Extreme tails for linear portfolio credit risk models[1]

André Lucas,[2,3] Pieter Klaassen,[2,4]
Peter Spreij[5] and Stefan Straetmans[6]

Abstract

We consider the extreme tail behaviour of the CreditMetrics model for portfolio credit losses. We generalise the model to allow for alternative distributions of the risk factors. We consider two special cases and provide alternative tail approximations. The results reveal that one has to be careful in applying extreme value theory for computing extreme quantiles efficiently. The applicability of extreme value theory in characterising the tail shape very much depends on the exact distributional assumptions for the systematic and idiosyncratic credit risk factors.

1. Introduction

The management of *market* risks by banks and other lending institutions - especially investment banks - has gained in importance in recent years due to growing proprietary trading portfolios on the banks' balance sheets; see, for example, the popularity of the value-at-risk (VaR) concept. However, *credit* risk management is perhaps even more important within the financial sector because it directly relates to a bank's core function of financial intermediation.

Until recently, the bulk of the credit risk literature mainly concentrated on assessing the credit risk of individual exposures in isolation, ie without taking into account the potential for credit quality comovements and defaults; see, for example, Altman (1983), Caouette et al (1998) or the Journal of Banking and Finance (2001, vol 25 (1)) as starting references. More recently a portfolio view on credit losses has emerged by recognising that changes in credit quality tend to comove over the business cycle and that one can diversify part of the credit risk by a clever composition of the loan portfolio across regions, industries and countries. Thus in order to assess the credit risk of a loan portfolio, a bank must not only investigate the creditworthiness of its customers, but also identify the concentration risks and possible comovements of risk factors in the portfolio.

Several approaches have been developed in order to determine the credit loss distribution at the portfolio level; see, for example, *CreditMetrics* by Gupton et al (1997), *CreditRisk+* by Credit Suisse (1997), *PortfolioManager* by KMV (Kealhofer (1995)) or *CreditPortfolio View* by McKinsey (Wilson (1997a,b)). Despite the apparent differences between these approaches, they exhibit a common underlying framework; see Koyluoglu and Hickman (1998) and Gordy (2000). In a recent paper we extended the one-factor *CreditMetrics* approach to allow for general dependencies on and distributions of credit risk factors; see, for example, Lucas et al (2001a). We also introduced a limit law to efficiently approximate loss quantiles for portfolios with a finite number of exposures; see Lucas et

1 We thank seminar participants at EURANDOM (Eindhoven) and risk conferences in Basel and Montreal for useful comments. Correspondence to: alucas@feweb.vu.nl, s.straetmans@berfin.unimaas.nl, pieter.klaassennl@abnamro.com or spreijwins@uva.nl.

2 Department of Finance and Financial Sector Management, Vrije Universiteit, De Boelelaan 1105, NL-1081HV Amsterdam, Netherlands.

3 Tinbergen Institute Amsterdam, Keizersgracht 482, NL-1017EG Amsterdam, Netherlands.

4 ABN AMRO Bank NV, Financial Markets Risk Management (HQ 2035), PO Box 283, NL-1000EA Amsterdam, Netherlands.

5 Korteweg-de Vries Institute, University of Amsterdam, Plantage Muidergracht 24, NL-1018TV Amsterdam, Netherlands.

6 Limburg Institute of Financial Economics (LIFE), Maastricht University, Tongersestraat 53, NL-6200MD Maastricht, Netherlands.

al (2001b) and Finger (1999). This limit law can be used in order to perform analyses of the sensitivity of the credit loss quantiles to changes in the exposure characteristics, such as credit quality, the degree of systematic risk, and the maturity profile.

Suppose, however, that a credit risk manager is also interested in calculating credit loss quantiles for very high confidence levels or, stated differently, for very low tail probabilities *q*. Such tail probabilities may be much smaller than the usual 5% or 1%. These extreme credit loss quantiles may be of interest for the sake of testing certain stress scenarios. An initial way of calculating these quantiles consists in using the closed-form expression of the credit loss limit law from Lucas et al (2001b). However, in order to be able to derive this expression, one has to choose a probability distribution for the latent variable triggering credit migrations and defaults in the *CreditMetrics* setup. It can be shown that the quantile calculations may be quite sensitive to varying this distributional choice for the latent variable.[7] Moreover, in case more than one systematic risk factor is present, analytical techniques may be unavailable. In such cases, the manager has to resort to simulations. If the desired tail probability is extremely small, an unduly large number of simulations might be called for.

