

SERIE RESEARCH MEMORANDA

Tail Behavior of Credit Loss Distributions

André Lucas, Stefan Straetmans, Pieter Klaassen, and Peter Spreij

Research Memorandum 1999-60

December 1999



Tail Behavior of Credit Loss Distributions*

André Lucas^{‡§}

Stefan Straetmans[†]

Pieter Klaassen^{††}

Peter Spreij[¶]

This version: December 15, 1999

Abstract

We derive the exact loss distribution for portfolios of bonds or corporate loans when the number of risks grows indefinitely. We show that in many cases this distribution lies in the maximal domain of attraction of the Weibull (Type III) limit law. Knowledge of the distribution and its tail behavior is important for risk management in order not to over- or underestimate the likelihood of extreme credit losses for the portfolio as a whole. Conform to the credit risk literature, we assume that bond (or loan) defaults are triggered by a latent variable model involving two stochastic variables: systematic and idiosyncratic risk of the bond. It is shown that the tail behavior of these two variables translates into the tail behavior of the whole credit loss distribution. Surprisingly, even if both variables are thin-tailed, the credit loss distribution can have a finite tail index. Moreover, if idiosyncratic risk exhibits heavier tails than the systematic risk factor the tail index of the credit loss distribution can become extremely high, giving rise to a non-conventional shape of the credit loss distribution.

Key words: credit risk; Value-at-Risk; tail events; tail index.

*Andre Lucas also thanks the Dutch Organization for Scientific Research (N.W.O.) for financial support. Correspondence to: alucas@econ.vu.nl, sstraetmans@econ.vu.nl, Pieter.Klaassen@nl.abnamro.com, or spreij@wins.uva.nl.

[‡]Dept. Finance and Financial Sector Management, Vrije Universiteit, De Boelelaan 1105, NL-1081HV Amsterdam, the Netherlands

[§]Tinbergen Institute Amsterdam, Keizersgracht 482, NL-1017EG Amsterdam, the Netherlands

[†] ABN-AMRO Bank NV Financial Markets Risk Management, P.O.Box 283, NL-1000EA Amsterdam, the Netherlands

[¶]Korteweg-de Vries Institute, University of Amsterdam, Plantage Muidergracht 24, NL-1018TV Amsterdam, the Netherlands

1 Introduction

Extreme events play an important role in financial risk management. Banks and other financial institutions must be able to meet all their financial obligations even in situations of extremely adverse market conditions. As an example, consider a pension fund that promises its members to pay a price indexed amount per year from retirement until death. In order to finance this scheme, the fund gathers contributions from the members and invests these in stocks and bonds. A popular optimality criterion for choosing the portfolio mix between stocks and bonds constitutes the minimization of the expected cost (in terms of present and future contributions) subject to a constraint on risk. The risk constraint may be thought of constituting a bound on the likelihood that a certain outflow of pension money is no longer sustainable due to e.g. extremely and thus unexpectedly low returns on stocks and bonds or high inflation. If the probability bound is chosen sufficiently low in determining the pension's optimal asset mix, the latter will be highly influenced by the extreme tail behavior of the pension fund's market risks in portfolio. Similar examples can be constructed for banks, which typically have a portfolio of assets and liabilities.

The likelihoods of extremal events are increasingly interpreted and used as risk measures as the above example illustrates, see Jorion (1997). In particular, people often use a concept called Value-at-Risk or VaR. VaR is an extreme quantile of the distribution of financial losses. Typically, one uses quantiles corresponding to probability levels between 5% and 0.1%. The widespread use of VaR has been spurred by at least two factors. First, there have been several large losses involving more complex financial instruments called derivatives, see for example Jorion (1995). These losses have made responsible managers aware of the fact that they need to have insight into the riskiness of current financial activities. VaR is a simple way of meeting this objective: it gives the maximum amount that can be lost given a certain level of confidence. Due to its simplicity, VaR is commonly used in the financial industry, even though there are some conceptual problems with it as a measure of risk, see Artzner, Delbaen, Eber, and Heath (1997). A second reason for the widespread use of VaR is its acceptance by regulatory authorities, see Basle Committee on Bank Supervision (1996). Regulators require banks to conduct a sound risk management practice, including the use of VaR as a quantitative risk management tool.

So far, we only discussed how to measure and manage the portfolio risk evolving from market risk, i.e., risk evolving from fluctuations in market prices -and thus holding returns- of quoted financial assets. The potential usefulness of extreme value theory for market risk and the estimation of

extreme quantiles has been demonstrated by, e.g., Danielsson and de Vries (1998a,b). Though the importance of market risk has been growing over the last decades, credit risk is still (and has been historically) the most important risk factor for banks. Credit risk may be defined as the likelihood that obligors will not be able to repay the principal amount (and interest) of an issued loan (default risk). It also involves financial risks due to changes in the creditworthiness of the obligor, see J.P. Morgan (1999). In the present paper, we will be mainly concerned with the default risk component of credit risk.

It is generally accepted that the distribution of credit losses is skewed and exhibits heavy tails. Because we are interested in assessing the likelihood of extreme credit losses for quantitative credit risk management, a correct specification of the distributional tail is important. The framework we present here enables one to derive the distribution of credit losses and its tail behavior. More specifically, we show that highest order statistics for credit loss observations lie in the domain of attraction of an extreme value distribution of the Weibull type. Moreover, the framework enables one to pin down the determinants of the tail index under a variety of assumptions. To be more precise, we model credit losses using a latent variable approach inspired on JP Morgan's CreditMetrics, J.P. Morgan (1999). The latent variables are driven by two stochastic variables reflecting the systematic and idiosyncratic risk component of a given loan. Surprisingly we find that credit losses exhibit a finite tail index even when the underlying variables that 'trigger' the default mechanism are thin-tailed, e.g., normal. Moreover, the tail index can be arbitrarily close to zero if idiosyncratic risk is sufficiently more thin-tailed than the systematic risk component. Finally, we show that the effect of joint default behavior, as expressed by the systematic risk factor, on the distributional tail depends on the relative probability mass in the tails of the systematic and idiosyncratic risk component.

The set-up of the paper is as follows. In Section 2 we provide the basic modeling framework and derive the distribution of credit loss distributions for large corporate loan or bond portfolios. We also use this section to provide some more background as to the different approaches used for modelling credit loss distributions. In Section 3, we derive expressions for the tail index of credit losses for homogenous loan portfolios and a single factor. The results are generalized in Section 4 to the case of heterogenous portfolios and multi-factor models. In Section 5, we make a first step in characterizing the other characteristics of tail behavior apart from the tail index. For a linear Gaussian factor model, we characterize the slowly varying function that co-determines the tail behavior. Section 5 suggest that there may be strong biases in estimates of the tail index in finite samples, as the slowly varying

function decreases to zero for extreme credit losses. We investigate this issue by means of simulation in Section 6. Section 7 concludes, while the Appendix gathers all the proofs.

2 Basic framework

We consider a portfolio containing n bonds or loans. Each bond has a specific price which is determined by current interest rates, the bond's initial credit rating, and its maturity. During the life time of the bond, its characteristics can change, giving rise to changes in the market price of the bond. For example, the firm issuing the bond may go bankrupt. In that case, the price of the bond falls to a level guaranteed by, e.g., the sale of the firm's collateral, or even to zero if no guarantees are built into the bond contract. To capture these features, we characterize each bond j , where $j = 1, \dots, n$, by a two-dimensional vector

$$(S_j, \pi_j(S_j)). \quad (1)$$

The first variable is a latent variable that is crucial in triggering a company's default or a change in its credit rating. A prime candidate for S_j is the company's 'surplus', i.e., the difference in market value of assets and liabilities. If this surplus falls below a certain threshold, default occurs. We assume that the portfolio exposures are driven by a common factor structure

$$S_j = g_j(f, \varepsilon_j), \quad (2)$$

where f is the common factor, ε_j is a firm-specific risk factor, and $g_j(\cdot, \bullet)$ defines the functional form of the factor model for the j th firm. The formulation in (2) comprises the usual factor models from the literature. For example, if we set $g_j(f, \varepsilon_j) = \beta_j f + \varepsilon_j$ for some factor loading $\beta_j \in \mathbb{R}$, we obtain the formulation of CreditMetrics, see J.P. Morgan (1999). If $g_j(f, \varepsilon_j) = \varepsilon_j / (\beta_j f)$, we obtain the Creditrisk⁺ specification, at least up to first order, see Gordy (1999) and Credit Suisse (1999). Throughout the present article, we consider the one-factor model only, i.e., $f \in \mathbb{R}^1$. This simplifies the derivations considerably while still allowing us to analyze a broad range of interesting properties of credit loss distributions. In Section 7, we comment on approaches to generalize the results to a multi-factor setting.

