaled martingale it holds that

$$\begin{aligned}\left\langle \frac{M^{n,\alpha}}{\sqrt{n}} \right\rangle_t &= \int_0^t \lambda_s^\alpha \left(1 - \frac{N_s^{n,\alpha}}{n}\right)\mathrm{d}s \\ &\underset{(n\to\infty)}{\overset{a.s.}{\to}} \int_0^t \lambda_s^\alpha \exp(-\Lambda_s^\alpha)\,\mathrm{d}s \\ &= 1 - \exp(-\Lambda_t^\alpha),\end{aligned}$$

where we have used dominated convergence and the conditional strong law of large numbers for the convergence $((N_s^{n,\alpha})/n) \overset{a.s.}{\to} \mathbb{E}[Y_s^1 | \mathcal{F}^Z] = 1 - \exp(-\Lambda_s^\alpha)$. It follows from the functional CLT for martingales with random quadratic variation to a conditional Gaussian martingale Liptser and Shiryaev [41], Theorem 4, p.567 that $M^{n,\alpha}$ converges to a continuous martingale B^α with $\langle B^\alpha \rangle_t = 1 - \exp(-\Lambda_t^\alpha)$.

One easily derives that $\hat{N}^{n,\alpha}$ is the solution to

$$\mathrm{d}\hat{N}_t^{n,\alpha} = -\lambda_t^\alpha \hat{N}_t^{n,\alpha}\,\mathrm{d}t + n^{-1/2}\,\mathrm{d}M_t^{n,\alpha},$$

implying that

$$\hat{N}_t^{n,\alpha} = n^{-1/2} \exp(-\Lambda_t^\alpha) \int_0^t \exp(\Lambda_s^\alpha)\,\mathrm{d}M_s^{n,\alpha}.$$

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It follows from the above, the validity of the P-UT condition for martingales, Jacod and Shiryaev [31], VI.6.13 and the weak convergence theorem for stochastic integrals, Jacod and Shiryaev [31], VI.6.22 that $\hat{N}^{n,\alpha}$ converges to a process $\hat{N}^{\alpha}$, given by

$$\hat{N}^{\alpha}_t = \exp(-\Lambda^{\alpha}_t) \int_0^t \exp(\Lambda^{\alpha}_s)\,\mathrm{d}B^{\alpha}_s, \tag{5.4}$$

which is the solution to Eq. (5.2).

We next consider the convergence of $\hat{N}^{\alpha}$ for $\alpha \to \infty$. From the ergodic theorem for Markov chains (see Eq. (2.5)), we obtain, for $\alpha \to \infty$, $\int_0^t Z^{\alpha}_s\,\mathrm{d}s \overset{a.s.}{\to} \pi t$, and hence $\Lambda^{\alpha}_t \to \lambda_\infty t$ and $\exp(\Lambda^{\alpha}_t) \to \exp(\lambda_\infty t)$ a.s. As these processes are increasing and the limit is continuous we can apply Jacod and Shiryaev [31], Thm VI.2.15(c) to find that this convergence is uniform on compact sets,

$$\sup_{s\le T} |\exp(\Lambda^{\alpha}_s) - \exp(\lambda_\infty s)| \overset{\mathbb{P}}{\to} 0 \quad \text{as} \quad n \to \infty. \tag{5.5}$$

Furthermore, we also obtain $\langle B^{\alpha}\rangle_t \overset{\mathbb{P}}{\to} 1 - \exp(-\lambda_\infty t)$. Hence by the CLT for martingales again, we have the weak convergence of B^{α} to a continuous martingale B with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$. By the same arguments as above, we have that the stochastic integral process in Eq. (5.4) converges to $\int_0^{\cdot} \exp(\lambda_\infty s)\,\mathrm{d}B_s$, and therefore $\hat{N}^{\alpha}$ converges to the process $\hat{N}$ given by $\hat{N}_t = \exp(-\lambda_\infty t)\int_0^t \exp(\lambda_\infty s)\,\mathrm{d}B_s$, which is the solution to Eq. (5.3). ■

THEOREM 5.2: *With the assumptions and notation of Theorem 5.1 we have, for $\alpha \to \infty$ that the counting processes $N^{n,\alpha}$ converge to the counting process N^n whose submartingale decomposition is*

$$\mathrm{d}N^n_t = \lambda_\infty (n - N^n_t)\,\mathrm{d}t + \mathrm{d}M^n_t, \quad N^n_0 = 0.$$

Equivalently, the centered processes $\hat{N}^{n,\alpha}$ converge weakly to $\hat{N}^n$ defined as the solution of the SDE

$$\mathrm{d}\hat{N}^n_t = -\lambda_\infty \hat{N}^n_t\,\mathrm{d}t + \mathrm{d}\hat{M}^n_t, \quad \hat{N}_0 = 0,$$

where $\hat{M}^n = n^{-1/2}M^n$.

Furthermore, we have that the process $\hat{N}^n$ converges weakly to $\hat{N}$ defined as the solution of the SDE

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t, \quad \hat{N}_0 = 0,$$

where B is a continuous Gaussian martingale with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$.

PROOF: The first assertion is shown in, for instance, the recent reference Mandjes and Spreij [43], Corollary 2. For the second step we find ourselves in the situation of Proposition 4.1, and if we apply this result the proof is complete. ■

Remark 5.3: It is striking that Theorems 5.1 and 5.2 tell that the order in which the limits are taken (first $n \to \infty$, then $\alpha \to \infty$ or vice versa) give the same limit for $\hat{N}^{n,\alpha}$. It is a priori not guaranteed that in the two situations the same limit results. Moreover, below we will investigate what happens if α and n jointly tend to infinity, see for example, Theorems 5.6, 5.11 and Proposition 5.13 where different limits will appear. Three different scenarios will be investigated, namely, α tends faster to infinity than n, the converse situation, and the balanced case, where the speeds to convergence are proportional, and in the latter case without loss of generality equal.

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Up to now, we investigated limit behavior, where limits have been taken in specified order. We continue with the case where α and n jointly tend to infinity. First, we do this when this happens at the same rate for both of them, w.l.o.g. we take them equal, $\alpha = n$, implying the scaling $Q \mapsto nQ$. We will take the asymptotic centering process ϱ, similar to the one in Eq. (4.1). We find this process by defining $\varrho_t := \lim_{n\to\infty}(1/n)\mathbb{E}N_t^n = 1 - \exp(-\lambda_\infty t)$. A differential equation for ϱ is given by

$$\dot{\varrho}_t = \lambda_\infty(1 - \varrho_t), \quad \varrho_0 = 0. \tag{5.6}$$

In this notation, we have in analogy to Eq. (3.3) that the process N^n is given by

$$\mathrm{d}N_t^n = \lambda^{\mathrm{T}} Z_t^n(n - N_t^n)\,\mathrm{d}t + \mathrm{d}M_t^n, \quad N_0^n = 0. \tag{5.7}$$

In the proof of Theorem 5.6 the following lemma turns out to be useful, of which we shall also use a stochastic version.

Lemma 5.4: *Consider a measurable space $(\Omega, \mathcal{F})$. Let (μ_n) be a sequence of signed measures, such that the total variations $||\mu_n||$ are bounded by a constant B and that are converging weakly to a signed measure μ, and let (f_n) be a sequence of measurable functions, converging uniformly to f. Then the integrals $\mu_n(f_n)$ converge to $\mu(f)$.*

Proof: Consider the inequalities

$$\begin{aligned}|\mu_n(f_n) - \mu(f)| &\le |\mu_n(f_n) - \mu_n(f)| + |\mu_n(f) - \mu(f)| \\ &\le ||f_n - f||_\infty ||\mu_n|| + |\mu_n(f) - \mu(f)| \\ &\le B||f_n - f||_\infty + |\mu_n(f) - \mu(f)|.\end{aligned}$$

By the assumptions made, both terms on the right in the above display tend to zero. ∎

Remark 5.5: Lemma 5.4 also has a stochastic version. If the functions f, f_n are random variables, the measures μ and μ_n are random as well (measurable in an appropriate way), the conclusion of the lemma under evidently modified conditions holds 'ω-wise, that is, almost surely.