In order to circumvent this risk of misspecification, one can also estimate credit loss quantiles by directly focusing upon the distributional tail of portfolio credit losses. It is now generally accepted as a stylised fact that the tail of credit loss distributions behaves fairly different from the tail of a normal distribution. In particular, the portfolio credit losses exhibit more probability mass in the tails than a normal distribution with identical mean and variance. In fact, using the toolkit of *extreme value theory (EVT)*, we have shown in our previous paper that the tail probabilities of portfolio credit losses are polynomially declining to zero whereas a normal distribution has a tail that declines at an exponential rate. Stated differently, extreme portfolio credit losses happen relatively more frequently than one would expect on the basis of a normally distributed random variable. As a result, common rules of thumb for calculating loss quantiles based on the normal paradigm no longer apply. For example, the 99.9% quantile may lie much more than three standard deviations above the distributional mean, which is the number one would expect for the normal distribution.

Distributions with a polynomial tail decay are also called heavy-tailed or fat-tailed distributions. The statistical theory of EVT shows that a wide class of distributional models all display polynomially declining tails. Stated otherwise, if one is only interested in the tail behaviour of an empirical process, one does not need to know the whole distribution. For statistical inference on the extreme quantiles, it is sufficient to know that the stochastic process exhibits heavy tails. Apart from providing statistical derivations of limit laws for sample maxima, EVT also provides various estimators for the rate of tail decay in the case of fat tails, the so-called tail index. Quantile estimators that use these tail index estimates as an input can then easily be formulated.

EVT has become increasingly popular in financial research as a tool for modelling the tail of return distributions with an eye towards calculating risk measures such as value-at-risk (VaR). Exploiting the empirical stylised fact of heavy-tailed financial returns (Mandelbrot (1952)), EVT provides extreme quantile estimates for confidence levels *q* typically beyond the tail of the empirical distribution function ($q < n^{-1}$ with *n* the sample size). Good starting references on applications of EVT in market risk management include Daníelsson and de Vries (2000), Longin (2000), Longin and Solnik (2001), Embrechts et al (1997) and Embrechts (2000). Diebold et al (1998) provide a discussion of pitfalls and opportunities in the use of extreme value analysis in financial risk management.

EVT techniques have reportedly been employed in empirical work to limit the number of simulations needed to reliably estimate far-out quantiles. This is especially relevant if multiple risk factors are present. To our knowledge, however, there are no theoretical papers on the applicability of EVT to estimating credit loss quantiles far out in the distributional tail. In this paper we investigate the accuracy (estimation error) of EVT techniques for credit loss distributions. More specifically we investigate how far one should go into the distributional tail in order to obtain extreme value quantile estimates that are reasonably close to their exact underlying values. The latter quantile values are calculated for two different parametric distributions of the factor model components triggering default: the factors are assumed to be either normally distributed or Student-t distributed. For both cases, we

7 Lucas et al (2001a) consider Gaussian (as in the *CreditMetrics* setup) and non-Gaussian parameterisations for the latent variable and find that minor changes in these distributional assumptions can have large effects on extreme credit loss quantiles.

find that the confidence levels q should be chosen extremely low in order to obtain an acceptable level of estimation risk. This evidence raises doubts over the practical use of extreme value analysis in the field of credit risk management.

The remainder of this article is structured as follows. In Section 2 we briefly review the *CreditMetrics* setup towards deriving the analytic distribution of portfolio credit losses. In Sections 3 and 4 we apply extreme value analysis to the tails of the portfolio credit loss distribution and compare the EVT quantile values with their true underlying counterparts in order to assess estimation risk. Concluding remarks are in Section 5.

2. Theoretical framework

Consider a credit portfolio consisting of n bonds. As we eventually want to focus upon the accuracy of extreme value analysis for estimating credit loss quantiles far into the tail, we keep the model setup relatively stylised to highlight the main issues. In particular, we consider bonds with identical characteristics (equal initial ratings, unit face values (1), equal default probabilities, etc). Moreover, we allow for only two end-of-period states for the bond: defaulted and not defaulted.[8]

In our benchmark setting, each bond j, where $j = 1, ..., n$, is characterised by a latent variable S_j triggering a bond's default. A logical, though not the only, candidate for S_j is the company's "surplus", ie the difference between the market value of assets and that of liabilities. Default occurs when the surplus falls below a threshold s^*. Given our assumption of a uniform default probability for the entire portfolio, s^* does not depend on j. The credit loss on individual exposures j is now given by the indicator variable

$1_{\{S_j < s^*\}}$.

