On Likelihood Inference for Stochastic Volatility Models

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Diffusion bridge

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t$$

A solution of in the interval $[t_1, t_2]$ such that $X_{t_1} = x_1$ and $X_{t_2} = x_2$ is called a (t_1, x_1, t_2, x_2)-bridge.

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Beskos, Papaspiliopoulos and Roberts (2006, 2007)

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Straightforward diffusion bridge simulation using e.g. the Euler scheme:

Bladt and Sørensen (2007)

Assumption: Ergodic diffusion

$$\int_{\ell}^{r} [s(x)\sigma^2(x)]^{-1} dx < \infty$$

where

$$s(x) = \exp\left(-2\int_{x^\#}^{x} \frac{\alpha(y)}{\sigma^2(y)} dy\right)$$

with $x^\# \in (\ell, r)$ (the state space)

$$\int_{x^\#}^{r} s(x)dx = \int_{\ell}^{x^\#} s(x)dx = \infty.$$

A one-dimensional ergodic diffusion is time-reversible

Diffusion bridge simulation

$$dX_t^i = \alpha(X_t^i)dt + \sigma(X_t^i)dW_t^i, \quad X_0^1 = a \quad \text{and } X_0^2 = b$$

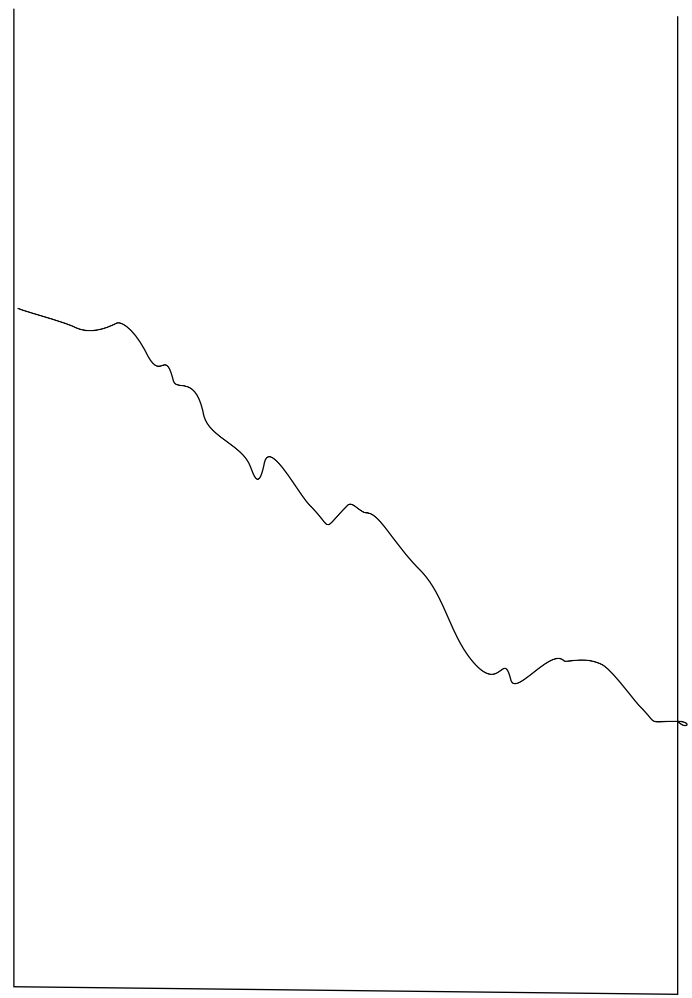
W^1 and W^2 independent standard Wiener processes

Define $\tau = \inf\{0 \leq t \leq 1 | X_t^1 = X_{1-t}^2\}$ ($\inf \emptyset = +\infty$) and

$$Z_t = \begin{cases} X_t^1 & \text{if } 0 \leq t \leq \tau \\ \\ X_{1-t}^2 & \text{if } \tau < t \leq 1. \end{cases}$$