Theorem 5.6: *Let N^n be given by Eq. (5.7) and ϱ by Eq. (5.6). Then the scaled and centered process $\hat{N}^n$ given by*

$$\hat{N}_t^n := n^{-1/2}(N_t^n - n\varrho_t),$$

converges weakly (as $n \to \infty$) to the solution of the following SDE

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t + \mathrm{d}G_t, \quad \hat{N}_0 = 0, \tag{5.8}$$

where G is a Gaussian martingale with

$$\langle G\rangle_t = \frac{V}{2\lambda_\infty}(1 - \exp(-2\lambda_\infty t)),$$

with $V = \lambda^T(\mathrm{diag}\{\pi\}D^T + D\mathrm{diag}\{\pi\})\lambda$ (D is the deviation matrix of the background Markov chain), and B is a Gaussian martingale with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$, independent of G.

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Proof: We divide the proof into a number of steps.

Step 1. We begin by rewriting the expression for $\hat{N}^n$. We have

$$\hat{N}^n_t = e^{-\lambda^{\mathrm{T}}\zeta^n_t}\left(\int_0^t e^{\lambda^{\mathrm{T}}\zeta^n_s} n^{1/2}\lambda^{\mathrm{T}}(Z^n_s - \pi)(1-\varrho_s)\,\mathrm{d}s + \int_0^t e^{\lambda^{\mathrm{T}}\zeta^n_s}\,\mathrm{d}n^{-1/2}M^n_s\right), \tag{5.9}$$

where $\zeta^n_s = \int_0^s Z^n_u\,\mathrm{d}u$. Now consider the process

$$X^n_t = \int_0^t e^{\lambda^{\mathrm{T}}\zeta^n_s} n^{1/2}\lambda^{\mathrm{T}}(Z^n_s - \pi)(1-\varrho_s)\,\mathrm{d}s + \int_0^t e^{\lambda^{\mathrm{T}}\zeta^n_s}\,\mathrm{d}n^{-1/2}M^n_s.$$

Define $\Psi^n_s = \exp(\lambda^{\mathrm{T}}\zeta^n_s)(1-\varrho_s)\lambda^{\mathrm{T}}D$ and recall from Eq. (2.2) that $Z^n_t - Z^n_0 - n\int_0^t QZ^n_s\,\mathrm{d}s = \tilde{M}^n_t$ for a martingale $\tilde{M}^n$. From Eq. (2.1) we obtain $DQ = \Pi - I$, $\Pi Z^n_t = \pi$. Hence, we can write

$$X^n_t = -\int_0^t \Psi^n_s\,\mathrm{d}n^{-1/2}Z^n_s + \int_0^t \Psi^n_s\,\mathrm{d}n^{-1/2}\tilde{M}^n_s + \int_0^t e^{\lambda^{\mathrm{T}}\zeta^n_s}\,\mathrm{d}n^{-1/2}M^n_s. \tag{5.10}$$

Step 2. In order to prove weak convergence of the process in Eq. (5.9) we prove joint weak convergence of Eq. (5.10) and $e^{-\lambda^{\mathrm{T}}\zeta^n_t}$. By using the same reasoning as in the proof of Theorem 5.1 in order to arrive at Eq. (5.5), we have $\exp(\lambda^{\mathrm{T}}\zeta^n_t) \overset{ucp}{\to} \exp(\lambda_\infty t)$ and the u.c.p. convergence of $\exp(-\lambda^{\mathrm{T}}\zeta^n_t)$ to $\exp(-\lambda_\infty t)$.

Step 3. In order to establish weak convergence of Eq. (5.10) we establish joint weak convergence of the terms. We begin with showing that the first term converges weakly to the zero process. Using the result from Step 2 and $1-\varrho_s = \exp(-\lambda_\infty s)$, we get from the continuous mapping theorem that

$$\Psi^n \overset{ucp}{\to} \Psi := \lambda^{\mathrm{T}}D. \tag{5.11}$$

We have that the processes $t \mapsto \exp(\lambda^{\mathrm{T}}\zeta^n_t)$ are of bounded variation on compact intervals, uniformly in n. Therefore, as $t \mapsto e^t$ is Lipschitz on compact sets and ϱ_s is of bounded variation, we also have that the Ψ^n are of bounded variation and have bounded total variation processes on compact sets uniformly in n. It follows that $n^{-1/2}\Psi^n$ converges u.c.p. to the zero process and so does its total variation process. To analyze the integral $\int_0^t \Psi^n_s\,\mathrm{d}Z^n_s$ we make the following observation, derived from Jansen [32]. Every component $Z^{n,i}$ of the process Z^n takes values in $\{0,1\}$, and hence $\Delta Z^{n,i}_s \in \{-1,+1\}$. Therefore the integral $\int_0^t \Psi^n_s\,\mathrm{d}Z^{n,i}_s$ is of the form $\sum_{\tau_i \le t} \pm\Psi^n_{\tau_i}$, where the τ_i are the jump times of $Z^{n,i}$. Hence $|\int_0^t \Psi^n_s\,\mathrm{d}Z^{n,i}_s|$ is bounded from above by the sum of the total variation of $Z^{n,i}$ and its sup-norm, see Jansen [32], Lemma 6.6.5. Therefore we have for the first term of Eq. (5.10), that

$$\int_0^t \Psi^n_s\,\mathrm{d}n^{-1/2}Z^n_s \overset{ucp}{\to} 0.$$

By Slutsky's theorem, one obtains joint convergence of the three terms in Eq. (5.10), as soon as the final two terms jointly converge weakly. This we show in the next step.

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Step 4. For the weak convergence of the remaining two terms of Eq. (5.10) consider the locally square-integrable martingale

$$\mathbf{M}_t^n = \begin{bmatrix} n^{-1/2}\tilde{M}_t^n \\ n^{-1/2}M_t^n \end{bmatrix}$$

which by Assumption 3.1 has quadratic variation given by

$$\langle \mathbf{M}^n \rangle_t = \int_0^t \begin{bmatrix} \tilde{V}_s^{n,*} & 0 \\ 0 & V_s^{n,*} \end{bmatrix} \mathrm{d}s,$$

where

$$\begin{aligned} \tilde{V}_s^{n,*} &= \mathrm{diag}\{QZ_s^n\} - Q\mathrm{diag}\{Z_s^n\} - \mathrm{diag}\{Z_s^n\}Q^{\mathrm{T}}, \\ V_s^{n,*} &= \lambda^{\mathrm{T}} Z_s^n (1 - n^{-1}N_s^n) \end{aligned}$$

because of Eq. (2.3). Therefore, by following Jacod and Shiryaev [31], section III.6a, the square integrable martingale

$$\mathbf{M}_t^{\zeta,n} = \begin{bmatrix} \displaystyle\int_0^t \Psi_s^n \,\mathrm{d}n^{-1/2}\tilde{M}_s^n \\ \displaystyle\int_0^t e^{\lambda^{\mathrm{T}}\zeta_s^n} \,\mathrm{d}n^{-1/2}M_s^n \end{bmatrix} \tag{5.12}$$

has quadratic variation

$$\langle \mathbf{M}^{\zeta,n} \rangle_t = \int_0^t \begin{bmatrix} \Psi_s^n \tilde{V}_s^{n,*} (\Psi_s^n)^{\mathrm{T}} & 0 \\ 0 & \exp(2\lambda^{\mathrm{T}}\zeta_s^n) V_s^{n,*} \end{bmatrix} \mathrm{d}s. \tag{5.13}$$

In order to prove weak convergence of the last two terms in Eq. (5.10) we aim to apply the MCLT to the martingale $\mathbf{M}^{\zeta,n}$ in Eq. (5.12). Thereto we need that (i) the jumps of the martingale on compact sets disappear and that (ii) the quadratic variation converges to a deterministic continuous function in probability.