We make the following assumption for the components in (2).

Assumption 1 (i) $\{\varepsilon_j\}_{j=1}^{\infty}$ is sequence of independent random variates that is independent of f .

(ii) $g_j(f, \varepsilon_j)$ is monotonically increasing in f and ε_j for all j .

(iii) Let $F_j(\cdot)$ denote the (at most everywhere continuously differentiable) distribution function of ε_j , and $\bar{F}_j(\cdot) = 1 - F_j(\cdot)$. Moreover, let $\bar{\varepsilon} = \inf\{\varepsilon | F(\varepsilon) = 1\}$ and $\underline{\varepsilon} = \sup\{\varepsilon | F(\varepsilon) = \mathbf{0}\}$. If $\bar{\varepsilon} < \infty$, $F_j(\cdot)$ has a right-hand tail expansion

$$\bar{F}_j(\bar{\varepsilon} - 1/\varepsilon) = \varepsilon^{-\nu_1^\varepsilon} \cdot \exp(p_2^\varepsilon(\varepsilon)) \cdot L_\varepsilon(\varepsilon) \quad (3)$$

for ε sufficiently large, where $p_2^\varepsilon(\varepsilon) = \nu_3^\varepsilon \cdot \varepsilon^{\nu_2^\varepsilon} \cdot (1 + o(1))$ and $L(\cdot)$ is a slowly varying function. Conversely, if $\bar{\varepsilon} = \infty$, the tail expansion takes the same form as in (3), but with the left-hand side replaced by $\bar{F}_j(\varepsilon)$. Analogous tail expansions apply for $F_f(\cdot)$, the distribution of f , with the expansion being valid for the left-hand tail.

Part (i) of the assumption is standard in the credit risk literature. It asserts that the surplus variables S_j , which trigger a firm's default, have a systematic risk component f , and a so called firm-specific or idiosyncratic risk component ε_j . The systematic risk component allows for defaults to occur in clusters. For example, average default rates are much higher during recessions than during booms of the economy. This can be modelled by letting f have more mass in the lower tails during recessions. Part (ii) places restrictions on the functional form of the factor model (2). In particular, we only consider cases where we can always uniquely retrieve an element from the vector (S_j, f, ε_j) given the other two elements. Note that both the linear CreditMetrics model and the multiplicative CreditRisk⁺ model satisfy this criterion, see Gordy (1999) and the model specifications presented earlier. The focus on *increasing* $g(\cdot, \cdot)$ is not very restrictive per se. For example, if $g_j(f, \varepsilon_j)$ is decreasing in f , we can transform variables and consider $g_j(\tilde{f}, \varepsilon_j)$ with $\tilde{f} = -f$. Part (iii) of the assumption places restrictions on the tail behavior of the idiosyncratic and systematic risk components. If $p_2^\varepsilon(\varepsilon) \equiv 0$, we allow for tails that lie in the domain of attraction of a Fréchet or a Weibull law, see Embrechts et al. (1997). We also allow for tails that have an exponential decay. Though our formulation is not as general as that in Theorem 3.3.26 of Embrechts et al. (1997), we still cover all distributions that are commonly used in empirical exercises, e.g., the normal and the Gamma distributions.

The second characteristic of bond j in (1) is the credit loss $\pi_j(\cdot)$ associated with bond j . The loss depends on several factors. First, the surplus variable S_j may trigger a credit event such as default or a credit rating migration. This results in a monetary loss.¹ The severity of the loss is determined by

¹ We assume that the exposures are marked to market, such that a credit rating migration generally causes a change in market value due to a change in the (market) credit

(inter alia) the initial credit rating, as the ratings are directly related to the bond's market discount factors. The amount lost may also depend on the state of the economy. This can be modeled by incorporating a common factor ψ in $\pi_j(\bullet)$. For example, the level and shape of the term structure of interest rates may change over time, resulting in higher or lower losses in the event of default. Similarly, there may be an additional idiosyncratic risk component η_j , for example, if the bond is a convertible bond. In the present paper we refrain from introducing the additional complications of dependence of $\pi_j(\cdot)$ on ψ or η_j . Finally, the loss functions $\pi_j(\cdot)$ may differ in general over different bonds or firms. Easy examples are cases in which the sizes of the loans or their maturities differ over firms.

We make the following assumption for the loss functions.

Assumption 2 $\sup_j E [\pi_j(S_j)^2 | f] < \infty$ almost surely.

The assumption makes the application of a law of large numbers possible at a later stage. Note that the potential loss $\pi_j(\cdot)$ may still be unbounded, as long as the conditional squared expectation is bounded uniformly in j . This comprises most cases of practical interest, e.g., portfolios of bonds, convertible bonds, interest rate derivatives like swaps, etc. As mentioned, for reasons of simplicity we do not allow $\pi_j(\cdot)$ to depend on other stochastic variables than S_j . Such extensions can be useful if one wants to study credit risk and market risk in an integrated framework, see the remarks in Lucas, Klaassen, Spreij, and Straetmans (1999).

The credit loss for a portfolio of size n is now given by

$$C_n = \sum_{j=1}^n \pi_j(S_j), \quad (4)$$

i.e., the sum of the individual losses. Instead of considering the distribution of C_n directly, we follow Lucas et al. (1999) and consider the case $n \rightarrow \infty$. Define

$$C = \lim_{n \rightarrow \infty} C_n/n. \quad (5)$$

The advantage of considering the distribution of credit losses for (infinitely) large portfolios only, is that the number of stochastic components is limited considerably. This facilitates the study of the tail behavior of credit losses. As was shown in Lucas et al. (1999), empirically relevant quantiles of C , e.g., 99% or 99.9%, can be used to construct good approximations to quantiles of C_n for n in the range 300-500 or larger. These values of n are quite small

spread.

given the usually large numbers of exposures in typical bank portfolios. We have the following theorem.

Theorem 1 (Williams)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \pi_j(S_j, k_j) - E[\pi_j(S_j, k_j) | f] \xrightarrow{a.s.} 0.$$

Using Assumption 2, the proof of this theorem follows directly from Theorem 12.13 of Williams (1991), see also Lucas et al. (1999). Consequently, from now on we only consider

$$C = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n E[\pi_j(S_j, k_j) | f], \quad (6)$$

which does only depend on one stochastic variable, f , and not on $\{\varepsilon_j\}_{j=1}^\infty$. As argued in the introduction, we are especially interested in the tail behavior of C , as this is very important from a credit risk management perspective. The most straightforward way to study this behavior is by using extreme value theory. In the next two sections we start by deriving the rate of tail decay in the form of the tail index for portfolios of increasing complexity.

3 Tail index for homogenous portfolios

For expositional purposes, we first derive the tail index of the credit loss distribution for a homogenous portfolio. This portfolio contains exposures which exhibit the same systematic risk, initial credit rating, and credit loss in case of default. In particular, for all j we set $g_j(f, \varepsilon_j) = g(f, \varepsilon_j)$, $F_j(\mathbf{E}) = F_\varepsilon(\varepsilon)$, and $\pi_j(S_j) = 1_{\{S_j < s\}}(S_j)$ for some relevant $s \in \mathbb{R}$, where $1_A(\cdot)$ is the indicator function for the set A . So all bonds in the portfolio have the same characteristics. In case of default ($S_j < s$), one money unit is lost, whereas there is no loss if default does not occur. Using this stylized portfolio, (6) simplifies to

$$C = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n E[1_{\{S_j < s\}}(S_j) | f] = P[g(f, \varepsilon_j) < s | f]. \quad (7)$$

Given the monotonicity conditions in Assumption 1, we can define the partial inverse functions $g^{-\varepsilon}(f, S_j)$ and $g^{-f}(S_j, \varepsilon_j)$ such that

$$g(f, g^{-\varepsilon}(f, S_j)) = g(g^{-f}(S_j, \varepsilon_j), \varepsilon_j) = g(f, \varepsilon_j) = S_j. \quad (8)$$

In addition to Assumption 1, we postulate the following requirements for the factor model (2).