Then the distribution of $\{Z_t\}_{0 \leq t \leq 1}$ conditional on the event $\{\tau \leq 1\}$ equals the conditional distributions of $\{X_t\}_{0 \leq t \leq 1}$ given that $X_0 = a$ and $X_1 = b$, i.e. Z is a $(0, a, 1, b)$-bridge

Diffusion bridge simulation



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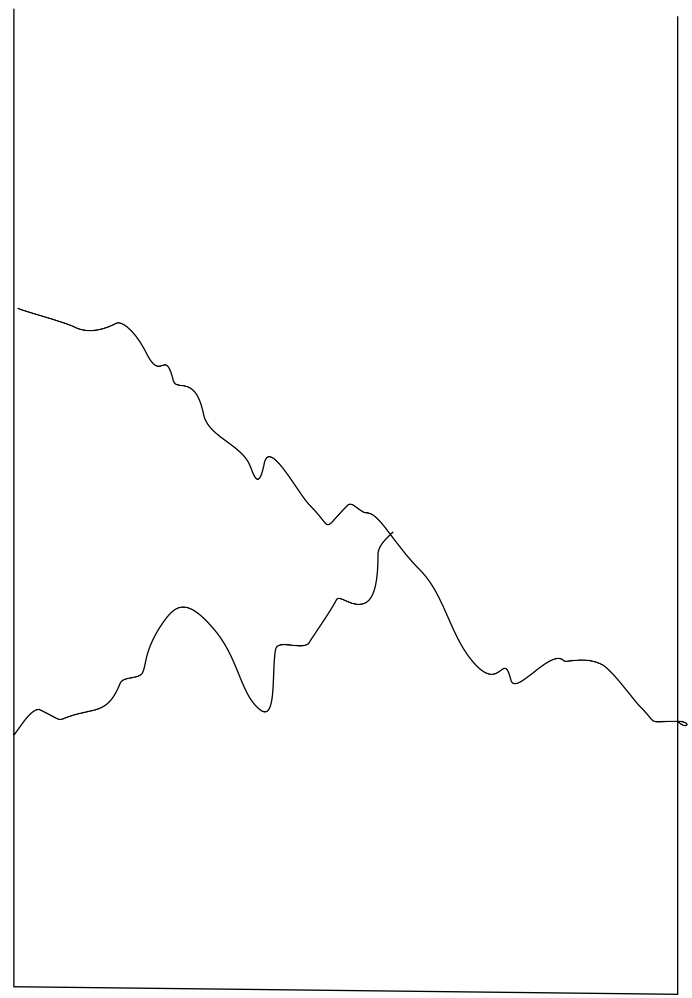

Implementation

$$Y^1_{\delta i}, Y^2_{\delta i}, \quad i = 0, 1, \ldots, N, \qquad Y^1_0 = a, Y^2_0 = b$$

independent simulations of X^1 and X^2 in $[0, \Delta]$ with step size $\delta = \Delta/N$

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Keep simulating Y^1 and Y^2 until there is a ν such that either

$$Y^1_{\delta\nu} \geq Y^2_{\delta(N-\nu)} \text{ and } Y^1_{\delta(\nu+1)} \leq Y^2_{\delta(N-(\nu+1))}$$

or

$$Y^1_{\delta\nu} \leq Y^2_{\delta(N-\nu)} \text{ and } Y^1_{\delta(\nu+1)} \geq Y^2_{\delta(N-(\nu+1))}$$

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$$B_{\delta i} = \begin{cases} Y^1_{\delta i} & \text{for } i = 0, 1, \ldots, \nu \\ \\ Y^2_{\delta(N-i)} & \text{for } i = \nu + 1, \ldots N \end{cases}$$

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$$E(|Z_t - B_t|) \leq C\delta^\gamma \quad \text{for } t \in (0, 1)$$

Ornstein-Uhlenbeck bridge

$$X_{t_i} = e^{-\theta(t_i - t_{i-1})} X_{t_{i-1}} + W_i, \qquad W_i \sim N\left(0, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta(t_i - t_{i-1})}\right)\right)$$