For (i) the proof is that both integrals in Eq. (5.12) have continuous integrands. Therefore the stochastic integral with respect to M^n ($\tilde{M}^n$ can be treated in the same way) we have

$$\max_{0\le t\le T} \left\{ |\Delta \int_0^t \Psi_s^n \,\mathrm{d}n^{-1/2}M_s^n| \right\} \le \|\Psi^n\|_\infty n^{-1/2} \to 0,$$

where $\|\Psi^n\|_\infty$ denotes the supremum-norm of Ψ^n on $[0, T]$ which is finite as Ψ^n is bounded on compact intervals uniformly in n. For (ii) we check the convergence of the two non-zero entries in the quadratic variation separately.

First, we consider $\int_0^t \Psi_s^n \tilde{V}_s^{n,*}(\Psi_s^n)^{\mathrm{T}}\,\mathrm{d}s$. We know that Ψ^n converges u.c.p. to Ψ, see Eq. (5.11), and from Eq. (2.5) that $\int_0^t Z_s^n\,\mathrm{d}s$ converges a.s. to πt. Below we apply the almost sure version of Lemma 5.4 to the elements of the matrix $\int_0^t \Psi_s^n \tilde{V}_s^{n,*}(\Psi_s^n)^{\mathrm{T}}\,\mathrm{d}s$. Take the ij-th element of this matrix. It is, in obvious notation, a sum over k and l of integrals $\int_0^t (\Psi_s^n)_{ik}(\tilde{V}_s^{n,*})_{kl}(\Psi_s^n)_{jl}\,\mathrm{d}s$, where those integrals are of the form $\int_0^t (\Psi_s^n)_{ik}(\Psi_s^n)_{jl}\mu_{kl}(\mathrm{d}s)$, with $\mu_{kl}(\mathrm{d}s) = (\tilde{V}_s^{n,*})_{kl}\,\mathrm{d}s$. Using that $\int_0^t (\tilde{V}_s^{n,*})\,\mathrm{d}s \to (\mathrm{diag}\{\pi\}D^{\mathrm{T}} + D\mathrm{diag}\{\pi\})\,t$, we see

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that an application of Lemma 5.4 results in

$$\begin{aligned}
&\int_0^t \Psi_s^n \left(\mathrm{diag}\{QZ_s^n\} - Q\mathrm{diag}\{Z_s^n\} - \mathrm{diag}\{Z_s^n\}Q^{\mathrm{T}}\right)(\Psi_s^n)^{\mathrm{T}}\,\mathrm{d}s \qquad \mathbf{(5.14)}\\
&\overset{ucp}{\to} -\int_0^t \Psi_s(Q\mathrm{diag}\{\pi\} + \mathrm{diag}\{\pi\}Q^{\mathrm{T}})\Psi_s^{\mathrm{T}}\,\mathrm{d}s\\
&= \lambda^{\mathrm{T}}(\mathrm{diag}\{\pi\}D^{\mathrm{T}} + D\mathrm{diag}\{\pi\})\lambda\, t =: V\,t,
\end{aligned}$$

since $DQ = \Pi - I$ and $Q\pi = 0$, see Eq. (2.1). Note that V is nonnegative in view of Eq. (2.4).

Next, we consider the other non-zero entry in the quadratic variation. Recall $V_t^{n,*} = \lambda^{\mathrm{T}} Z_t^n(1 - n^{-1}N_t^n)$. We first show u.c.p. convergence of $\int_0^t V_s^{n,*}\,\mathrm{d}s$ to a continuous function. This requires a couple of steps.

The first step is to show that $n^{-1}N_t^n$ converges in probability to $\varrho_t = 1 - \exp(\lambda_\infty t)$, as a matter of fact we show that the convergence is in the L^2-sense. Recall that, conditional on $\mathcal{F}^Z$, N_t^n is binomial with parameters n and $p_t^n = 1 - \exp(-\int_0^t \lambda^{\mathrm{T}} Z_s^n\,\mathrm{d}s)$. Therefore we have $\mathbb{E}(n^{-1}N_t^n - \varrho_t) = \mathbb{E}p_t^n - \varrho_t \to 0$. Hence, writing $\mathbb{E}(n^{-1}N_t^n - \varrho_t)^2 = (\mathbb{E}(n^{-1}N_t^n - \varrho_t))^2 + \mathrm{Var}(n^{-1}N_t^n)$, we only have to prove that the variance tends to zero. By the law of total variation we have

$$\begin{aligned}
\mathrm{Var}(n^{-1}N_t^n) &= n^{-2}\mathbb{E}\,\mathrm{Var}(N_t^n|\mathcal{F}^Z) + \mathrm{Var}(\mathbb{E}[n^{-1}N_t^n|\mathcal{F}^Z])\\
&= n^{-1}\mathbb{E}[p_1^n(1-p_t^n)] + \mathrm{Var}\,p_t^n.
\end{aligned}$$

As the p_t^n are bounded and $p_t^n \to 1 - \exp(-\lambda_\infty t)$, we get $\mathrm{Var}(n^{-1}N_t^n) \to 0$ by application of the dominated convergence theorem.

Now we are ready for the final step. Write

$$\begin{aligned}
\int_0^t \exp(2\lambda^{\mathrm{T}}\zeta_s^n)V_s^{n,*}\,\mathrm{d}s &= \int_0^t \exp(2\lambda^{\mathrm{T}}\zeta_s^n)\lambda^{\mathrm{T}}Z_s^n(1-\varrho_s)\,\mathrm{d}s\\
&\quad + \int_0^t \exp(2\lambda^{\mathrm{T}}\zeta_s^n)\lambda^{\mathrm{T}}Z_s^n(\varrho_s - n^{-1}N_s^n)\,\mathrm{d}s. \qquad \mathbf{(5.15)}
\end{aligned}$$

Consider the expectation of the absolute value of the last integral. By the Cauchy-Schwartz inequality, its square is less than

$$\mathbb{E}\int_0^t (\exp(2\lambda^{\mathrm{T}}\zeta_s^n)\lambda^{\mathrm{T}}Z_s^n)^2\,\mathrm{d}s \times \int_0^t \mathbb{E}(\varrho_s - n^{-1}N_s^n)^2\,\mathrm{d}s.$$

In this product, the first factor is bounded, whereas the second factor tends to zero by the above L^2-convergence of $n^{-1}N_s^n$ to ϱ_s and by application of the monotone convergence theorem.