Assumption 3 (i) Let \mathcal{S}_j , $\mathcal{3}$, and \mathcal{E}_j denote the support of S_j , f , and ε_j respectively. Then $g^{-\varepsilon}(f, s)$ and $g_j^{-f}(s, \varepsilon_j)$ are well defined for all j , $s \in \mathcal{S}_j$, $f \in \mathcal{3}$, and $\varepsilon_j \in \mathcal{E}_j$. Also,

$$\{g_j^{-\varepsilon}(f, s) | s \in \mathcal{S}_j\} = \mathcal{E}_j$$

for all $f \in \mathcal{3}$ and

$$\{g^{-f}(s, \mathcal{E}_j) | s \in \mathcal{S}_j\} = \mathcal{F}.$$

for all $\varepsilon_j \in \mathcal{E}_j$. (ii) If $p_2^\varepsilon(\cdot) \equiv 0$ and $p_2^f(\cdot) \equiv 0$, $\lim_{\varepsilon \uparrow \bar{\varepsilon}} \ln |g^{-f}(s, \varepsilon)| / \ln |\varepsilon| = \zeta$. Else $\lim_{\varepsilon \uparrow \bar{\varepsilon}} g^{-f}(s, \varepsilon) / \varepsilon = \zeta$.

Part (i) of the assumption requires that the factor model (2) is balanced in the systematic risk component f and the idiosyncratic risk ε_j . In particular, any realization of f can be compensated by a suitable realization of ε_j to produce the same value of S_j . This implies that if ε_j is pushed to the upper end of its support, f has to be pushed to the lower edge of its support in order to realize the same value of S_j . The intuition in economic terms of this condition is as follows. Consider the borderline case where a firm j is almost pushed into bankruptcy. If the common risk factor f is extremely adverse, then firm specific conditions (ε_j) have to be highly advantageous to prevent the firm from going bankrupt. Typical examples excluded by Assumption 3 are when firms are pushed into bankruptcy for a given realization of f no matter their firm specific risk factor realization ε_j . Part (ii) of the assumption is again a condition on the balancedness of the factor model (2). In particular, the rate at which systematic risk components have to change in order to maintain a fixed level of surplus $S_j = s$ has to be related to the rate of change of the idiosyncratic risk factor. Again, one can easily check that the condition is met for the linear and multiplicative factor model specifications commonly studied in the credit risk literature. Also note that one can often use a transformation of variables to achieve the required balancedness. For example, instead of considering $f \cdot \exp(\varepsilon_j)$ one can consider $f \cdot \tilde{\varepsilon}_j$ with $\varepsilon_j = \exp(\varepsilon_j)$. Such transformations can alter the tail behavior of the random variates, such that one has to make sure that Assumption 1 is still met.

The following theorem is proved in the Appendix.

Theorem 2 Let Assumptions 1 through 3 be satisfied.

- If $p_2^f(\cdot) = p_2^\varepsilon(\cdot) \equiv 0$, C lies in the maximum domain of attraction of the Weibull distribution with tail index $\nu_C = \zeta \nu_1^f / \nu_1^\varepsilon$.

- Otherwise, if $\nu_2^f = \nu_2^\varepsilon$, C lies in the maximum domain of the Weibull distribution with tail index $\nu_C = \nu_3^f \zeta^{\nu_2^f} / \nu_3^\varepsilon$.

Theorem 2 directly reveals how the tail index of the credit loss distribution depends on the tail indices of the latent factors (f and ε_j) and on the factor model $g(\cdot)$. The dependence on the factor model only enters through ζ , which is controlled by the balancedness condition (ii) in Assumption 3. If the tails of f and ε_j are of the Frechet or Weibull type, see Embrechts et al. (1997), the theorem shows that the tail index of the credit loss distribution is directly proportional to the ratio of the tail index of f to that of ε_j . The tail index of C can thus be very low if ν_1^ε is much higher than ν_1^f . Put differently, the tails of the credit loss distribution may be very fat if the idiosyncratic risk is much lighter tailed than the systematic risk, at least if both risks have Frechet type tails. This makes economic sense. If bad realizations of f occur more often because f has fatter tails than ε_j , simultaneous default of large portions of the portfolio as opposed to isolated firms is more likely, resulting in a higher probability of extreme realizations of C .

An interesting implication of the second part of Theorem 2 is that the tail index of credit losses can be finite even if the underlying risk factors f and ε_j are both thin-tailed, see also Lucas et al. (1999) and Figure 1 below. As an example, take the linear factor model $S_j = \beta f + \varepsilon_j$, with f and ε_j both standard normally distributed and $\beta > 0$. Clearly, $\nu_3^\varepsilon = \nu_3^f = -1/2$, $\nu_2^\varepsilon = \nu_2^f = 2$, and $\zeta = \beta^{-2}$. As a result we obtain $\nu_C = \beta^{-2}$. This confirms the results in Lucas et al. (1999). So higher systematic risk in terms of a higher β (and thus a lower ζ) transforms into a lower tail index of C .

To illustrate all the above findings, we present some credit loss distributions with different tail indices. We consider the linear factor model $S_j = \beta f + \varepsilon_j$ with $\beta = 1$. We further assume that f and ε_j follow a Student t distribution and that the probability of default is 5%. The resulting tail indices are given in Figure 1.

The first thing to see in Figure 1 is that for $\nu_1^\varepsilon > \nu_1^f$ the distribution function of C approaches 1 for $C \uparrow 1$ with increasing steepness. This implies that the density of C will be increasing in C near the maximum credit loss of 1. The fact that such phenomena have so far never been observed in the literature is not surprising. Up to now, the focus has only been on thin-tailed risk factors, e.g., normal or Gamma. The above results for fat-tailed risk factors, therefore, constitute a new contribution to the literature. The results also have a practical edge for credit risk management. The likelihood of extreme credit losses is increased if the common risk factor has fatter tails than the idiosyncratic risk factor. As it can be difficult to reliably estimate the tail-fatness of f and ε_j from the empirical data that is typically available,

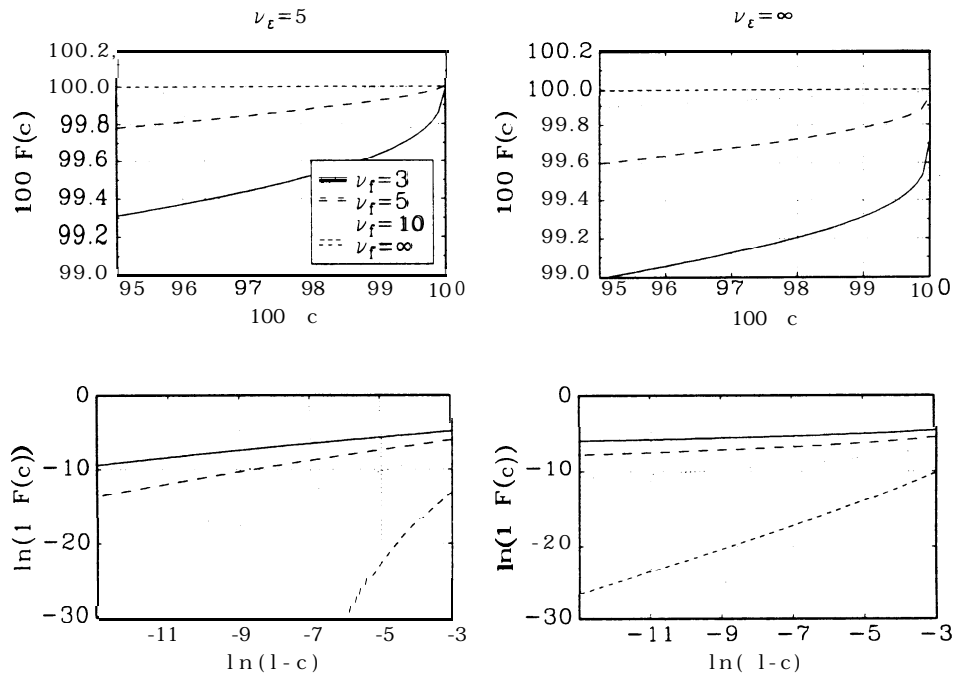


Figure 1: Credit loss distributions with different tail indices

The figure contains the credit loss densities for a homogenous portfolio (top row) and the log-exceedance probabilities for extreme credit losses. The underlying factor model is linear, $g(\mathbf{f}, \varepsilon_j) = \mathbf{f} + \varepsilon_j$, and the default probability is 5%. All ε_j are identically distributed. The risk-factors \mathbf{f} and ε_j both follow a Student t distribution with ν_1^f and ν_1^ε degrees of freedom, respectively.

a more conservative approach than that based on normally distributed risk factors can be warranted for prudent risk management.