$i = 1, \ldots, n+1$, $t_0 = 0$, $X_0 = a$

$$Z_{t_i} = X_{t_i} + (b - X_{t_{n+1}}) \frac{e^{\theta t_i} - e^{-\theta t_i}}{e^{\theta t_{n+1}} - e^{-\theta t_{n+1}}}, \quad i = 0, \ldots, n+1$$

$(0, a, t_{n+1}, b)$- O.U. bridge

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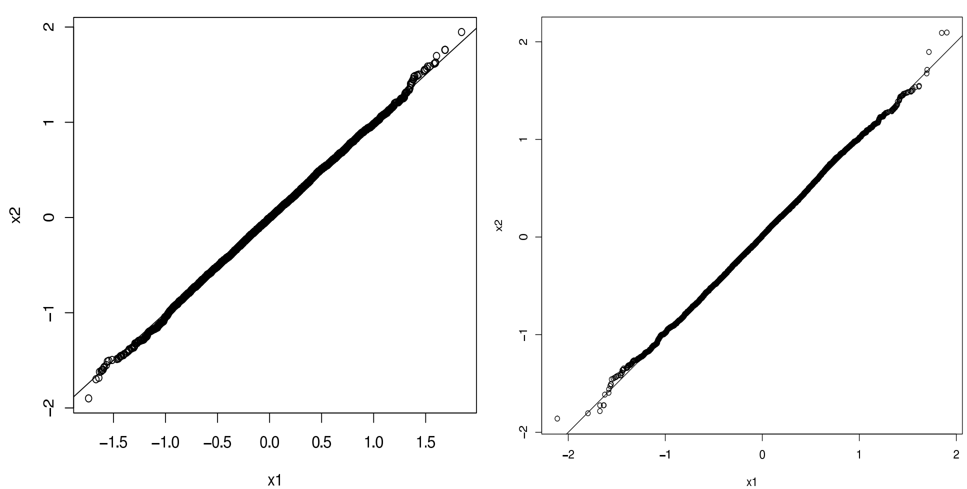
$(0, a, t_{n+1}, b)$- O.U. bridge

$(0,0,1,0)$ and $(0,-2,1,2)$ Ornstein–Uhlenbeck bridges ($\theta = 0.5$, $\sigma = 1.0$)



Ornstein-Uhlenbeck bridge

$a \mapsto b$	CPU (sec.)	rejection prob.	probability of move
$0 \mapsto 0$	0.5	0.17	
$0 \mapsto 1$	0.7	0.41	0.1
$0 \mapsto 2$	1.7	0.77	0.006
$-1 \mapsto 1$	1.9	0.80	0.02
$-1 \mapsto 2$	11.9	0.97	0.0005

Hyperbolic diffusion bridge

$$dX_t = -\frac{X_t}{\sqrt{1+X_t^2}}dt + dW_t$$

Barndorff-Nielsen (1978)

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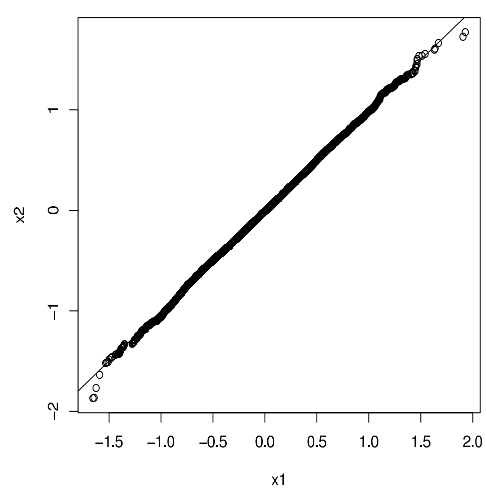
$(0,0,1,0)$-hyperbolic diffusion bridge



Hyperbolic diffusion bridge

$a \mapsto b$	CPU (sec.)	rejection prob.
$0 \mapsto 0$	0.6	0.14
$0 \mapsto 1$	0.8	0.36
$0 \mapsto 2$	2.1	0.77
$-1 \mapsto 1$	2.0	0.76
$-1 \mapsto 2$	12.6	0.96