We assume that the company surplus variable S_j obeys the linear factor model

$$S_j = \rho f + \sqrt{1-\rho^2}\varepsilon_j \tag{1}$$

with f and ε_j representing systematic influences (business cycle conditions, stock market fluctuations) and firm-specific shocks, respectively. Non-linear extensions of this model can be found in Lucas et al (2001a). The systematic and idiosyncratic shocks are assumed to follow stationary distributions $f \sim G(\cdot)$ and $\varepsilon_j \overset{iid}{\sim} F(\cdot)$. These distributional assumptions imply that our model encompasses the Gaussian *CreditMetrics* setup.

It can now easily be shown that

$$\begin{aligned} C &= \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} 1(S_j < s^*) \overset{a.s.}{=} P\left[S_j < s^* \,\middle|\, f\right] \qquad (2) \\ &= P\left[\varepsilon_j < \frac{s^* - \rho f}{\sqrt{1-\rho^2}} \,\middle|\, f\right] \\ &= F\left(\frac{s^* - \rho f}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

Notice that equation (2) is equivalent to Theorem 1 in Lucas et al (2001b). By conditioning on the common factor f in (2), one effectively averages out all idiosyncratic risk ε_j just as in the case of linear portfolio theory. Indeed, within the CAPM model only systematic risk persists when the number of assets increases. The limit law in (2) generalises this feature to the non-linear context of credit risk

8 The effects of portfolio heterogeneity on the credit loss distribution and its tail are discussed in Lucas et al (2001a,b).

management. Moreover, it is important to realise that the above limit law holds irrespective of the precise distributional assumptions on f and ε_j.

Knowledge of the limit law's analytic expression enables risk managers to calculate the loss distribution's quantiles for given confidence levels q without the need to resort to simulations. This follows from the following chain of equalities:[9]

$$
\begin{aligned}
P[C > c] &= P\left[F\left(\frac{s^* - \rho f}{\sqrt{1-\rho^2}}\right) > c\right] \\
&= P\left[f < \frac{s^* - \sqrt{1-\rho^2}\,F^{-1}(c)}{\rho}\right] \\
&= G\left(\frac{s^* - \sqrt{1-\rho^2}\,F^{-1}(c)}{\rho}\right) \\
&= q, \qquad (3)
\end{aligned}
$$

which can be rewritten as

$$c = F\left[\frac{s^* - \rho G^{-1}(q)}{\sqrt{1-\rho^2}}\right]. \qquad (4)$$

Clearly the use of the analytic quantile formula (4) requires knowledge of $G(\cdot)$ and $F(\cdot)$. However, a credit risk manager might only be interested in knowing the credit-at-risk for very low values of q for the sake of, for example, stress testing. In the next section we investigate to what extent extreme value analysis might be of use to the credit risk manager in order to calculate these extreme credit risk quantiles.

3. Analysing extreme tails

It has been established previously that portfolio credit losses in (2) exhibit a heavy tail; see Lucas et al (2001a,b) for a formal proof. This property can be expressed analytically as:

$$P(C > c) = (1 - c)^{\alpha} L(1/(1 - c)), \qquad (5)$$

where α is the tail index of C governing the tail decay towards zero and $L(\cdot)$ stands for a slowly varying function, ie $\lim_{x\to\infty} L(tx)/L(t) = 1$, for $t > 0$. Examples are $L(x) = \ln(x)$ and $L(x) = K$ for some constant K. Clearly, the lower the tail index, the more likely extreme credit losses become.

It can be easily shown that there is a direct relation between the tail properties of the factors f and ε_j in (1) and the value of α. For example, if (f, ε_j) are standard normally distributed, then $\alpha = (1 - \rho^2)/\rho^2$. For Student-t distributed risk factors with corresponding degrees of freedom μ and ν for f and ε_j, respectively, we have $\alpha = \mu/\nu$; see Lucas et al (2001a). The tail result for the Gaussian case might appear somewhat counterintuitive at first sight, as normally distributed (thin-tailed) risk factors lead to a portfolio credit loss distribution with a polynomially (ie "fat") tail. However, the result simply reflects that a higher degree of systematic risk ρ implies a stronger domino effect of individual loans defaulting simultaneously in a credit portfolio. This effect makes the tail of the portfolio losses relatively fatter (lower α). The Student-t result leads to the observation that the tails of the credit loss distribution may be very fat if the idiosyncratic risk factor has thinner tails than the systematic risk factor ($\nu > \mu$). This makes economic sense. If f has fatter tails than ε_j, extreme realisations of S_j occur relatively more

[9] Analytic quantile calculations for linear *multifactor* models are more complicated, but there are still advantages over pure simulation in that the number of stochastic variables is reduced significantly by n.

often due to bad realisations of f than bad realisations of ε_j. Consequently, it is much more likely that large portions of the portfolio default simultaneously (due to systematic risk). Because of this clustering effect, extreme realisations of portfolio credit losses also become more likely, resulting in a lower rate of tail decay.