4 Heterogenous portfolios

So far we have concentrated on a homogenous portfolios for a one-factor model. It is much more interesting, however, to study heterogenous portfolios and multi-factor models. For simplicity, we focus on a portfolio consisting of \mathbf{m} homogenous groups, where \mathbf{m} can be arbitrarily large. Using (6), we have

$$C = \sum_{i=1}^m \lambda_i. \quad P[g_i(f, \varepsilon_i) < s | f] = \sum_{i=1}^m \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)),$$

where, with a slight abuse of notation, we have replaced the firm index j by the group index i . The constants λ_i denote the i th exposure size multiplied by the fraction of firms present in group i .

In order to study extreme credit losses, we need several additional definitions. First define the upper bound of the support of C . Let

$$c^* = \sup_{f \in \mathcal{F}} \sum_{i=1}^m \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)),$$

then C^* defines our maximum credit loss. Note that C^* does not equal the maximum credit loss for a portfolio of finite size, as it may be the case that $C^* < \sum_{i=1}^m \lambda_i$. Firms can be pushed into default by either the systematic (f) or the idiosyncratic (ε_j) risk factors. Therefore, the maximum credit loss for a portfolio of finite size is always $\sum_{i=1}^m \lambda_i$. In the limit, however, the idiosyncratic risk can be diversified, meaning in statistical terms that Theorem 1 applies. It is common to exclude the diversifiable risk from an analysis on the management and pricing of risk, cf. Markowitz (1952). Therefore, we concentrate on C^* as the extremum of interest.

Let \mathcal{M} be a collection of sets defined by

$$\left\{ M \subset \{1, \dots, m\} \mid \sup_{f \in \mathcal{F}} \sum_{i \in M} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) = C^* \wedge \nexists M^s \subset M, \right. \\ \left. M^s \neq M, \text{ such that } \sup_{f \in \mathcal{F}} \sum_{i \in M^s} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) = C^* \right\}. \quad (9)$$

In words, \mathcal{M} is the collection smallest subsets of the portfolio that can give

rise to the maximum credit loss C^* .

Along with each set $A4 \in \mathcal{M}$, we define a collection of (disjoint) arcs $\hat{\mathcal{F}}(M) \subseteq \mathcal{F}$, such that for f along any $\hat{f} \in \hat{\mathcal{F}}(M)$ the portfolio results in its supremum credit loss. The collection $\hat{\mathcal{F}}(M)$ is assumed to be complete in the sense that for any arc \hat{f} leading to C^* for subgroup M , we have that all f -values along the arc also belong to an arc in $i(M)$.

Along each arc \hat{f} , we assume there exists indices $\iota(\hat{f})$ and constants $\zeta(\xi, \hat{f})$ such that

$$\frac{F_i(g_i^{-\varepsilon}(f, s))}{F_{\iota(\hat{f})}(g_{\iota(\hat{f})}^{-\varepsilon}(f, s))} \leq \zeta(\xi, \hat{f})$$

for all $i \in M$, $\xi > 0$ sufficiently small, and f uniformly in $f \in \hat{f}$. We also assume $0 \leq \zeta(\xi, \hat{f}) < \infty$. The fact that ζ depends on ξ denotes that we only consider that part of the arc \hat{f} that brings the credit loss within a ξ distance of C^* . We also define $\zeta(\hat{f}) = \lim_{\xi \downarrow 0} \inf \zeta(\xi, \hat{f})$ as the uniform upper bound (infinitely) close to C^* . Also define

$$\bar{\lambda}_{\iota(\hat{f})} = \sum_{i \in M} \lambda_i \cdot \zeta(\hat{f}).$$

To complete the set of definitions, let

$$\hat{f}^*(M, \xi) = \arg \operatorname{ess\,sup}_{\hat{f} \in \hat{\mathcal{F}}(M)} P \left[\bar{\lambda}_{\iota(\hat{f})} [1 - F_{\iota(\hat{f})}(g^{-\varepsilon}(f, s))] < \xi \mid \hat{f} \right] P[\hat{f}],$$

$$\hat{f}^*(M) = \lim_{\xi \downarrow 0} \hat{f}^*(M, \xi),$$

$$M^*(\xi) = \arg \operatorname{ess\,sup}_{M \in \mathcal{M}} P \left[\bar{\lambda}_{\iota(\hat{f}^*(M))} [1 - F_{\iota(\hat{f}^*(M))}(g^{-\varepsilon}(f, s))] < \xi \mid \hat{f}^*(M) \right] P[\hat{f}^*(M)],$$

and finally

$$M^* = \lim_{\xi \downarrow 0} M^*(\xi).$$

Given all these definitions, we can now present the following theorem on the tail index of credit losses for heterogenous multi-factor models. The theorem is proved in the appendix.

Theorem 3

$$\lim_{\xi \downarrow 0} \frac{\ln P[C > c^* - \xi]}{\ln \xi} = \lim_{\varepsilon \uparrow \bar{\varepsilon}} \frac{\ln P \left[g_{\iota(\hat{f}^*(M^*))}(f, \varepsilon) < s_1 \right]}{\ln [1 - F_{\iota(\hat{f}^*(M^*))}(\varepsilon)]}. \quad (10)$$

The main result of this theorem is threefold. First, the theorem states that in order to look at the extreme tail behavior of C we do not have to

take the whole portfolio into account. Only subgroups M that can produce the maximum credit loss matter. Second, given the focus on a particular M , we do not have to consider all possible realizations of the common factors f that push subgroup M into default. Instead, we only have to consider one critical path, $\hat{f}^*(M^*)$ for one particular $M = M^*$. Finally, given the critical path $\hat{f}^*(M^*)$ and the subportfolio characterized by M^* , the extreme tail behavior is only determined by one particular original group of the portfolio, characterized by $i = \iota(f^*(M^*))$. So a study of the tail behavior of C for heterogeneous multi-factor models after stepwise simplification boils down to a study of the tail behavior of a homogenous single-factor model. Homogeneity is assured by the focus on the specific (homogenous) portfolio group $i = \iota(f^*(M^*))$, while a single-factor approach suffices because we only consider realizations along $f^*(M^*)$.

It is easy to see that Theorem 3 comprises the result in Theorem 2. For a homogenous portfolio and a single-factor model, we can skip the subindex ι in (10). The numerator and denominator in (10) are then obviously directly related to the left-hand tail index of f and the right-hand tail index of ε , respectively. This corroborates the statements in Theorem 2 and leads to very similar expressions in case of multi-factor models and heterogeneous portfolios.

Apart from its positive contribution in characterizing the extreme credit loss tails, Theorem 3 also points to a potential problem of the application of extreme value theory to credit loss problems. The limiting credit loss is determined by a single subgroup of the total portfolio only. For example, if a specific group only constitutes an arbitrarily small portion of the total portfolio, it can still be the case that this group completely determines the extreme tail behavior. The small size of the group, however, makes it immaterial for the tail shape of credit losses near empirically relevant quantiles. We discuss this issue and related issues pertaining to a cautious application of extreme value theory in the credit risk context in the next two subsections.