Stochastic volatility model

$$dX_t = (\beta_1 + \beta_2 V_t)dt + \sqrt{V_t}d\bar{W}_t$$

$$V_t = V_t^{(1)} + V_t^{(2)}$$

$$dV_t^{(i)} = -\alpha_{i1}(V_t^{(i)} - \alpha_{i2})dt + \sigma_i(V_t^{(i)}; \alpha_{i3})dB_t^{(i)}, \quad i = 1, 2.$$

$$\bar{W}_t = \sqrt{1 - \rho_1^2 - \rho_2^2} W_t + \rho_1 B_t^{(1)} + \rho_2 B_t^{(2)}$$

W, $B^{(1)}$ and $B^{(2)}$ independent standard Wiener processes

$|\rho_i| < 1$, $i = 1, 2$, $\alpha_{11} > \alpha_{21} > 0$

Bibby & Sørensen (2003)

Barndorff-Nielsen & Shephard (2001)

Bibby, Skovgaard & Sørensen (2005)

Likelihood conditional on volatility

Strategy follows Chib, Pitt & Shephard (2006)

Data: $X_{t_0}, X_{t_1}, \ldots X_{t_n}$

Conditionally on the volatility process:

$$Y_i = X_{t_i} - X_{t_{i-1}} \sim N\left(\beta_1 \Delta_i + \beta_2 S_i + \rho_1 Z_i^{(1)} + \rho_2 Z_i^{(2)}, (1 - \rho_1^2 - \rho_2^2) S_i\right)$$

$\Delta_i = t_i - t_{i-1}$,

$$S_i = \int_{t_{i-1}}^{t_i} (V_t^{(1)} + V_t^{(2)}) dt$$

$$Z_i^{(j)} = \int_{t_{i-1}}^{t_i} \sqrt{V_t^{(1)} + V_t^{(2)})} dB_t^{(j)}, \quad j = 1, 2$$

Likelihood with volatility observation

Likelihood if we had observed $V^{(1)}$ and $V^{(2)}$ continuously in $[t_0, t_n]$

$$U_t^{(i)} = h_i(V_t^{(i)}; \alpha_{i3})$$

$$h_i(x; \alpha_{i3}) = \int^x \frac{1}{\sigma_i(y; \alpha_{i3})} dy$$

$$dU_t^{(i)} = b_i(U_t^{(i)}; \alpha_{i*})dt + dB_t^{(i)}, \quad i = 1, 2$$

$$b_i(x; \alpha_{i1}, \alpha_{i2}, \alpha_{i3}) = -\alpha_{i1} \frac{h_i^{-1}(x; \alpha_{i3}) - \alpha_{i2}}{\sigma_i(h_i^{-1}(x; \alpha_{i3}); \alpha_{i3})} - \frac{1}{2}\sigma_i'(h_i^{-1}(x; \alpha_{i3}); \alpha_{i3})$$

Likelihood with volatility observation

$$\log L(\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \rho_1, \rho_2) =$$

$$\sum_{i=1}^{n} \log \varphi \left(Y_i; \beta_1 \Delta_i + \beta_2 S_i(\alpha_{*3}) + \rho_1 Z_i^{(1)}(\alpha_{*3}) + \rho_2 Z_i^{(2)}(\alpha_{*3}), (1 - \rho_1^2 - \rho_2^2) S_i(\alpha_{*3}) \right)$$

$$+ \sum_{j=1}^{2} \left(a_j(U_{t_n}^{(j)}; \alpha_{j*}) - a_j(U_{t_0}^{(j)}; \alpha_{j*}) - \frac{1}{2} \int_{t_0}^{t_n} \left(b_j(U_t^{(j)}; \alpha_{j*})^2 + b_j'(U_t^{(j)}; \alpha_{j*} \right) dt \right)$$

$$a_j(x; \alpha_{j*}) = \int^x b_j(u; \alpha_{j*}) du.$$

$$S_i(\alpha_{*3}) = \int_{t_{i-1}}^{t_i} \left[h_i^{-1}(U_t^{(1)}; \alpha_{13}) + h_i^{-1}(U_t^{(2)}; \alpha_{23}) \right] dt$$