In order to evaluate the accuracy of extreme value analysis for the sake of extreme quantile estimation, we compare exact tail quantiles for specific choices of $F(\cdot)$ and $G(\cdot)$ with tail quantiles calculated by means of (4). The exact analytic quantiles are calculated by means of the quantile formula (4). We consider two specific choices of $F(\cdot)$ and $G(\cdot)$. Our first choice is the standard CreditMetrics model with $F(\cdot)$ and $G(\cdot)$ both standard normal. Second, we also consider a fat-tailed alternative where $F(\cdot)$ and $G(\cdot)$ are Student-t distributions with 3 and 5 degrees of freedom, respectively. The Student-t distributions are rescaled to have unit variance. These numbers for the degrees of freedom parameters are not unreasonable given empirical work on the tail behaviour of stock returns. Moreover, this choice of parameters ensures that the portfolio credit loss density does not diverge towards the edges of its support; see Lucas et al (2001a). We set the value of the asset correlation parameter ρ^2 to 20%, which is the value prescribed for corporate loans in the Basel proposals for the New Capital Accord; see Basel Committee on Banking Supervision (2001).

Table1

ML estimate of α for different tail probabilities *q*

q	**$\mu = \infty, \nu = \infty$**		**$\mu = 5, \nu = 3$**	
	ML	**EVT**	**ML**	**EVT**
10^{-1}	45.1	4	44.2	1.67
10^{-2}	29.8	4	9.3	1.67
10^{-3}	22.7	4	1.9	1.67
10^{-4}	18.7	4	0.9	1.67
10^{-5}	16.1	4	1.1	1.67
10^{-6}	14.2	4	1.3	1.67
10^{-7}	13.0	4	1.4	1.67
10^{-8}	12.0	4	1.5	1.67

Note: The table contains the ML estimate of the tail index α in the Weibull approximation of the tail obtained by minimising Kullback-Leibler distance in the tail, ie conditional on $c > c^*$ with c^* the $(1 - q)$-quantile of the exact credit loss distribution. The model is the CreditMetrics model with a 1% unconditional default probability, Student-t(5) distributed systematic risk factor f, and Student-t(3) distributed idiosyncratic risk factor ε_j. The correlation parameter is $\rho^2 = 20\%$. The EVT column contains the exact (limiting) EVT tail index.

Taking the tail expression in (5) as a point of departure, EVT analysis of the credit loss tail naturally starts by considering a linear (1st) Taylor approximation of the credit loss tail around the upper bound of the distributional support $c = 1$:

$$P(C > c) \approx K \cdot (1 - c)^{\alpha} \tag{6}$$

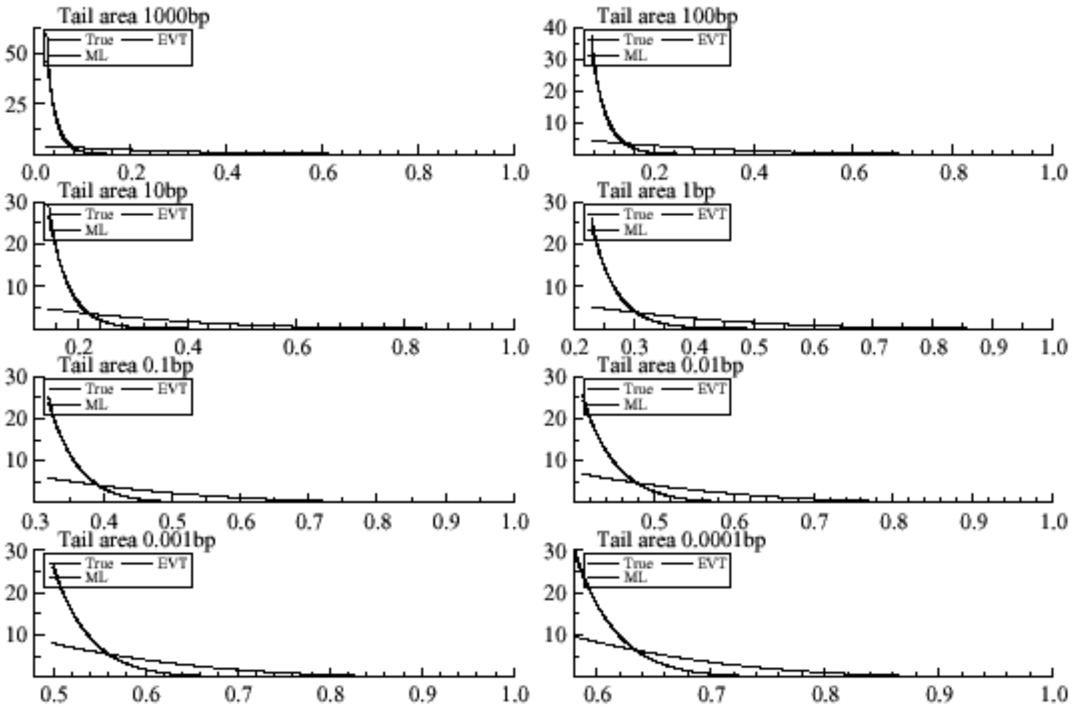
for some constants K and α, and with c close to 1. Thus we assume that the slowly varying function $L(1/(1 - c))$ is approximately constant for large c. First, we calculate extreme tail probabilities using (6) using the exact values of the tail index, ie $\alpha = (1 - \rho^2) / \rho^2 = 4$ for the Gaussian model, and $\alpha = \mu / \nu = 5/3$ for the Student-t model. Second, we estimate α by a Maximum Likelihood (ML) procedure which consists in minimising the Kullback-Leibler distance between (6) and (5) over the range $[c^*,1]$, where c^* is the $(1 - q)$-quantile of the credit loss tail for small values of q.[10]

[10] The Maximum Likelihood (ML) procedure based upon the Kullback-Leibler distance is asymptotically equivalent to applying the Hill (1975) estimator to a set of historical credit losses. This is because the Hill estimator is the ML estimator for a Pareto distribution. Note that $(1 - C)^{-1}$ has a regularly varying, ie a Pareto-type tail. Conditional upon knowledge of α (either the true value or an estimate), the scaling constant K in (6) can easily be calibrated from $\int_{c^*}^{1} K\alpha(1-c)^{\alpha-1}dc = q$ in order to ensure that the probability mass under the approximating pdf equals that under the exact pdf.

Let us now turn to the results for the linear tail approximation case. Table 1 gives the ML estimates of α for decreasing tail probabilities q, ie the lower q, the larger the corresponding tail quantile.

Figure 1

Tail approximations for the Gaussian model



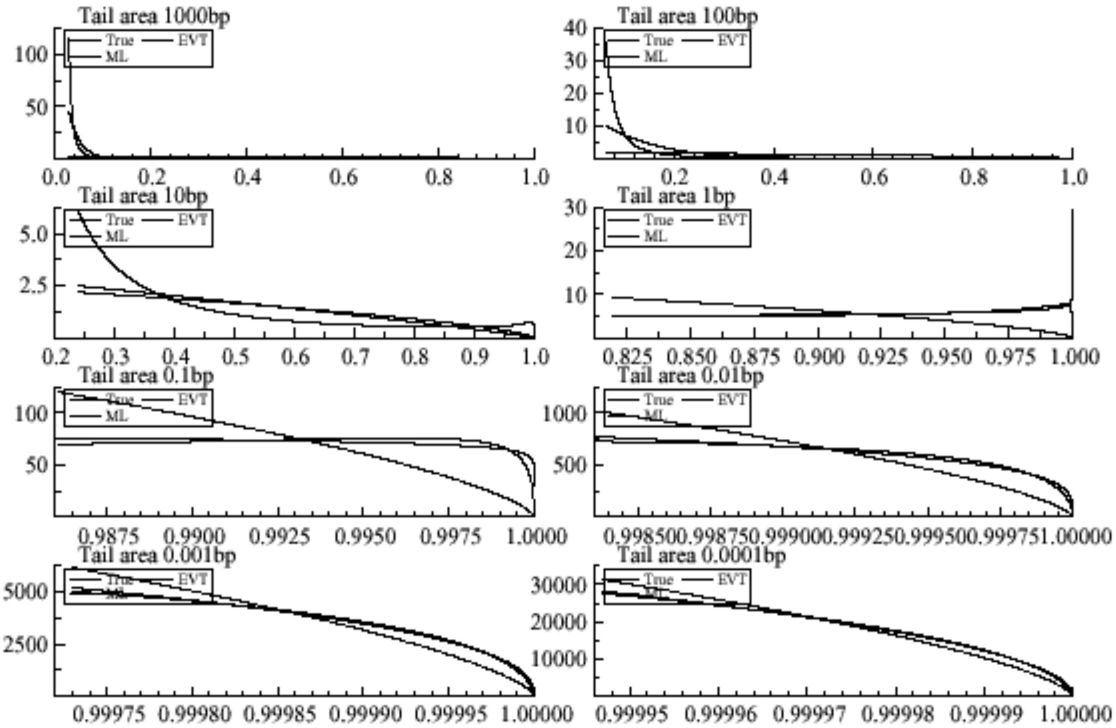
The model is the Gaussian CreditMetrics model with a 1% unconditional default probability. The title gives the tail area over which the credit loss distribution is plotted. EVT is the EVT tail approximation, while ML is the Weibull fit obtained after maximum likelihood or minimum Kullback-Leibler. The correlation parameter is $\rho^2 = 20\%$.