5 Specific example: complete tail behavior

The fact that the credit loss distribution for (infinitely) large portfolios lies in the maximum domain of attraction of a Weibull law might suggest that extreme value theory can be very useful for computing extreme credit loss quantiles. As these quantiles are used in risk management exercises, they are of clear economic interest.

One example of how extreme value theory can be useful is as follows. If a specific quantile of the credit loss distribution is required, one can simulate from the underlying factor model. These simulations transform into

simulated credit losses through (4). Using a set of N simulated credit losses, the q th quantile can be estimated by the qN -th simulated order statistic. A much more efficient estimate can however be obtained by exploiting the precise tail shape of the credit loss distribution. Examples of such estimators are given in . . . One usually proceeds by estimating the rate of decay of the tail of C using only the extreme order statistics. Subsequently, using estimates of scale and location constants, the tail can be inverted to arrive at the relevant quantile.

This brings us to an important point. Usually, much attention is paid to the type of tail behavior (Gumbel, Fréchet, Weibull) and to reliable estimation of the tail index. Much less attention is paid to the scaling constants that are needed to make the tail approximations operational in empirical applications. A similar approach was taken in the present article. In Sections 3 and 4 we focused on obtaining closed-form expressions for the tail index for a range of distributions for f and ε_j . In this section, we study the scaling constants in more detail, as these are important for the practical applicability of extreme value theory.

As an aside, note that we showed in Section 4 that the rate of decay of the tail may differ substantially over different ranges of the credit loss support. This may limit the applicability of extreme value theory, as different tail shapes may dominate for different sizes of the credit loss. We come back to this issue using simulations in Section 6.

In the present section, we use an analytic approach instead of simulations. We derive the complete tail behavior of credit losses for a specific example and point out tail shape inhomogeneity over the support of credit losses even in case of a homogenous portfolio and a single-factor model. This warrants the cautious use of EVT, if it is to be used at all.

We consider the Gaussian linear factor model for a homogenous portfolio, see also Belkin, Suchower, and Forest (1998). We have

$$S_j = \rho f + \sqrt{1 - \rho^2} \varepsilon_j,$$

where f and ε_j are standard normal, and $0 < \rho < 1$. This set-up has the basic ingredients of the CreditMetrics approach. We only consider default loss, and the default probability and bond/loan size is the same for all exposures in the portfolio. We normalize the loan sizes by letting the maximum credit loss $c^* = 1$. We now have the following result.

Theorem 4 Given the model above, C has a tail expansion of the form

$$P[C > 1-\xi] = \xi^{(1-\rho^2)/\rho^2} \frac{\rho |\Phi^{-1}(\xi)|^{\rho^{-2}-2}}{|s + \Phi^{-1}(\xi)\sqrt{1-\rho^2}|} \exp \left[-\frac{s^2}{2\rho^2} - \frac{s\Phi^{-1}(\xi)\sqrt{1-\rho^2}}{\rho} \right] \cdot (1+o(1)), \quad (11)$$

for $\xi \downarrow 0$. The leading terms in this tail expansion are

$$P[C > 1-\xi] = \xi^{(1-\rho^2)/\rho^2} \frac{\rho(-\ln(\xi^2))^{\frac{1-3\rho^2}{2\rho^2}}}{\sqrt{1-\rho^2}} \exp \left[-\frac{s^2}{2\rho^2} - \frac{s\sqrt{-\ln(\xi^2)}\sqrt{1-\rho^2}}{\rho} \right] \cdot (1+o(1)). \quad (12)$$

Note that $\exp(|\ln(\xi)|^{1/2})$ is slowly varying in the sense of Karamata, as

$$\lim_{\xi \downarrow 0} \frac{\exp(|\ln(t\xi)|^{1/2})}{\exp(|\ln(\xi)|^{1/2})} = \lim_{\xi \downarrow 0} \exp\left(\frac{1}{2} \ln |t| / |\ln(\xi)|^{1/2}\right) = 1.$$

Define

$$L(1/\xi) = \frac{\rho}{\sqrt{1-\rho^2}} e^{-s^2/(2\rho^2)} (-\ln(2\pi\xi^2))^{-\frac{2-3}{2}} \Big/ \exp \left[\frac{s\sqrt{1-\rho^2}}{\rho} \sqrt{-\ln(2\pi\xi^2)} \right] \cdot (1+o(1)), \quad (13)$$

then $L(\cdot)$ is slowly varying and $\lim_{\xi \downarrow 0} L(\cdot) = 0$ for $s < 0$, which is typically the case. We have

$$P[C > 1-\xi] = \xi^{(1-\rho^2)/\rho^2} \cdot L(1/\xi),$$

with $L(\cdot)$ as in (13).

The above results illustrate some of the potential pitfalls in the application of extreme value theory for even simple homogenous credit portfolios. By concentrating only on the rate of tail decay, i.e., the tail index, we are effectively concentrating on the factor $\xi^{(1-\rho^2)/\rho^2}$ in **(11)**. The slowly varying function $L(1/\xi)$, however, is also important for characterizing the complete tail behavior of C for empirically relevant quantiles. The fact that $\lim_{\xi \downarrow 0} L(1/\xi) = 0$ signals that tail probabilities will be overestimated if we only concentrate on $\xi^{(1-\rho^2)/\rho^2}$. The (neglected) presence of $L(\cdot)$ may also cause biases in standard estimates of the tail index, thus further limiting the applicability of extreme value theory. To investigate these issues in more depth, we use simulations in the next section.

6 Simulated examples: tail inhomogeneity

In this section we make a first step in analyzing the usefulness of extreme value theory for credit loss distribution and credit risk management. As in the previous sections, we concentrate on the estimation of the tail index. In a future paper we plan to consider the fit of tail expansions based on extreme value theory for typical credit loss quantiles of empirical interest, such as 95%, 99% or even higher.

In our simple simulation set-up, we focus on the linear factor model of CreditMetrics,

$$S_j = \rho_j f + \sqrt{1 - \rho_j^2} \varepsilon_j, \quad (14)$$

where f and ε_j both have mean zero and unit variance. The factors either follow a thin-tailed Gaussian distribution or a fatter-tailed Student $t(5)$ distribution. The higher the value of ρ_j , the larger the systematic risk. We consider $\rho_j = 0.1, 0.25, 0.5$. We assume a default probability of 1%, which corresponds to a portfolio of BB rated bonds, see J.P. Morgan (1999).

As seen in Sections 3 and 4, the tail index is determined by only one part of the portfolio. It is therefore interesting to study the effect of the relative magnitude of different parts of the portfolio on the estimated tail index. To do this, we consider a portfolio consisting of two homogenous groups. Each group is characterized by its value of ρ_j^2 , ρ_1^2 and ρ_2^2 for group 1 and 2, respectively. Furthermore, group 1 and 2 constitute 100λ and $100(1 - \lambda)$ percent of the portfolio, respectively.

To estimate the tail index, several estimators can be thought of. We concentrate on the one most well-known, namely the Hill estimator, see Hill (1975). A crucial step in estimating the tail index is the selection of the number of order statistics to be taken into account. As we have an analytic expression for the tail index and know the data generating process of credit losses, we can choose the number of order statistics that minimizes the mean-squared error (MSE) of the tail index estimator. This procedure cannot be implemented for empirical data, but gives an indication of what one might best expect. A disadvantage of the Hill estimator is that it was originally developed to estimate the tail index of distributions of type II, i.e., distributions that lie in the maximum domain of attraction of a Fréchet law. We have proved earlier that the credit loss distribution lies in the maximum domain of attraction of a Weibull. This might cause complications for the validity of the Hill estimation procedure. To check this, we compute the Hill estimator both for the raw simulated credit losses, C , and for the transformed credit losses $(1 - C)^{-1}$. If C lies in the maximum domain of attraction of a Weibull with index α , $(1 - C)^{-1}$ lies in the maximum domain of attraction

of the Fréchet with the same index α .

The results of our simulation experiment are presented in Table 1. The note below the table describes the simulation set-up in more detail.