We now focus on the first term on the RHS of Eq. (5.15). By the ucp-convergence of the exponential term $\exp(2\lambda^{\mathrm{T}}\zeta_s^n)(1-\varrho_s)$ to $\exp(2\lambda_\infty s)(1-\varrho_s) = \exp(\lambda_\infty s)$ (as in Step 2), we can again apply the almost sure version of Lemma 5.4, to arrive at $\int_0^t \exp(2\lambda^{\mathrm{T}}\zeta_s^n)\lambda^{\mathrm{T}}Z_s^n(1-\varrho_s)\,\mathrm{d}s \overset{a.s.}{\to} \int_0^t \exp(\lambda_\infty s)\lambda_\infty\,\mathrm{d}s = \exp(\lambda_\infty t) - 1$. Summarizing all these intermediate results we get convergence in probability of the quadratic variation, that is,

$$\langle \mathbf{M}^{\zeta,n}\rangle_t \overset{\mathbb{P}}{\to} \begin{bmatrix} Vt & 0 \\ 0 & e^{\lambda_\infty t} - 1 \end{bmatrix}.$$

The MCLT allows us to deduce that $\mathbf{M}^{\zeta,n}$ converges weakly to a two-dimensional Gaussian martingale with the independence of the components.

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Step 5: By the weak convergence of $\mathbf{M}^{\zeta,n}$ and weak convergence of the first term in Eq. (5.10) to the zero process, we deduce by application of the continuous mapping theorem, weak convergence of Eq. (5.10) to a Gaussian martingale with quadratic variation $\int_0^t (V + \lambda_\infty e^{\lambda_\infty s})\,\mathrm{d}s$. Therefore $\mathbf{M}^{\zeta,n}$ has the limit distribution of

$$\begin{bmatrix} \int_0^t \sqrt{V}\,\mathrm{d}B_s^1 \\ \int_0^t \sqrt{\lambda_\infty e^{\lambda_\infty s}}\,\mathrm{d}B_s^2 \end{bmatrix}, \tag{5.16}$$

where B^1 and B^2 are independent standard Brownian motions and thus we have weak convergence of the sum in Eq. (5.10) to the Gaussian martingale in Eq. (5.16).

Step 6: In conclusion, we showed that $\hat{N}^{n,\alpha}$ can be written as the product of processes in Eq. (5.9). In Step 2 we show u.c.p. convergence of the first process and in Steps 3–5 we showed weak convergence of the second process. Therefore we have joint weak convergence of the processes in Eq. (5.9). Since multiplication is continuous at continuous limits in the Skorohod topology (c.f. Whitt [54], Thm 4.2) we have weak convergence of $\hat{N}^n$ to the process $\hat{N}$ given by ($\tilde{B}$ is a standard Brownian motion)

$$\hat{N}_t = e^{-\lambda_\infty t}\int_0^t \sqrt{V + \lambda_\infty e^{\lambda_\infty s}}\,\mathrm{d}\tilde{B}_s,$$

which solves the SDE

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + e^{-\lambda_\infty t}\sqrt{V + \lambda_\infty e^{\lambda_\infty t}}\,\mathrm{d}\tilde{B}_t.$$

Alternatively, we can represent this SDE as Eq. (5.8),

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t + \mathrm{d}G_t,$$

where B and G are independent Gaussian martingales, with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$ and $\langle G\rangle_t = (V/(2\lambda_\infty))(1 - \exp(-2\lambda_\infty t))$. ■

Remark 5.7: Let us compare Proposition 4.1 and Theorem 5.6. The Brownian motion B of Theorem 5.6 is as B in Proposition 4.1. The Brownian motion G in the theorem comes as an 'extra' compared with the situation of the proposition. If we apply Theorem 5.6 to the non-modulated case, which happens if the vector λ is a constant λ_∞ times $\mathbb{1}$, we have $\langle G\rangle_t = -\lambda_\infty^2 \mathbb{1}^{\mathrm{T}}(\mathrm{diag}\{\pi\}D^{\mathrm{T}} + D\mathrm{diag}\{\pi\})\mathbb{1}\, t$, which is indeed zero in view of the property $D\pi = 0$, see Eq. (2.1). So this theorem in the non-modulated case reduces to Proposition 4.1, as it should.

The centering in Theorem 5.6 is with the function $n\varrho_t$. In the proposition below we compare $\varrho_t = 1 - \exp(-\lambda_\infty t)$ with $\varrho_t^n = 1 - \exp(-\lambda^{\mathrm{T}}\zeta_t^n)$, and we shall see the result of alternative centering with ϱ_t^n in Proposition 5.9.

PROPOSITION 5.8: *It holds that $\hat{H}_t^n := \sqrt{n}(\varrho_t^n - \varrho_t)$ converges weakly to the process $\hat{H}$ given by $\hat{H}_t = \exp(-\lambda_\infty t)G_t^H$, where G^H is a Brownian motion with variance parameter $V = \lambda^T(diag\{\pi\}D^T + D\,diag\{\pi\})\lambda$, so $\langle G^H\rangle_t = Vt$.*

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Proof: We compute

$$\begin{aligned}\varrho_t^n - \varrho_t &= \exp(-\lambda_\infty t) - \exp(-\lambda^{\mathrm{T}}\zeta_t^n)\\ &= \int_0^t \exp(-\lambda^{\mathrm{T}}\zeta_s^n)\lambda^{\mathrm{T}}\zeta_s^n\,\mathrm{d}s - \int_0^t \exp(-\lambda_\infty s)\lambda_\infty\,\mathrm{d}s\\ &= \int_0^t (\exp(-\lambda^{\mathrm{T}}\zeta_s^n) - \exp(-\lambda_\infty s))\lambda^{\mathrm{T}} Z_s^n\,\mathrm{d}s + \int_0^t \exp(-\lambda_\infty s)\lambda^{\mathrm{T}}(Z_s^n - \pi)\,\mathrm{d}s\\ &= -\int_0^t (\varrho_s^n - \varrho_s)\lambda^{\mathrm{T}} Z_s^n\,\mathrm{d}s + \int_0^t \exp(-\lambda_\infty s)\lambda^{\mathrm{T}}(Z_s^n - \pi)\,\mathrm{d}s.\end{aligned}$$

For $\hat{H}_t^n = \sqrt{n}(\varrho_t^n - \varrho_t)$ we then obtain

$$\hat{H}_t^n = -\int_0^t \hat{H}_s^n \lambda^{\mathrm{T}} Z_s^n\,\mathrm{d}s + \sqrt{n}\int_0^t \exp(-\lambda_\infty s)\lambda^{\mathrm{T}}(Z_s^n - \pi)\,\mathrm{d}s.$$

Solving this equation yields

$$\hat{H}_t^n = \exp(-\lambda^{\mathrm{T}}\zeta_t^n)\sqrt{n}\int_0^t \exp(\lambda^{\mathrm{T}}\zeta_s^n - \lambda_\infty s)\lambda^{\mathrm{T}}(Z_s^n - \pi)\,\mathrm{d}s.$$

The latter equation has the same structure as Eq. (5.9), but with the martingale term missing. For the limit behavior we can therefore copy the relevant parts of the proof of Theorem 5.6, which yields the assertion. ■

Now we revisit Theorem 5.6, by replacing the centering $n\varrho_t$ by $n\varrho_t^n$. This leads to

Proposition 5.9: *Let $\hat{K}_t^n = n^{-1/2}(N_t^n - n\varrho_t^n)$. Then $\hat{K}^n$ converges weakly to the solution to the SDE $\mathrm{d}\hat{K}_t = -\lambda_\infty \hat{K}_t\,\mathrm{d}t + \mathrm{d}B_t$, where B is a continuous Gaussian martingale with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$.*

Proof: We follow the line of reasoning of the proof of Theorem 5.6. Parallel to the first step of that proof we now obtain

$$\hat{K}_t^n = n^{-1/2}M_t^n - \int_0^t \lambda^{\mathrm{T}} Z_s^n \hat{K}_s^n\,\mathrm{d}s, \tag{5.17}$$

equivalent to

$$\hat{K}_t^n = \exp(-\lambda^{\mathrm{T}}\zeta_t^n)\int_0^t n^{-1/2}\exp(\lambda^{\mathrm{T}}\zeta_s^n)\,\mathrm{d}M_s^n,$$

which is a simpler version of Eq. (5.9). Following the steps in the proof of the theorem, we arrive at the weak convergence of the stochastic integral to a Gaussian martingale $\tilde{B}$ with quadratic variation $\exp(\lambda_\infty t) - 1$ and of the process $\hat{K}^n$ to $\hat{K}$ given by $\hat{K}_t = \exp(-\lambda_\infty t)\tilde{B}_t$. The latter being equivalent to $\hat{K}$ solving

$$\mathrm{d}\hat{K}_t = -\lambda_\infty \hat{K}_t\,\mathrm{d}t + \mathrm{d}B_t,$$

where B is a Gaussian martingale with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$. ■

Putting the conclusions of Propositions 5.8 and 5.9 together and comparing them with Theorem 5.6, we can provide an illuminating explanation of the result of the theorem.