EM algorithm

$\theta = (\beta_1, \beta_2, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \rho_1, \rho_2)$

$\tilde{\theta}$ any value of the parameter vector

(1) (E–step) Generate sample paths $V^{(1,k)}$ and $V^{(2,k)}$, $k = 1, \ldots, M$ conditional on $Y_1, \ldots, Y_n$ using the parameter value $\tilde{\theta}$, and calculate

$$g(\theta) = \frac{1}{M - M_0} \sum_{k=M_0+1}^{M} \log L(\theta; Y_1, \ldots, Y_n, (h_1(V_t^{(1,k)}; \tilde{\alpha}_{13}), h_2(V_t^{(2,k)}; \tilde{\alpha}_{23})), t \in [t_0, t_n])$$

(for a suitable burn-in period M_0)

(2) (M–step) $\tilde{\theta} =$ argmax $g(\theta)$

(3) GO TO (1)

Conditional volatility simulation

Chib, Pitt & Shephard (2006)

Simulate two independent unrestricted stationary sample paths of $V^{(1)}$ and $V^{(2)}$ in $[t_0, t_n]$

Repeat the following

1) Randomly draw $\nu_2 < \cdots < \nu_{K+1}$ from the set $\{1, 2, \ldots, n-1\}$ and set $\nu_1 = 0$, $\nu_{K+2} = n$, $\tau_j = t_{\nu_{j+1}}$, $j = 0, \ldots, K+1$

2) In each interval $[\tau_{j-1}, \tau_j]$ update by simulating diffusion bridges conditional on the values of the volatility proceses at the times τ_{j-1} and τ_j obtained in the previous iteration and on $Y_{t_{\nu_j}}, Y_{t_{\nu_j}+1}, \ldots, Y_{t_{\nu_{j+1}}}$

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Step 2): modification of Chib, Pitt & Shephard (2006)

Conditional bridge simulation

(1) Set k = 1. Simulate two independent $(\tau_{j-1}, v_{i0}, \tau_j, v_{i1})$-diffusion bridges, $V^{(i,0)}$, $i = 1, 2$

(2) Propose two new sample paths by sampling two new independent $(\tau_{j-1}, v_{i0}, \tau_j, v_{i1})$-diffusion bridges $V^{(i,k)}$, $i = 1, 2$

(3) Accept the proposed diffusion bridges with probability

$$\min\left(1, \prod_{i=1}^{n_j} \frac{\varphi(Y_{\nu_j+i}; \beta_1 \Delta_{\nu_j+i} + \beta_2 S^{(k)}_{\nu_j+i} + \rho_1 Z^{(1,k)}_{\nu_j+i} + \rho_2 Z^{(2,k)}_{\nu_j+i}, (1-\rho_1^2-\rho_2^2) S^{(k)}_{\nu_j+i})}{\varphi(Y_{\nu_j+i}; \beta_1 \Delta_{\nu_j+i} + \beta_2 S^{(k-1)}_{\nu_j+i} + \rho_1 Z^{(1,k-1)}_{\nu_j+i} + \rho_2 Z^{(2,k-1)}_{\nu_j+i}, (1-\rho_1^2-\rho_2^2) S^{(k-1)}_{\nu_j+i})}\right)$$

where $S_i^{(\ell)}$, $Z^{(1,\ell)}$ and $Z^{(2,\ell)}$ are calculated on the basis of the ℓth simulated pair of volatility sample paths. Otherwise $V^{i,k} = V^{i,k-1}$, $i = 1, 2$

(4) Set $k = k + 1$ and GO TO (2)

Generalized Heston model

$$dX_t = \beta_1 dt + \sqrt{V_t} d\bar{W}_t$$

$$dV_t^{(i)} = -\alpha_{i1}(V_t^{(i)} - \alpha_{i2})dt + \alpha_{i3}\sqrt{V_t^{(i)}} dB_t^{(i)}, \quad i = 1, 2$$

$$\bar{W}_t = \sqrt{1 - 2\rho^2} W_t + \rho(B_t^{(1)} + B_t^{(2)})$$

$\alpha_{i3}^2 = 2\alpha_{i1}\alpha_3$,   ($\alpha_3 > 0$)