We first concentrate on the tail index estimates of the untransformed credit losses C . For Gaussian factors, the Hill estimator generally underestimates the true tail index. The discrepancy is smaller for smaller values of the tail index. Based on Section 4 one can prove that for Gaussian factors the tail index is given by $\max_{i=1,2}(\mathbf{1} - \rho_i^2)/\rho_i^2$. By inspecting the entries in Table 1, we see that the bias in the Hill estimator is larger for smaller sizes of that part of the portfolio characterized by the smallest ρ_i^2 . This is intuitively clear. If the portion of the portfolio determining the extreme tail behavior is only small, the tail shape governed by the remaining exposures in the portfolio will heavily influence the less extreme tails, thus influencing the Hill estimates for finite sample sizes. The biases are much less noticeable if the smaller part of the portfolio has a larger ρ_i^2 .

If the factors follow a Student $t(5)$ distribution, the true tail index is generally lower. The bias in the Hill estimator is also generally lower for ρ_i^2 equal to 0.25 and 0.50. For less systematic risk, $\rho_i^2 = 0.10$, the bias is still about 30% if the tail behavior is determined by the smallest part of the portfolio.

A peculiar finding of the present simulation experiment is seen if we consider the transformed credit losses. As mentioned, $(1 - C)^{-1}$ has Pareto type tails. With a few exceptions for the Student $t(5)$, however, the Hill estimates are all too high, both for Gaussian and Student t factors. At first sight, one might be tempted to explain this from the presence of the slowly varying function in the expression for the tail. As proved in Section 5 for the Gaussian case, this function decreases to zero, probably causing a bias towards thinner tails in finite samples. It is then unclear, however, why the same reasoning does not apply for the untransformed credit losses where a downward rather than an upward bias was seen. Clearly, more research has to be put in studying the behavior of different tail index estimators for credit loss distributions in order to understand these results more thoroughly.

7 Concluding remarks

In this paper, we followed the conditional approach to credit risk management. Using a latent factor model, we (nonlinearly) decompose credit risk into a systematic and an idiosyncratic risk factor. We allow for different tail behavior of both risk components and a general factor structure. With these ingredients, we prove that under a wide variety of circumstances, the distri-

Table 1: Hill Estimates of the Credit Loss Tail Index

λ	Tail C									Tail $(1 - C)^{-1}$														
	Gaussian factors						Student t(5) factors			Gaussian factors						Student t(5) factors								
	$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)	$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)	$\hat{\alpha}$ (α)		$\hat{\alpha}$ (α)					
	ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2			
		10	25	50				10	25	50				10	25	50				10	25	50		
1	10	6.2(9.0)	5.3(9.0)	5.5(9.0)			10	3.1(3.0)	1.8(3.0)	1.9(3.0)			10	84.2(9.0)	27.5(9.0)	9.4(9.0)			10	56.6(3.0)	10.8(3.0)	2.6(3.0)		
	25	6.5(9.0)	2.6(3.0)	2.6(3.0)			2 5	3.1(3.0)	1.8(1.7)	1.4(1.7)			2 5	80.9(9.0)	27.9(3.0)	9.2(3.0)			25	50.6(3.0)	12.0(1.7)	2.3(1.7)		
	5 0	6.1(9.0)	2.8(3.0)	1.0(1.0)			5 0	3.1(3.0)	1.8(1.7)	1.1(1.0)			50	79.5(9.0)	27.4(3.0)	9.1(1.0)			5 0	50.0(3.0)	13.1(1.7)	1.9(1.0)		
	ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2			
		10	25	50				10	25	50				10	25	50				10	25	50		
10	10	5.7(9.0)	6.2(9.0)	4.5(9.0)			10	3.1(3.0)	1.9(3.0)	2.0(3.0)			10	83.0(9.0)	28.3(9.0)	10.2(9.0)			10	60.3(3.0)	11.8(3.0)	3.3(3.0)		
	25	6.1(9.0)	2.8(3.0)	2.7(3.0)			2 5	2.9(3.0)	1.8(1.7)	1.6(1.7)			2 5	68.0(9.0)	29.5(3.0)	8.9(3.0)			2 5	34.5(3.0)	9.4(1.7)	2.3(1.7)		
	5 0	5.9(9.0)	3.0(3.0)	1.0(1.0)			5 0	2.6(3.0)	1.7(1.7)	1.1(1.0)			5 0	48.6(9.0)	22.8(3.0)	9.1(1.0)			5 0	21.7(3.0)	8.0(1.7)	2.9(1.0)		
	ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2			
		10	25	50				10	25	50				10	25	50				10	25	50		
50	10	6.2(9.0)	5.9(9.0)	4.5(9.0)			10	3.1(3.0)	2.3(3.0)	2.2(3.0)			10	84.2(9.0)	42.8(9.0)	17.0(9.0)			10	57.2(3.0)	16.7(3.0)	6.3(3.0)		
	25	5.8(9.0)	2.7(3.0)	2.7(3.0)			2 5	2.3(3.0)	1.8(1.7)	1.4(1.7)			2 5	41.7(9.0)	27.7(3.0)	14.1(3.0)			2 5	21.4(3.0)	10.3(1.7)	4.4(1.7)		
	5 0	5.6(9.0)	2.7(3.0)	1.0(1.0)			50	1.7(3.0)	1.4(1.7)	1.1(1.0)			50	18.2(9.0)	12.7(3.0)	8.2(1.0)			50	6.4(3.0)	4.5(1.7)	2.3(1.0)		
	ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2			
		10	25	50				10	25	50				10	25	50				10	25	50		
90	10	6.4(9.0)	5.4(9.0)	5.9(9.0)			10	3.1(3.0)	2.9(3.0)	2.6(3.0)			10	85.1(9.0)	72.9(9.0)	52.8(9.0)			10	57.4(3.0)	44.2(3.0)	21.6(3.0)		
	25	5.6(9.0)	2.6(3.0)	2.7(3.0)			25	1.9(3.0)	1.8(1.7)	1.7(1.7)			25	28.6(9.0)	27.2(3.0)	23.4(3.0)			25	10.0(3.0)	10.5(1.7)	8.5(1.7)		
	50	4.9(9.0)	2.7(3.0)	1.0(1.0)			5 0	2.0(3.0)	1.4(1.7)	1.1(1.0)			50	9.8(9.0)	8.5(3.0)	8.3(1.0)			50	2.6(3.0)	2.6(1.7)	2.3(1.0)		
	ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2				ρ_1^2		ρ_2^2			
		10	25	50				10	25	50				10	25	50				10	25	50		
99	10	6.1(9.0)	6.1(9.0)	5.9(9.0)			10	3.1(3.0)	3.1(3.0)	3.0(3.0)			10	85.5(9.0)	79.9(9.0)	79.4(9.0)			10	51.9(3.0)	52.7(3.0)	51.5(3.0)		
	25	5.1(9.0)	2.8(3.0)	2.7(3.0)			25	1.8(3.0)	1.8(1.7)	1.8(1.7)			2 5	27.5(9.0)	28.2(3.0)	27.6(3.0)			25	11.0(3.0)	9.7(1.7)	10.7(1.7)		
	50	4.5(9.0)	2.6(3.0)	1.0(1.0)			5 0	2.0(3.0)	1.6(1.7)	1.0(1.0)			50	9.8(9.0)	8.7(3.0)	8.3(1.0)			50	2.5(3.0)	2.4(1.7)	2.1(1.0)		

18

The table contains estimates of the tail index of the credit loss distribution based on the Hill estimator. The factor model is linear, $S_j = \gamma_j f + (1 - \rho_j^2)^{1/2} \varepsilon_j$, and the factors f and ε_j either have a standard normal, or a *unit variance* Student $t(5)$ distribution. The values of $\rho_j^2/100$ instead of ρ_j^2 are presented in the table. The loan portfolio consists of 2 groups, where group 1 (characterized by ρ_1^2) constitutes $\lambda\%$ of the portfolio. The tail index estimates are obtained as follows. For a series of 1000 observations, we compute the Hill estimator as a function of the number of order statistics. This is repeated 100 times to obtain an estimate of the variance of the Hill estimator. As we know the tail index analytically, we can compute the mean-squared error (MSE) as a function of the number of order statistics using the estimated variance (mentioned earlier) and the squared bias. The latter is computed using the average of the Hill estimates (per number of order statistics) over the 100 replications. The final estimate is the average Hill estimate for the number of order statistics that minimizes the MSE. As the Hill estimator is typically for Fréchet laws and we have a distribution in the maximum domain of attraction of a Weibull, we estimate the index using the original credit losses (tail C) and transformed credit losses (tail $(1 - C)^{-1}$), where we note that $(1 - C)^{-1}$ has Fréchet tails if C has Weibull tails.