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Write $\hat{N}^n_t = \hat{H}^n_t + \hat{K}^n_t$, and recall the following weak limits. We have seen that the limit process $\hat{H}$ satisfies $\mathrm{d}\hat{H}_t = -\lambda_\infty \hat{H}_t\,\mathrm{d}t + \exp(-\lambda_\infty t)\,\mathrm{d}G^H_t$, with $\langle G^H\rangle_t = Vt$. And we have also seen that the limit process $\hat{K}$ is a Gaussian process satisfying $\mathrm{d}\hat{K}_t = -\lambda_\infty \hat{K}_t\,\mathrm{d}t + \mathrm{d}B_t$, with $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$. Adding up these limits (justified by the proof of the theorem) yields that the limit process $\hat{N}$ satisfies $\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t\,\mathrm{d}t + \mathrm{d}B_t + \exp(-\lambda_\infty t)\,\mathrm{d}G^H_t$. With $G_t = \int_0^t \exp(-\lambda_\infty s)\,\mathrm{d}G^H_s$, we have $\langle G\rangle_t$ as in the theorem, and the SDE Eq. (5.8) follows again.

Summarizing, the result of Theorem 5.6 can be explained by decoupling $\hat{N}^n$ into $\hat{H}^n$ and $\hat{K}^n$ and their limits results according to Propositions 5.8 and 5.9. From a distributional point of view, the result of Theorem 5.6 is more appealing than Proposition 5.9, since the latter involves centering with a random process. In line with the common view on a central limit theorem, one can interpret the statement of the theorem by saying that asymptotically, for fixed t, the random variable $\hat{N}^n_t$ has a normal distribution with (nonrandom) mean $n\varrho_t$ and variance $n\exp(-2\lambda_\infty t)(Vt + \exp(\lambda_\infty t) - 1)$.

Note that the $\hat{K}_t$ in Proposition 5.9 is the same limiting process as the limiting process $\hat{N}_t$ in Theorem 5.1 and Theorem 5.2. In continuation of our discussion above we can explain the equivalence of these limits via the centering process. The process $\hat{K}^n$ is centered with the stochastic process $n\varrho^n$ (and likewise we use for $\hat{N}^{n,\alpha}$ centering with the stochastic process $n\varrho^\alpha$). Centering in this way, as opposed to centering with $n\varrho_t$, removes the first term in Eq. (5.9), which in Theorem 5.6 converges to the Gaussian term G with $\langle G\rangle_t = \frac{V}{2\lambda_\infty}(1 - \exp(-2\lambda_\infty t))$. Intuitively one cancels out the 'extra' variation, due to the first term Eq. (5.9) which results in a situation where the order of limits does not matter anymore. This situation is to some extent similar to the case for the process given by Eq. (5.2). But note also the difference between the two cases, the martingale in Eq. (5.2) is continuous and Gaussian, whereas the martingale in Eq. (5.17) is a (scaled) compensated jump martingale, although with a continuous Gaussian limit.

Next, we investigate the limit behavior of N_t when we speed up the Markov chain with n^β for some $\beta > 0$. Note that before we had $\beta = 1$. The Propositions 5.8 and 5.9 now take a different form, but the proofs of the results in Proposition 5.10 below are similar to the previous ones, and are therefore omitted. Let us write, in order to express the dependence on β, $\varrho^{n,\beta} = 1 - \exp(-\lambda^{\mathrm{T}}\zeta^{n,\beta}_t)$ with $\zeta^{n,\beta}_t = \int_0^t Z_{n^\beta s}\,\mathrm{d}s$.

Proposition 5.10:

(i) *Let* $\hat{H}^{n,\beta}_t := n^{\beta/2}(\varrho^{n,\beta}_t - \varrho_t)$*. Then* $\hat{H}^{n,\beta}$ *converges weakly to the process* $\hat{H}$ *given by* $\hat{H}_t = \exp(-\lambda_\infty t)G^H_t$*, where* G^H*, as before, is a Brownian motion with variance parameter* $V = \lambda^T(diag\{\pi\}D^T + D\,diag\{\pi\})\lambda$*.*

(ii) *Let* $\hat{K}^{n,\beta}_t = n^{-1/2}(N^n_t - n\varrho^{n,\beta}_t)$*. Then* $\hat{K}^{n,\beta}$ *converges weakly to the solution to the SDE* $\mathrm{d}\hat{K}_t = -\lambda_\infty \hat{K}_t\,\mathrm{d}t + \mathrm{d}B_t$*,*

where B *is a continuous Gaussian martingale with* $\langle B\rangle_t = 1 - \exp(-\lambda_\infty t)$*.*

As a consequence of this proposition, we have the following extension of Theorem 5.6.

Theorem 5.11: *Let* N^n *be given by Eq. (5.7) and* ϱ *by Eq. (5.6). Then the scaled and centered process* $\hat{N}^{n,\beta}$ *given by*

$$\hat{N}^{n,\beta}_t := n^{-1/2(1+(1-\beta)^+)}(N^n_t - n\varrho_t),$$

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converges weakly (as $n \to \infty$) to the solution of the following SDE

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t \,\mathrm{d}t + \mathbb{1}_{\{\beta \le 1\}} \,\mathrm{d}G_t + \mathbb{1}_{\{\beta \ge 1\}} \,\mathrm{d}B_t, \quad \hat{N}_0 = 0, \tag{5.18}$$

where G is a Gaussian martingale with

$$\langle G \rangle_t = \frac{V}{2\lambda_\infty}(1 - \exp(-2\lambda_\infty t)),$$

with $V = \lambda^T(\mathrm{diag}\{\pi\} D^T + D\mathrm{diag}\{\pi\})\lambda$ (D is the deviation matrix of the background Markov chain), and B is a Gaussian martingale with $\langle B \rangle_t = 1 - \exp(-\lambda_\infty t)$, independent of G.

Alternatively, we have the representation

$$\mathrm{d}\hat{N}_t = -\lambda_\infty \hat{N}_t \,\mathrm{d}t + \left(\mathbb{1}_{\{\beta \le 1\}} V \exp(-2\lambda_\infty t) + \mathbb{1}_{\{\beta \ge 1\}} \lambda_\infty \exp(-\lambda_\infty t)\right)^{1/2} \mathrm{d}W_t,$$

where W is a standard Brownian motion.

Proof: We only have to consider the cases $\beta < 1$ and $\beta > 1$, as the case $\beta = 1$ is covered by Theorem 5.6. For $\beta < 1$ we have $\hat{N}^{n,\beta} = n^{-1+\beta/2}(N_t^n - n\varrho_t) = n^{(\beta-1)/2}\hat{K}_t^{n,\beta} + \hat{H}^{n,\beta}$. From Proposition 5.10 we obtain that $\hat{N}^{n,\beta}$ has $\hat{H}$ as the limit process. For $\beta > 1$ we have $\hat{N}^{n,\beta} = n^{-1/2}(N_t^n - n\varrho_t) = \hat{K}_t^{n,\beta} + n^{(1-\beta)/2}\hat{H}^{n,\beta}$. From Proposition 5.10 we now obtain that $\hat{N}^{n,\beta}$ has $\hat{K}$ as the limit process.