The ratio $\zeta = \alpha_{21}/\alpha_{11}$ is known

Exponential family of stochastic processes, Küchler & Sørensen (1997)

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Re-parametrization:

$$\begin{aligned} \gamma_1 &= \beta_1, \;\; \gamma_2 = \rho\sqrt{\alpha_3\alpha_{11}} \;\; \omega = (1 - 2\rho^2)\alpha_3\alpha_{11} \\ \kappa_1 &= \alpha_{11} \;\; \kappa_2 = (\alpha_{12}/\alpha_3 - \tfrac{1}{2}) \;\; \kappa_3 = (\alpha_{22}/\alpha_3 - \tfrac{1}{2}) \end{aligned}$$

$$\gamma = (\gamma_1, \gamma_2)^T, \quad \kappa = (\kappa_1, \kappa_2, \kappa_3)^T$$

Generalized Heston model: the M-step

$$\tilde\gamma = (\bar D^T \bar D)^{-1} \bar D^T \tilde Y \qquad \tilde\omega = \left(\sum_{i=1}^{n} \tilde Y_i^2 - \tilde Y^T \bar D (\bar D^T \bar D)^{-1} \bar D^T \tilde Y \right) / (n-2)$$

$$\tilde Y_i = Y_i T_i, \qquad \tilde Y = (\tilde Y_1, \dots, \tilde Y_n)^T \qquad \bar D = \begin{pmatrix} \Delta_1 T_i & \bar Z_1 \\ \vdots & \vdots \\ \Delta_n T_i & \bar Z_n, \end{pmatrix}$$

$$T_i \quad = \quad \overline{\overline{1/\sqrt{\frac{1}{\tilde\alpha_3 \tilde\alpha_{11}} \int_{t_{i-1}}^{t_i} (V_t^{(1,k)} + V_t^{(2,k)}) dt}}}$$

$$\bar Z_i \quad = \quad \overline{\frac{\int_{t_{i-1}}^{t_i} \sqrt{V_t^{(1,k)} + V_t^{(2,k)}} d(B^{(1,k)} + B^{(2,k)})_t}{\sqrt{\int_{t_{i-1}}^{t_i} (V_t^{(1,k)} + V_t^{(2,k)}) dt}}}$$

$$\overline{H_k} = \frac{1}{M - M_0} \sum_{k=M_0+1}^{M} H_k$$

Generalized Heston model: the M-step

$$\tilde{\kappa} = \bar{M}^{-1}\bar{C}$$

$$\bar{C} = \begin{pmatrix} \frac{1}{4}(t_n - t_0)(1+\zeta) + \frac{1}{2\tilde{\alpha}_3\tilde{\alpha}_{11}}\left(\overline{V_{t_0}^{(1,k)} + V_{t_0}^{(2,k)} - V_{t_n}^{(1,k)} - V_{t_n}^{(2,k)}}\right) \\ \\ \frac{1}{2}\overline{\log\left(\frac{V_{t_n}^{(1,k)}}{V_{t_0}^{(1,k)}}\right)} + \frac{1}{\tilde{\alpha}_3\tilde{\alpha}_{11}}\overline{\int_{t_0}^{t_n} V_s^{(1,k)} ds} \\ \\ \frac{1}{2}\overline{\log\left(\frac{V_{t_n}^{(2,k)}}{V_{t_0}^{(2,k)}}\right)} + \frac{1}{c\tilde{\alpha}_3\tilde{\alpha}_{11}}\overline{\int_{t_0}^{t_n} V_s^{(2,k)} ds} \end{pmatrix}$$