Clearly ML tail index estimates vary considerably with the chosen tail probability. For the Student-t model, the variation is even non-monotonic. It appears that the direction of convergence is ultimately towards its theoretical limit. But even for tail probabilities equal to a basis point of a basis point, the distance between the EVT and the ML α may be substantial, see the Gaussian model. For given values of α, K, and c (close to 1), the corresponding cumulative probabilities and densities can be derived. Conditional tail densities ($h(c|c > c^*)$) for different tail areas are shown in Figures 1 (Gaussian case) and 2 (Student-t case). Each figure contains three density curves: the exact density calculated by using the limit law for portfolio credit losses in (4), and two approximating densities $\hat{h}(c|c > c^*) = K\alpha(1-c)^{\alpha-1}$. The two approximations are the EVT, which uses the exact EVT value for α, and ML, which uses the α that minimises the Kullback-Leibler distance between the approximation $\hat{h}(\cdot)$ and the exact density $h(\cdot)$. For the ML density we impute the tail index estimates from Table 1.

At first sight, the ML fit does remarkably well in approximating the exact density. For the Gaussian models, the ML fit and exact density overlap for all practical purposes for given tail probability q. It may still be the case, however, that for varying q the approximation becomes worse. We investigate this issue in the next section in more detail. The EVT fit appears to approximate the true densities in the extreme tail, meaning that its shape resembles the extreme right-hand part of the exact density in each of the plots. For a given tail probability q, however, the EVT fit over the range $[c^*,1]$ is appallingly bad compared to the ML fit, unless one considers the Student-t case and $q = 10^{-8}$. This means that to recover the exact or limiting EVT tail shape from the exact credit loss density, one has to go *really extremely* far out into the tails. One may wonder whether credit risk managers want to know loss quantiles for $q \leq 10^{-4}$, which appears to be necessary for (exact) EVT to start to work.

Figure 2

Tail approximations for the Student-t model



The model is the CreditMetrics model with a 1% unconditional default probability, Student-t(5) distributed systematic risk factor *f*, and Student-t(3) distributed idiosyncratic risk factor ε_j. The title gives the tail area over which the credit loss distribution is plotted. EVT is the EVT tail approximation, while ML is the Weibull fit obtained after maximum likelihood or minimum Kullback-Leibler. The correlation parameter is ρ^2 = 20%.

Hitherto, we have compared the exact and approximate densities of credit losses on the basis of first-order tail approximation in (6). Note, however, that the linear approximation may be very imprecise because it assumes that the slowly varying function $L(\cdot)$ is approximately constant far out in the tails. As we have shown in previous work, this does not hold for the Gaussian model; see Lucas et al (2001a). This may partly explain the poor fit of the EVT approximation for moderately extreme quantiles. For the Student-t model, we can go even further. There, it can be shown analytically that the EVT fit is very poor for empirically relevant quantiles, but ultimately correct.