Putting these two cases (combined with $\beta = 1$) together we see that we obtain for $\hat{N}^{n,\beta}$ the limit process $\mathbb{1}_{\{\beta \le 1\}}\hat{H} + \mathbb{1}_{\{\beta \ge 1\}}\hat{K}$. Therefore we also have

$$\begin{aligned}\mathrm{d}\hat{N}_t^{n,\beta} &= -\lambda_\infty \hat{N}_t^{n,\beta} \,\mathrm{d}t + \mathbb{1}_{\{\beta \le 1\}} \exp(-\lambda_\infty t) \,\mathrm{d}G_t^H + \mathbb{1}_{\{\beta \ge 1\}} \,\mathrm{d}B_t \\ &= -\lambda_\infty \hat{N}_t^{n,\beta} \,\mathrm{d}t + \mathbb{1}_{\{\beta \le 1\}} \,\mathrm{d}G_t + \mathbb{1}_{\{\beta \ge 1\}} \,\mathrm{d}B_t,\end{aligned}$$

which completes the proof. ■

Remark 5.12: From the quadratic variation of the Brownian terms in the limit of Theorem 5.11 one sees that the quadratic variation process converges to $(V/(2\lambda_\infty))\mathbb{1}_{\{\beta \le 1\}} + \mathbb{1}_{\{\beta \ge 1\}}$ if $t \to \infty$. Note that then also the quadratic variation $\langle \hat{N} \rangle_t \to (V/(2\lambda_\infty))\mathbb{1}_{\{\beta \le 1\}} + \mathbb{1}_{\{\beta \ge 1\}}$, as it is completely determined by the quadratic variation of the martingale part of $\hat{N}$. Therefore, the Brownian terms converge to a Gaussian random variable with expectation zero for $t \to \infty$ and variance $(V/(2\lambda_\infty))\mathbb{1}_{\{\beta \le 1\}} + \mathbb{1}_{\{\beta \ge 1\}}$. It follows that the limiting process $\hat{N}$ in Theorem 5.11 is Gaussian with vanishing variance for $t \to \infty$, and therefore behaves as the constant zero process for large t.

The previous limit theorems were based on a fixed value of λ, which is possibly something to relax. Consider a modulated process, but with default intensity $n^{-\gamma}\lambda^\top Z_t$, for some $\gamma > 0$. In financial terms, we consider a market with many obligors whose individual default rate tends to zero and we are still interested in the total number of defaults. As before, we scale $Q \mapsto n^\beta Q$ for some $\beta > 0$, which speeds up the background process. We omit the dependence on β in the notation as it will turn out that the diffusion limit we derive is independent of β. So, we consider

$$\mathrm{d}N_t^{n,\gamma} = n^{-\gamma}\lambda^\top Z_t^n (n - N_t^{n,\gamma}) \,\mathrm{d}t + \mathrm{d}M_t^{n,\gamma}, \quad N_0^{n,\gamma} = 0.$$

If $\gamma = 1$, we know that the limit process (for $n \to \infty$) is a Poisson process with intensity λ_∞. The case $\gamma > 1$ is not very interesting, one certainly has $N_t^{n,\gamma} \stackrel{a.s.}{\to} 0$ for fixed t. But

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for $0 < \gamma < 1$ there is something to do; the case $\gamma = 0$, we have already encountered in Proposition 4.1. Write $\lambda_t^{n,\gamma}$ for $n^{-\gamma}\lambda^\top Z_t^n$ and $\Lambda_t^{n,\gamma} = \int_0^t \lambda_s^{n,\gamma}\,\mathrm{d}s$. Put $\varrho_t^{n,\gamma} = 1 - \exp(-\Lambda_t^{n,\gamma})$.

Proposition 5.13: *Let $\beta > 0$. Consider for $\gamma \in (0,1)$ the process $\hat{N}^{n,\gamma}$ given by*

$$\hat{N}_t^{n,\gamma} = n^{(\gamma-1)/2}\left(N_t^{n,\gamma} - n\varrho_t^{n,\gamma}\right).$$

As $n \to \infty$, this process converges weakly to a Brownian motion with variance parameter λ_∞.

Proof: One checks that

$$\mathrm{d}\hat{N}_t^{n,\gamma} = -\lambda_t^{n,\gamma}\hat{N}_t^{n,\gamma}\,\mathrm{d}t + \mathrm{d}\hat{M}_t^{n,\gamma},$$

where $\hat{M}_t^{n,\gamma} = n^{(\gamma-1)/2}M_t^{n,\gamma}$, with

$$\langle \hat{M}^{n,\gamma}\rangle_t = \int_0^t \lambda^\top Z_s^n\left(1 - \frac{N_s^{n,\gamma}}{n}\right)\mathrm{d}s.$$

Note that the jumps of $\hat{M}^{n,\gamma}$ disappear for $n \to \infty$, as $\gamma < 1$, and that

$$\left(1 - \frac{N_t^{n,\gamma}}{n}\right)\int_0^t \lambda^\top Z_s^n\,\mathrm{d}s \le \langle \hat{M}^{n,\gamma}\rangle_t \le \int_0^t \lambda^\top Z_s^n\,\mathrm{d}s.$$

As for each $t \ge 0$, $\mathbb{E}((N_t^{n,\gamma})/n) \to 0$ and $N_t^{n,\gamma} \ge 0$ it holds that $((N_t^{n,\gamma})/n) \xrightarrow{\mathbb{P}} 0$ and thus $\langle \hat{M}^{n,\gamma}\rangle_t \xrightarrow{\mathbb{P}} \lambda_\infty t$. Consequently, $\hat{M}^{n,\gamma}$ weakly converges to $\sqrt{\lambda_\infty}B$, where B is a standard Brownian motion. As $\hat{N}_t^{n,\gamma} = \exp(-\Lambda_t^{n,\gamma})\int_0^t \exp(\Lambda_s^{n,\gamma})\,\mathrm{d}\hat{M}_s^{n,\gamma}$, by previous arguments and using that $\Lambda_t^{n,\gamma} \xrightarrow{\mathbb{P}} 0$ since $\gamma > 0$, $\hat{N}^{n,\gamma}$ converges to $\sqrt{\lambda_\infty}B$ as well. ■

6. SOME ILLUSTRATING SIMULATIONS

In this section, we will show some graphs of simulations, illustrating some of the results proven in this paper. To illustrate all the results would require too much space, so we will show two intuitive results, namely, the first part of Theorem 5.2 where we only speed up the underlying Markov process, and Theorem 5.6.

We simulate $N_t^{n,\alpha}$ as in Eq. (5.1) and $\hat{N}^{n,\beta}$ as in Theorem 5.11 for a couple of parameter settings of α and n on a time interval $[0,T]$. We take $T = 3$ for the first and $T = 10$ for the second simulation, to illustrate the interesting phenomona corresponding to the theorems. We take a state space of three elements for the Markov chain

$$Q = \begin{bmatrix} -5 & 1 & 5 \\ 2 & -2 & 5 \\ 3 & 1 & -10 \end{bmatrix},$$

and the different values of the intensity are summarized by the vector $\lambda = \begin{bmatrix}0.1 & 1 & 3\end{bmatrix}^{\mathrm{T}}$, a fixed choice in all simulations.