Generalized Heston model: the M-step

$$\bar{M} = \begin{pmatrix} \frac{1}{2\tilde{\alpha}_3\tilde{\alpha}_{11}}\overline{\int_{t_0}^{t_n}\left(V_s^{(1,k)} + V_s^{(2,k)}\right) ds} & -\frac{1}{2}(t_n - t_0) & -\frac{1}{2}(t_n - t_0) \\ -\frac{1}{2}(t_n - t_0) & \frac{\tilde{\alpha}_3\tilde{\alpha}_{11}}{2}\overline{\int_{t_0}^{t_n}\frac{1}{V_s^{(1,k)}} ds} & 0 \\ -\frac{1}{2}(t_n - t_0) & 0 & \frac{\tilde{\alpha}_3\tilde{\alpha}_{11} c}{2}\overline{\int_{t_0}^{t_n}\frac{1}{V_s^{(2,k)}} ds} \end{pmatrix}$$

Markov chain Monte Carlo

Gibbs sampler:

(1) Draw $\tilde{\theta}$ from the prior distribution

(2) Simulate, for the parameter value $\tilde{\theta}$, sample paths of $V^{(1)}$ and $V^{(2)}$ conditionally on the data $Y_1, \ldots, Y_n$ by the methods described previously

(3) Draw $\tilde{\theta}$ from the posterior distribution conditional on $Y_1, \ldots, Y_n$, $V^{(1)}$ and $V^{(2)}$

(4) GO TO (2)

Generalized Heston model: MCMC

Prior:

• γ, ω and κ are independent

• γ 2-dimensional normal distributed with mean γ_0 and covariance matrix Σ (conditional on $\gamma_2 > 0$)

• ω inverse gamma distributed with shape parameter 1 and scale parameter 1

• κ 3-dimensional normal distributed with expectation κ_0 and covariance matrix V (conditional on $\kappa_1 > 0$ and $\kappa_j > -\frac{1}{2}$, $j = 1, 2$)

Generalized Heston model: MCMC

The posterior distribution:

• γ given the full data set and ω is 2-dimensional normal distributed with mean $(\Sigma^{-1} + \omega^{-1} D^T D)(\Sigma^{-1}\gamma_0 + \omega^{-1} D^T D\hat{\gamma}$ and covariance matrix $(\Sigma^{-1} + \omega^{-1} D^T D)^{-1}$ (conditional on $\gamma_2 > 0$)

• ω given the full data set and γ is inverse gamma distributed with shape parameter 1.5 and scale parameter $(1 + \frac{1}{2}(\gamma - \hat{\gamma})^T D^T D(\gamma - \hat{\gamma}))^{-1}$

$$\hat{\gamma} = (D^T D)^{-1} D^T \tilde{Y} \qquad \tilde{Y}_i = Y_i / \sqrt{S_{ik}} \qquad D = \begin{pmatrix} \frac{\Delta_1}{\sqrt{S_1}} & Z_1 \\ \vdots & \vdots \\ \frac{\Delta_n}{\sqrt{S_n}} & Z_n, \end{pmatrix}$$

$$S_{ik} = \frac{1}{\tilde{\alpha}_3 \tilde{\alpha}_{11}} \int_{t_{i-1}}^{t_i} (V_t^{(1,k)} + V_t^{(2,k)}) dt \qquad Z_i = \frac{\int_{t_{i-1}}^{t_i} \sqrt{V_t^{(1)} + V_t^{(2)}} d(B^{(1)} + B^{(2)})_t}{\sqrt{\int_{t_{i-1}}^{t_i} (V_t^{(1)} + V_t^{(2)}) dt}}$$

Generalized Heston model: MCMC

The posterior distribution:

• κ given the full data set 3-dimensional normal distributed with expectation $(V^{-1}+M)^{-1}(V^{-1}\kappa_0+C)$ and covariance matrix $(V^{-1}+M)^{-1}$ (conditional on $\kappa_1>0$ and $\kappa_j>-\frac{1}{2}$, $j=1,2$)

$$C=\begin{pmatrix} \frac{1}{4}(t_n-t_0)(1+c)+\frac{1}{2\tilde{\alpha}_3\tilde{\alpha}_{11}}\left(V_{t_0}^{(1)}+V_{t_0}^{(2)}-V_{t_n}^{(1)}-V_{t_n}^{(2)}\right) \\ \\ \frac{1}{2}\log\left(\frac{V_{t_n}^{(1)}}{V_{t_0}^{(1)}}\right)+\frac{1}{\tilde{\alpha}_3\tilde{\alpha}_{11}}\int_{t_0}^{t_n}V_s^{(1)}ds \\ \\ \frac{1}{2}\log\left(\frac{V_{t_n}^{(2)}}{V_{t_0}^{(2)}}\right)+\frac{1}{c\tilde{\alpha}_3\tilde{\alpha}_{11}}\int_{t_0}^{t_n}V_s^{(2)}ds \end{pmatrix}$$