bution of portfolio credit losses lies in the maximum domain of attraction of a Weibull law. For a homogenous portfolio and a single factor, the tail index is related directly to the factor model structure and the tail indices of systematic and idiosyncratic risk. For multi-factor models and heterogeneous portfolios, the expression for the tail index is much more involved. The key result, however, is that only an arbitrarily small subset of the complete portfolio determines the extreme tail behavior. Moreover, only one particular realization of the systematic risk factors matters for the tail index. In words, the thickness of the extreme tail is determined by the realization of risk factors that in a sense produces the worst possible culmination of potential losses. If the idiosyncratic risk is much thinner tailed than the systematic risk, the tail index of credit losses is very small. In particular, the density of credit losses may then be increasing towards the upper end of its support. In some circumstances, the increasing part of the density already starts before extreme quantiles of empirical interest, e.g., 99%. This means that very extreme credit losses may show up with a much larger probability than based on a factor model with both Gaussian systematic and idiosyncratic risk.

For a special case of a linear Gaussian factor model, we were able to determine the tail behavior analytically in more much detail. We proved that in the setting studied the algebraically declining tail shape has to be complemented by a slowly varying function that decreases to zero. As a result, in finite samples tails may appear to be more rapidly declining than the in fact are. To study the magnitude of the potential bias, we conducted a small-scale simulation experiment. Some of the theoretical findings were supported in the experiment. Tail index estimates are generally biased, and the bias is larger if the part of the portfolio that determines the tail index is smaller. The simulation experiment also produced some interesting topics for further research. It turned out that the tail index estimators behaved completely differently if the credit losses were transformed or not. The transformation considered was that which makes the extreme tail shape of credit losses lie in the maximum domain of attraction of the Fréchet rather than the Weibull. Though one would expect intuitively that the Hill estimator, which is designed to estimate the tail index of “a type II limiting law, performs better for the transformed credit losses, it turned out that the estimates for the untransformed losses were much closer their true values.

Several interesting topics for future research remain. First, a further study into the properties of different tail index estimators for different transformations of credit losses seems warranted. This will also be a first step for the second topic of research, namely an assessment of the adequacy of extreme quantiles estimated using extreme value theory. Though the biases and properties in general of tail indices are interesting in themselves, the fit

of extreme tails and quantiles based on extreme value theory seems more important from a practical point of view. This boils down to studying the applicability and relevance of extreme value theory for practical credit risk management.

Appendix: Proofs

We first prove the following lemma for the homogenous portfolio of Sections 3.

Lemma A1 *Given Assumptions 1 through 3, we have*

$$\lim_{\xi \downarrow 0} \frac{\ln P[C > 1 - \xi]}{\ln \xi} = \lim_{\varepsilon \uparrow \bar{\varepsilon}} \frac{\ln [F_f(g^{-f}(s, \varepsilon))]}{\ln [1 - F_\varepsilon(\varepsilon)]}. \quad (\text{A1})$$

Proof: First note that $g^{-\varepsilon}(f, s)$ is decreasing in f . Moreover, the inverse of $g^{-\varepsilon}(f, s)$ with respect to f is given by $g^{-f}(s, \varepsilon_j)$. From (7), we have

$$C = P[\varepsilon_j < g^{-\varepsilon}(f, s) | f] = F_\varepsilon[g^{-\varepsilon}(f, s)], \quad (\text{A2})$$

where the inequality is preserved because $g(f, \varepsilon_j)$ is increasing in ε_j . Therefore,

$$\begin{aligned} P[C > 1 - \xi] &= P[g^{-\varepsilon}(f, s) > F_\varepsilon^{-1}(1 - \xi)] \\ &= P[f < g^{-f}(s, F_\varepsilon^{-1}(1 - \xi))] \\ &= F_f[g^{-f}(s, F_\varepsilon^{-1}(1 - \xi))], \end{aligned} \quad (\text{A3})$$

where the inequality is reversed because $g^{-\varepsilon}(f, s)$ is decreasing in ε . Using the substitution $\varepsilon = F_\varepsilon^{-1}(1 - \xi)$ we obtain the desired result. \blacksquare

The importance of this lemma lies in the fact that it allows us to compute the tail index of the distribution of C . From Corollary 3.3.13 of Embrechts, Kliippelberg, and Mikosch (1997) it follows that if (A1) equals $\nu_C \neq 0$, then C lies in the maximal domain of attraction of a Weibull law with (right) tail index ν_C .

Proof of Theorem 2: We first prove the first half of the theorem. If $\bar{\varepsilon} = \infty$, we have from the tail conditions in Assumption 1 and the result in Lemma A1 that

$$\begin{aligned} \lim_{\xi \downarrow 0} \frac{\ln[P(C > 1 - \xi)]}{\ln \xi} &= \lim_{\varepsilon \uparrow \infty} \frac{\ln [(g^{-f}(s, \varepsilon))^{-\nu_1^f} \cdot L_f(g^{-f}(s, \varepsilon))]}{\ln [\varepsilon^{-\nu_1^\varepsilon} \cdot L_\varepsilon(\varepsilon)]} \\ &= \zeta \nu_1^f / \nu_1^\varepsilon, \end{aligned}$$

where $\zeta = \lim_{\varepsilon \uparrow \infty} \ln[g^{-f}(s, \varepsilon)] / \ln[\varepsilon]$. Similarly, if $\bar{\varepsilon} < \infty$,

$$\begin{aligned} \lim_{\xi \downarrow 0} \frac{\ln[P(C > 1 - \xi)]}{\ln \xi} &= \lim_{\varepsilon \uparrow \bar{\varepsilon}} \frac{\ln [(\bar{f} - g^{-f}(s, \varepsilon))^{\nu_1^f} \cdot L_f([\bar{f} - g^{-f}(s, \varepsilon)]^{-1})]}{\ln [(\bar{\varepsilon} - \varepsilon)^{\nu_1^\varepsilon} \cdot L_\varepsilon([\bar{\varepsilon} - \varepsilon]^{-1})]} \\ &= \zeta \nu_1^f \nu_1^\varepsilon, \end{aligned}$$

where ζ now equals $\lim_{\varepsilon \uparrow \infty} \ln[\bar{f} - g^{-f}(s, \varepsilon)] / \ln[\bar{\varepsilon} - \varepsilon]$.

For the second half of the theorem, note that for $\bar{\varepsilon} = \infty$

$$\lim_{\xi \downarrow 0} \frac{\ln[P(C > 1 - \xi)]}{\ln \xi} = \lim_{\varepsilon \uparrow \infty} \frac{\nu_3^f (g^{-f}(s, \varepsilon))^{\nu_2^f}}{\nu_3^\varepsilon \varepsilon^{\nu_2^\varepsilon}} = \frac{\nu_3^f \zeta^{\nu_2^f}}{\nu_3^\varepsilon}$$

if $\nu_2^f = \nu_2^\varepsilon$, and 0 or ∞ if $\nu_2^f > \nu_2^\varepsilon$ and $\nu_2^f < \nu_2^\varepsilon$, respectively. A similar derivation can be set up for the case of finite $\bar{\varepsilon}$. \blacksquare

To prove Theorem 3, we first introduce the following lemma.

Lemma A2 For any $M_1, M_2 \in M$, $M_1 \neq M_2$, define

$$\mathcal{F}_1 = \{f \mid \sum_{i \in M_1} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) > C^* - \xi\} \quad (\text{A4})$$

and

$$\mathcal{F}_2 = \{f \mid \sum_{i \in M_2} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) > C^* - \xi\}. \quad (\text{A5})$$

Then $\lim_{\xi \downarrow 0} P(\mathcal{F}_1 \cap \mathcal{F}_2) = 0$.