We start by simulating $N_t^{n,\alpha}$ and $\lambda^{\mathrm{T}}Z_t$ for $n = 1000$ fixed and varying values of $\alpha \in \{1, 10, 100, 10000\}$ to illustrate the first part of Theorem 5.2. The sample paths of these simulations are shown in Figures 1 and 2.

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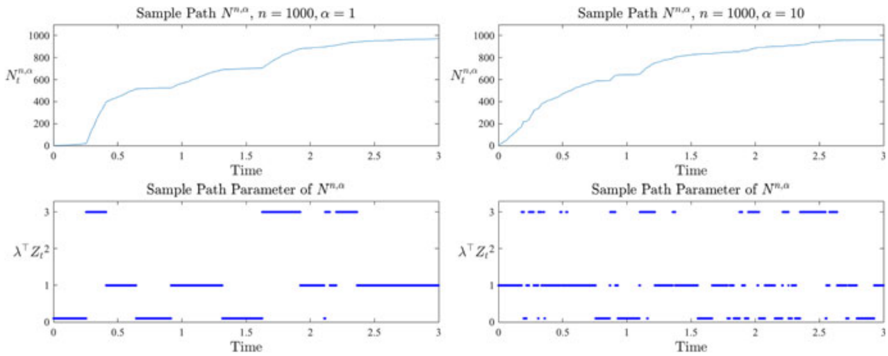

FIGURE 1. Sample Paths with $\alpha = 1$ (left) and $\alpha = 10$ (right)

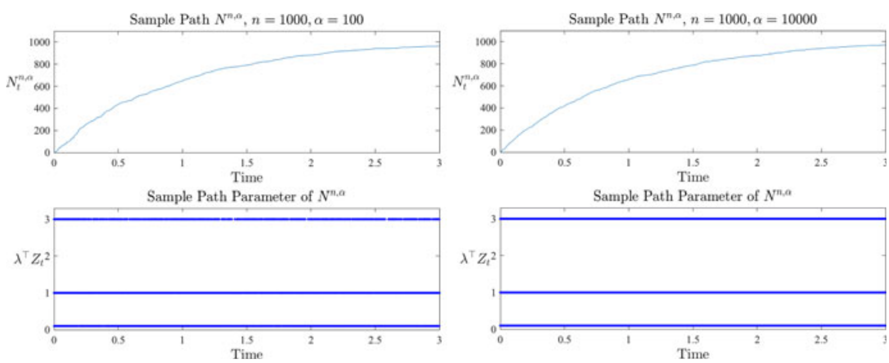

FIGURE 2. Sample Paths with $\alpha = 100$ (left) and $\alpha = 10,000$ (right)

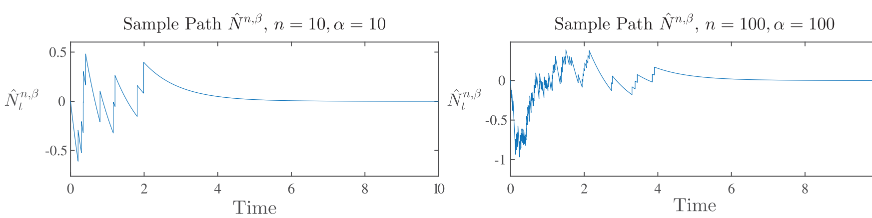

FIGURE 3. CLT illustration $n = \alpha = 10$ (left) and $n = \alpha = 100$ (right)

One can see the effect from the MM default rate in Figure 1. The contents of the first part of Theorem 5.2 is that this modulating effect should disappear and the default rate becomes a deterministic constant λ_∞ in the limit. This is visible in Figure 2, where this modulating effect disappears and a constant default rate appears due to the Markov chain jumping very fast.

Next, we simulate the centered and scaled process $\hat{N}^{n,\beta}$, for $\beta = 1$. We then have $\alpha = n$ and we choose $n \in \{10, 100, 1,000, 10,000\}$ in order to illustrate Theorem 5.11. The sample paths are shown in Figures 3 and 4.

Figures 3 and 4 illustrate how the process $\hat{N}^{n,\beta}$ converges to a continuous process, which fluctuates like a Gaussian martingale. We chose for the time scale $T = 10$ to show that the quadratic variation $\langle \hat{N} \rangle_t$ of the limiting process $\hat{N}$ tends to a constant as $t \to \infty$. So these figures also illustrate the observations made in Remark 5.12.

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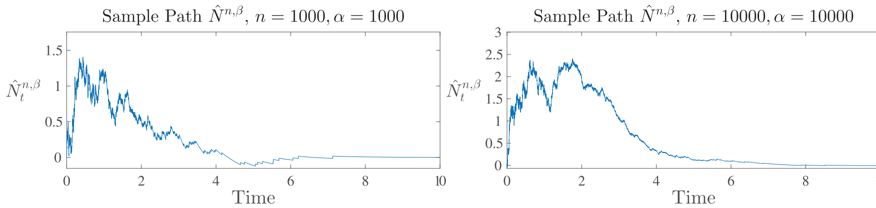

FIGURE **4.** CLT illustration $n = \alpha = 1,000$ (left) and $n = \alpha = 10,000$ (right)

7. INCLUSION OF RECOVERY

The process N of Eq. (3.2) counts the number of defaults of companies (as one of the interpretations). After a default, a company disappears from the market. Alternatively, one might think of recovery of defaulted companies. In this section, we present a few generalizations of previous results. As the proofs are similar to previous ones, but somewhat more involved, we only sketch them.

Supposing first that recovery happens at a constant rate μ per company and that Markov-modulation does not take place, we are dealing with a birth-death process N whose semimartingale decomposition is, instead of Eq. (3.2), now given by

$$\mathrm{d}N_t^n = (\lambda(n - N_t^n) - \mu N_t^n)\,\mathrm{d}t + \mathrm{d}M_t^n, \quad N_0^n = 0. \tag{7.1}$$

It is possible to show that N^n is a Markov chain on $\{0, 1, \ldots, n\}$ whose transition rates are $j\mu$ if N jumps from j to $j-1$ and $(n-j)\lambda$ if N jumps from j to $j+1$, whereas other transitions have rate zero. It follows that now N_t has a $\mathrm{Bin}(n, n\varrho_t)$ distribution, where ϱ satisfies the differential equation

$$\dot{\varrho} = \lambda(1 - \varrho) - \mu\varrho, \quad \varrho_0 = 0.$$

The solution to this equation is

$$\varrho_t = \frac{\lambda}{\lambda + \mu}(1 - \exp(-(\lambda + \mu)t).$$

To compute $\langle M^n \rangle_t = \langle N^n \rangle_t$, we first look at the optional quadratic variation process $[N^n]$. As $[N^n]_t = \sum_{s \leq t} (\Delta N_s^n)^2$, and a nonzero ΔN_s^n is either plus or minus 1, which happens with rates $\lambda(n - N_t^n)$ and μN_t^n, respectively, it follows that $(\mathrm{d}/\mathrm{d}t)\langle M^n \rangle_t = \lambda(n - N_t^n) + \mu N_t^n$.