Generalized Heston model: MCMC

$$M = \begin{pmatrix} \frac{1}{2\tilde{\alpha}_3\tilde{\alpha}_{11}} \int_{t_0}^{t_n} \left(V_s^{(1)} + V_s^{(2)}\right) ds & -\frac{1}{2}(t_n - t_0) & -\frac{1}{2}(t_n - t_0) \\ -\frac{1}{2}(t_n - t_0) & \frac{\tilde{\alpha}_3\tilde{\alpha}_{11}}{2} \int_{t_0}^{t_n} \frac{1}{V_s^{(1)}} ds & 0 \\ -\frac{1}{2}(t_n - t_0) & 0 & \frac{\tilde{\alpha}_3\tilde{\alpha}_{11}c}{2} \int_{t_0}^{t_n} \frac{1}{V_s^{(2)}} ds \end{pmatrix}$$

Likelihood inference for a diffusion process

$$dX_t = b_\alpha(X_t)dt + \sigma_\beta(X_t)dW_t$$

EM-algorithm ((α_0, β_0) initial parameter value)

(1) (E–step) Calculate the function

$$q(\alpha,\beta) \quad = \quad g_{\alpha,\beta}(x_n) - g_{\alpha,\beta}(x_1) - \frac{1}{2}\sum_{i=2}^{n}[h_\beta(x_i) - h_\beta(x_{i-1})]^2/(t_i - t_{i-1})$$

$$- \sum_{i=2}^{n} \log(\sigma_\beta(x_i)) - \frac{1}{2}\sum_{i=2}^{n} E_{Z^{(i,\alpha_0,\beta_0)}}\left(\int_{t_{i-1}}^{t_i}\left[\mu'_{\alpha,\beta}(Y_t^*(\beta,\beta_0)) + \mu_{\alpha,\beta}(Y_t^*(\beta,\beta_0))^2\right]dt\right)$$

(2) (M–step) $(\alpha_0, \beta_0) =$ argmax$_{\alpha,\beta} q(\alpha,\beta)$.

(3) GO TO (1).

Modification of Beskos, Papaspiliopoulos & Roberts (2006)

Likelihood inference for a diffusion process

$$Y_t^*(\beta, \beta_0) =$$

$$Z_t^{(i,\alpha_0,\beta_0)} + \frac{(t_i - t)(h_\beta(x_{i-1}) - h_{\beta_0}(x_{i-1})) + (t - t_{i-1})(h_\beta(x_i) - h_{\beta_0}(x_i))}{t_i - t_{i-1}}$$

for $t_{i-1} \le t \le t_i$, $i = 2, \ldots, n$

$Z_t^{(i,\alpha_0,\beta_0)}$ is the $(t_{i-1}, h_{\beta_0}(x_{i-1}), t_i, h_{\beta_0}(x_i))$-bridge for the with parameter values α_0 and β_0

$Z_t^{(i,\alpha_0,\beta_0)}$, $i = 2, \ldots, n$ are independent

Likelihood inference for a diffusion process

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$$h_\beta(x) = \int_{x^*}^{x} \frac{1}{\sigma_\beta(y)} dy \qquad \mu_{\alpha,\beta}(y) = \frac{b_\alpha(h_\beta^{-1}(y))}{\sigma_\beta(h_\beta^{-1}(y))} - \tfrac{1}{2}\sigma_\beta' \left(h_\beta^{-1}(y)\right)$$

$$g_{\alpha,\beta}(x) = \int_{x^*}^{x} \frac{b_\alpha(z)}{\sigma_\beta^2(z)} dz - \tfrac{1}{2}\log(\sigma_\beta(x))$$