From Figure 2, the Student-t model produces a tail such that the conditional (tail) density starts up, goes down, then remains fairly constant over a certain range, and then slowly increases to sharply decline towards zero for *c* very close to the maximum loss 1. The exact EVT fit shows a conditional tail density that starts up and then decreases towards zero for $c \uparrow 1$. This is precisely the shape of the true density. To understand why the EVT tail approximation fits so badly, we consider the tail shape in more detail. In the case of a Student-t(3), the inverse cdf of $F(\cdot)$ can be approximated by

$$F^{-1}(c) \approx \sqrt[3]{\frac{1.1}{1-c}},$$

see Abramowitz and Stegun (1970), equation (26.7.7). As a result, we obtain

$$h(c) = H'(c) = \sqrt{\frac{1-\rho^2}{\rho^2}} \frac{g\left(\frac{s^* - \sqrt{1-\rho^2}F^{-1}(c)}{\rho}\right)}{f(F^{-1}(c))} \tag{7}$$

$$= \sqrt{\frac{1-\rho^2}{\rho^2}}\,\frac{\frac{\Gamma(3)}{\Gamma(2.5)\sqrt{3\pi}}\left[1+\frac{(s^*-\sqrt{1-\rho^2}F^{-1}(c))^2}{3\rho^2}\right]^{-3}}{\frac{\Gamma(2)}{\Gamma(1.5)\sqrt{\pi}}\left[1+(F^{-1}(c))^2\right]^{-2}}, \tag{8}$$

where the Student-t densities are parameterised to have zero mean and unit variance. Using further standard Taylor expansions, we obtain

$$h(c) = K_h \tilde{a}_0^{-3}\left(\frac{1-c}{1.1}\right)^{2/3}\sum_{k=0}^{\infty} c_k \left(\frac{1-c}{1.1}\right)^{k/3} \tag{9}$$

for c near 1, where

$$c_k = d_k + 2d_{k-2} + d_{k-4}, \tag{10}$$

$$d_k = \begin{cases} \sum_{j=\lceil k/2 \rceil}^{k} \binom{j+2}{j}\binom{j}{k-j}\left(-\tilde{a}_0\right)^{-j}\tilde{a}_1^{2j-k}\tilde{a}_2^{k-j} & \text{for } k \geq 0, \\ 0 & \text{for } k < 0, \end{cases}$$

$$\tilde{a}_0 = a_1^2, \tag{11}$$

$$\tilde{a}_1 = 2a_0 a_1, \tag{12}$$

$$\tilde{a}_2 = 1 + a_0^2, \tag{13}$$

$$a_0 = s^* / (\rho\sqrt{3}), \tag{14}$$

$$a_1 = -\sqrt{\frac{1-\rho^2}{3\rho^2}}, \tag{15}$$

$$K_h = \frac{4\sqrt{3(1-\rho^2)}}{9\rho}; \tag{16}$$

see the appendix. It is clear from (9) that near $c = 1$ the density of credit losses indeed has a Weibull expansion with $\alpha - 1 = 2/3$, or $\alpha = 5/3 = \mu/\nu$. The expression for $d_k (k \geq 0)$ is equivalent to

$$d_k = \left(-\tilde{a}_0\tilde{a}_2\right)^{k/2}\binom{2+\frac{1}{2}k}{\frac{1}{2}k}\,{}_2F_1\left(\frac{-k}{2}, 3+\frac{k}{2}; \frac{1}{2}; \frac{\tilde{a}_1^2}{4\tilde{a}_0\tilde{a}_2}\right),$$

where ${}_2F_1(a, b; c; z)$ is a hypergeometric function; see Abramowitz and Stegun (1970), Chapter 15. For $k \to \infty$, $|d_k|$ diverges. However,

$$\lim_{k\to\infty} \frac{\ln|d_k|}{k} = a_2$$

for some constant a_2 that does not depend on c, see equation (17) in Section 2.3.2 of Erdélyi (1953). Note, therefore, that $d_k(1-c)^k$ converges to zero for $(1-c)$ for sufficiently small values of $(1-c)$. A plot of $\ln|c_k|/k$ is given in Figure 3.

It is clear that higher-order terms in (9) than $(1-c)^{2/3}$ will be smaller in magnitude than $K > 0$ if