Proof: Let $M_1, M_2 \in M$, $M_1 \neq M_2$. If the lemma is false, then for a given ξ with ξ sufficiently small, there exists a region $\mathcal{F}^* \subset \mathcal{F}$ such that $\mathcal{F}^* \subset \mathcal{F}_1$, $\mathcal{F}^* \subset \mathcal{F}_2$, and $P(f \in \mathcal{F}^*) > 0$. As M_1 is the smallest subset of the portfolio giving rise to the maximum credit loss,

$$\sum_{i \in M_1 \setminus M_2} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) > k,$$

for all $f \in \mathcal{F}^*$ and some constant $k > 0$. Using this and (A5), we have

$$\sum_{i \in M_1 \cup M_2} \lambda_i \cdot \tilde{F}_i(g_i^{-\varepsilon}(f, s)) > C^* - \xi + k,$$

for all $f \in \mathcal{F}^*$. As ξ can be chosen arbitrarily small, this contradicts the definition of C^* as the supremum credit loss and thus proves the lemma. \blacksquare

Proof of Theorem 3: Given Lemma A2, we have to consider

$$\sum_{M \in \mathcal{M}} \int P \left[\sum_{i \in M} \lambda_i [1 - F_i(g_i^{-\varepsilon}(f, s))] < \xi \mid \hat{f} \right] p(\hat{f}) d\hat{f}, \quad (\text{A6})$$

which follows by noting that

$$P \left[\sum_{i \in M} \lambda_i \cdot F_i(g_i^{-\varepsilon}(f, s)) > C^* - \xi \mid \hat{f} \right] = P \left[\sum_{i \in M} \lambda_i \cdot [1 - F_i(g_i^{-\varepsilon}(f, s))] < \xi \mid \hat{f} \right]. \quad (\text{A7})$$

Conditional on a given arc $\hat{f} \in \mathcal{F}(M)$, the uniform boundedness in the definition of $\iota(\hat{f})$ ensures that the term with $i = \iota(\hat{f})$ dominates the other terms for $\xi \downarrow 0$. As a result, we

can replace the sum over i by the single term indexed $\iota(\hat{f})$. A similar argument can be repeated, leading us to select the arc \hat{f}^* and subset M^* that produce the fattest tail in (A6). The proof is completed by noting that

$$\begin{aligned} \lim_{\xi \downarrow 0} \frac{\ln P[\bar{\lambda}_{\iota(\hat{f}^*(M^*))}[1 - F_{\iota(\hat{f}^*(M^*))}(g^{-\varepsilon}(f, s))] < \xi]}{\ln \xi} &= \lim_{\xi \downarrow 0} \frac{\ln P[g^{-\varepsilon}(f, s) > F_{\iota(\hat{f}^*(M^*))}^{-1}(1 - \xi)]}{\ln \xi} \\ &= \lim_{\xi \downarrow 0} \frac{\ln P[s > g(f, F_{\iota(\hat{f}^*(M^*))}^{-1}(1 - \xi))]}{\ln \xi} \\ &= \lim_{\varepsilon \uparrow \varepsilon} \frac{\ln P[g(f, \varepsilon) < s]}{\ln[1 - F_{\iota(\hat{f}^*(M^*))}^{-1}(\varepsilon)]}, \end{aligned} \quad (\text{A8})$$

which proves the theorem. \blacksquare

Proof of Theorem 4: Using the fact that for $x \downarrow -\infty$ we have $\Phi(x) = \phi(x)/|x|(1 + O(|x|^{-2}))$, we obtain

$$\begin{aligned} P[C > 1 - \xi] &= \Phi\left(\frac{s + \Phi^{-1}(\xi)\sqrt{1 - \rho^2}}{\rho}\right) \\ &\stackrel{a}{=} \frac{\phi\left(\frac{s + \Phi^{-1}(\xi)\sqrt{1 - \rho^2}}{\rho}\right)}{\frac{|s + \Phi^{-1}(\xi)\sqrt{1 - \rho^2}|}{\rho}} \\ &= e^x \mathbb{P}\left(\frac{s^2}{\rho^2} - \frac{s\Phi^{-1}(\xi)\sqrt{1 - \rho^2}}{\rho}\right) \left[\frac{\phi(\Phi^{-1}(\xi))}{|\Phi^{-1}(\xi)|}\right]^{\frac{1 - \rho^2}{\rho^2}} \frac{|\Phi^{-1}(\xi)|^{(1 - \rho^2)/\rho^2}}{|s + \Phi^{-1}(\xi)\sqrt{1 - \rho^2}|} \\ &\stackrel{a}{=} e^x \mathbb{P}\left(\frac{s^2}{\rho^2} - \frac{s\Phi^{-1}(\xi)\sqrt{1 - \rho^2}}{\rho}\right) [\xi]^{\frac{1 - \rho^2}{\rho^2}} \frac{|\xi|^{(1 - \rho^2)/\rho^2}}{|s + \Phi^{-1}(\xi)\sqrt{1 - \rho^2}|}. \end{aligned} \quad (\text{A9})$$

Let $\hat{\Phi}(x) = \phi(x)/|x|$, then

$$\hat{\Phi}^{-1}(\xi) = \frac{-\exp[-\frac{1}{2}LW(1/(2\pi\xi^2))]}{\sqrt{2\pi\xi^2}},$$

with $LW(\cdot)$ the Lambert-W function, i.e., the solution to

$$LW(x) \cdot \exp[LW(x)] = x.$$

For large positive x , we have asymptotically that

$$LW(x) = \ln(x) - \ln(\ln(x)) + o(\ln(\ln(x))),$$

such that

$$\hat{\Phi}^{-1}(\xi) \stackrel{\xi \downarrow 0}{\approx} -\sqrt{-\ln(2\pi\xi^2)}. \quad (\text{A10})$$

Substituting $\hat{\Phi}^{-1}(\xi)$ in (A9) by (A10), we obtain the desired result. \blacksquare

References

- Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1997). Definition of coherent measures of risk. mimeo.
- Basle Committee on Bank Supervision (1996). Amendment to the capital accord to incorporate market risks. Report 24, Bank of International Settlements, Basle.
- Belkin, B., S. Suchower, and L. Forest (1998). The effect of systematic credit risk on loan portfolio value-at-risk and loan pricing. *CreditMetrics Monitor* (1st Quarter), 17-28. <http://www.creditmetrics.com>.
- Credit Suisse (1999). *CreditRisk⁺*. Downloadable: <http://www.csfp.csh.com>.
- Danielsson, J. and C. de Vries (1998a). Beyond the sample: extreme quantile and probability estimation. Technical Report 98-016/2, Tinbergen Institute, Rotterdam.
- Danielsson, J. and C. de Vries (1998b). Value-at-Risk and extreme returns. Technical Report 98-017/2, Tinbergen Institute, Rotterdam.
- Embrechts, P., C. Klüppelberg, and T. Mikosch (1997). *Modeling Extremal Events*, Volume 33 of *Applications of Mathematics; Stochastic Modelling and Applied Probability*. Heidelberg: Springer Verlag.
- Gordy, M. (1999). A comparative anatomy of credit risk models. *Journal Of Banking and Finance*, forthcoming.
- Hill, B. (1975). A simple general approach to inference about the tail of a distribution. *Annals of Statistics* 3, 1163-1173.
- Jorion, P. (1995). *Big bets gone bad: derivatives and bankruptcy in Orange County*. Orlando: Academic Press.
- Jorion, P. (1997). *Value at Risk — the New Benchmark for Controlling Risk*. New York: McGraw-Hill.
- J.P. Morgan (1999). *CreditMetrics* (4th ed.). <http://www.creditmetrics.com>.
- Lucas, A., P. Klaassen, P. Spreij, and S. Straetmans (1999). An analytic approach to credit risk of large corporate bond and loan portfolios. Technical Report 1999-18, Vrije Universiteit. revised.
- Markowitz, H. (1952). Portfolio selection. *Journal of Finance* 7, 77-91.
- Williams, D. (1991). *Probability with Martingales*. Cambridge: Cambridge University Press.