PROPOSITION 7.1: *Let $\lambda, \mu > 0$ be constants and let N^n be given by Eq. (7.1). Then the scaled and centered process*

$$\hat{N}_t^n := n^{-1/2}(N_t^n - n\varrho_t)$$

converges weakly to the solution of the following SDE,

$$\mathrm{d}\hat{N}_t = -(\lambda + \mu)\hat{N}_t\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, \quad \hat{N}_0 = 0$$

as $n \to \infty$. Here B is a standard Brownian motion and

$$\sigma(t)^2 = \lambda - \frac{\lambda(\lambda - \mu)}{\lambda + \mu}(1 - \exp(-(\lambda + \mu)t) = \lambda - (\lambda - \mu)\varrho_t.$$

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The proof of this proposition is similar to that of Proposition 4.1, so we only highlight the main differences. For the process $\hat{N}^n_t$ we now obtain

$$\mathrm{d}\hat{N}^n_t = -(\lambda+\mu)\hat{N}^n_t \,\mathrm{d}t + \mathrm{d}\hat{M}^n_t,$$

where the martingale $\hat{M}^n$ has quadratic variation satisfying (see above)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{M}^n\rangle_t = \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}t}\langle M^n\rangle_t = \lambda - (\lambda-\mu)\frac{N_t}{n} \to \lambda - (\lambda-\mu)\varrho_t = \sigma^2(t), \qquad \text{for} \;\; n\to\infty.$$

Obviously, the jumps of M^n are negligible for large n. The remainder of the proof is as before.

In Eq. (7.1) the rates λ and μ are constant. From now on we assume that regime switching will be present, so we have time-varying rate $\lambda_t = \lambda^{\mathrm{T}} Z_{t-}$ as before and likewise, in similar notation, $\mu_t = \mu^{\mathrm{T}} Z_{t-}$. Hence for the Markov modulated case, we now have, instead of Eq. (5.7),

$$\mathrm{d}N^n_t = \left(\lambda^{\mathrm{T}} Z_t(n-N^n_t) - \mu^{\mathrm{T}} Z_{t-}N^n_t\right)\mathrm{d}t + \mathrm{d}M^n_t, \tag{7.2}$$

where M^n is a martingale with respect to $\mathbb{F} = \{\mathcal{F}^Y_t \vee \mathcal{F}^Z_\infty, t\geq 0\}$.

Remark 7.2: In principle, the recovery rate μ_t could depend on another Markov chain $\tilde{Z}$, leading to a seemingly more general model. But combining the chains Z and $\tilde{Z}$ into a bivariate chain, would lead to a representation like Eq. (7.2) again with the matrix Q composed from the transition matrices of Z and $\tilde{Z}$.

One can then investigate the limit behavior of the process N^n given by Eq. (7.2) for $n\to\infty$ together with rapid switching of the Markov chain. We confine ourselves to a generalization of Theorem 5.11. We use notation introduced in previous sections and self-evident analogies. We will need the function ϱ satisfying

$$\dot{\varrho}_t = \lambda_\infty(1-\varrho_t) - \mu_\infty\varrho_t, \quad \varrho_0 = 0, \tag{7.3}$$

where $\mu_\infty = \mu^{\mathrm{T}}\pi$, and the functions σ_1 and σ_2 as specified after the statement of the theorem.

Theorem 7.3: *Let N^n be given by Eq. (7.2) and ϱ by Eq. (7.3). Then the scaled and centered process $\hat{N}^{n,\beta}$ given by*

$$\hat{N}^{n,\beta}_t := n^{-1/2(1+(1-\beta)^+)}(N^n_t - n\varrho_t),$$

converges weakly (as $n\to\infty$) to the solution of the following SDE

$$\mathrm{d}\hat{N}_t = -(\lambda_\infty+\mu_\infty)\hat{N}_t\,\mathrm{d}t + \mathbb{1}_{\{\beta\leq 1\}}\sigma_1(t)\,\mathrm{d}B^1_t + \mathbb{1}_{\{\beta\geq 1\}}\sigma_2(t)\,\mathrm{d}B^2_t, \quad \hat{N}_0 = 0, \tag{7.4}$$

where B^1 and B^2 are independent Brownian motions. Alternatively, we have the representation

$$\mathrm{d}\hat{N}_t = -(\lambda_\infty+\mu_\infty)\hat{N}_t\,\mathrm{d}t + \left(\mathbb{1}_{\{\beta\leq 1\}}\sigma_1(s)^2 + \mathbb{1}_{\{\beta\geq 1\}}\sigma_2(s)^2\right)^{1/2}\mathrm{d}B_t,$$

where B is a standard Brownian motion.

We close with a few remarks on the proof. For the case $\beta = 1$ it is along the same lines as the one for Theorem 5.6, but with more complicated expressions, although methodologically

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there are hardly any changes. One now writes $\hat{N}^n_t = e^{-(\lambda+\mu)^{\mathrm{T}}\zeta^n_t} X^n_t$, where X^n_t is given by an analog of Eq. (5.10),

$$X^n_t = -\int_0^t \Psi^n_s \, \mathrm{d}n^{-1/2} Z^n_s + \int_0^t \Psi^n_s \, \mathrm{d}n^{-1/2} \tilde{M}^n_s + \int_0^t e^{(\lambda+\mu)^{\mathrm{T}}\zeta^n_s} \, \mathrm{d}n^{-1/2} M^n_s, \tag{7.5}$$

with in the present situation

$$\Psi^n_s = e^{(\lambda+\mu)^{\mathrm{T}}\zeta^n_s}((1-\varrho_s)\lambda - \varrho_s\mu)^{\mathrm{T}} D.$$

Another main difference is the quadratic variation of the bivariate martingale $\mathbf{M}^{\zeta,n}$. One now obtains

$$\langle \mathbf{M}^{\zeta,n} \rangle_t = \int_0^t \begin{bmatrix} \Psi^n_s \tilde{V}^{n,*}_s (\Psi^n_s)^{\mathrm{T}} & 0 \\ 0 & e^{2(\lambda+\mu)^{\mathrm{T}}\zeta^n_s} V^{n,*}_s \end{bmatrix} \mathrm{d}s, \tag{7.6}$$

where $\tilde{V}^{n,*}_s$ is as before, but

$$V^{n,*}_s = \lambda^{\mathrm{T}} Z^n_s (1 - n^{-1}N^n_s) + \mu^{\mathrm{T}} Z^n_s n^{-1} N^n_s \to \lambda_\infty(1-\varrho_s) + \mu_\infty \varrho_s.$$

As a consequence, the limit of $\langle \mathbf{M}^{\zeta,n} \rangle_t$ is not an expression as simple as before, but can still be computed explicitly (it only involves integration of exponential functions). For reasons of brevity we just write

$$\int_0^t \Psi^n_s \tilde{V}^{n,*}_s (\Psi^n_s)^{\mathrm{T}} \, \mathrm{d}s \to \int_0^t e^{2(\lambda_\infty+\mu_\infty)s} \Phi_s(\mathrm{diag}\{\pi\}D^{\mathrm{T}} + D\mathrm{diag}\{\pi\})\Phi^{\mathrm{T}}_s \, \mathrm{d}s$$
$$=: \int_0^t e^{2(\lambda_\infty+\mu_\infty)s} \sigma_1(s)^2 \, \mathrm{d}s,$$

where

$$\Phi_s = (1-\varrho_s)\lambda^{\mathrm{T}}_\infty - \varrho_s\mu^{\mathrm{T}}_\infty, \quad \sigma_1(s)^2 = \Phi_s(\mathrm{diag}\{\pi\}D^{\mathrm{T}} + D\mathrm{diag}\{\pi\})\Phi^{\mathrm{T}}_s$$

and

$$\int_0^t e^{2(\lambda+\mu)^{\mathrm{T}}\zeta^n_s} V^{n,*}_s \, \mathrm{d}s \to \int_0^t e^{2(\lambda_\infty+\mu_\infty)s}(\lambda_\infty(1-\varrho_s) + \mu_\infty\varrho_s) \, \mathrm{d}s$$
$$=: \int_0^t e^{(2(\lambda_\infty+\mu_\infty)s} \sigma_2(s)^2 \, \mathrm{d}s.$$

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