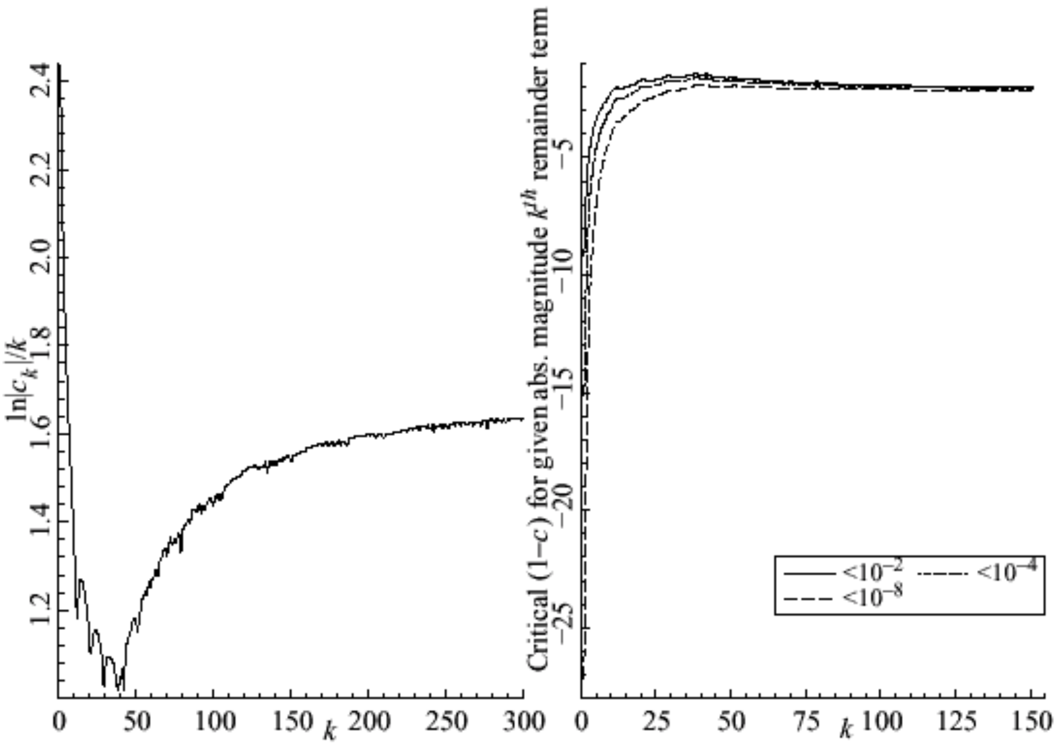
$$|c_k|\left(\frac{1-c}{1.1}\right)^{k/3} < K \Leftrightarrow 1-c < 1.1\left(\frac{K}{|c_k|}\right)^{3/k}.$$

A plot of the critical value of $1-c$ for different values of K is given in the right-hand panel of Figure 3. For example, if $K = 0.01$, we have for $k = 1$ that $(1-c)$ should be smaller than $7\cdot 10^{-10}$, which is about *7% of a basis point of a basis point*. Clearly, the Weibull tail expansion only appears to set in in the *really extreme* tail, and not before. This explains the tail shapes in Figure 2.

The main conclusion we draw from our present computations is that one has to be very careful in applying tail expansions stemming from extreme value theory in the credit risk context. Higher-order terms may be important because they decline to zero very late, like the $(1 - c)^{k/3}$ terms for k = 1,2 in the Student-t case. Moreover, the coefficients of the higher-order term may increase very steeply, also implying that one has to go further into the tails for the terms to become negligible. As a result, the extreme tail may start beyond quantiles of empirical interest. If this is the case, a different method of tail approximation might be called for altogether.

Figure 3

Coefficients $\ln|c_k|/k$ from (9) and following



The left-hand figure contains $\ln|c_k|/k$, where c_k are the tail expansion coefficients from (9) and following. The right-hand plot gives the critical value of $(1 - c)$ for which the kth-order term in the expansion is below K, where K is 10^{-2}, 10^{-4} or 10^{-8}. It is computed as $1.1(K/|c_k|)^{3/k}$.

4. Results

The ML fits in Figures 1 and 2 were reasonable for most tail areas. As this mimics the empirical application of EVT in practice to efficiently approximate a simulated version of $h(c)$, there is still some hope for the practical use of extreme value analysis in credit risk management. The applicability of EVT, however, hinges on the stability of the approximation over decreasing tail probabilities.

Of course, the estimate of α may differ for different tail areas (as shown in Table 1), but the real question is whether the fitted α produces estimates of credit loss quantiles or conditional expected credit losses that are adequate approximations to their true underlying values. To investigate this, we conducted the following experiment. Using the ML estimate of α for a specific tail probability q, we estimate the quantiles ($\hat{c}_1$ and $\hat{c}_2$) and conditional expected losses beyond those quantiles ($\hat{E}_1$ and $\hat{E}_2$) corresponding to tail probabilities of $q/10$ and $q/100$, respectively. We also calculated the percentage deviation (Δ) of the estimates from their true values. The results are in Table 2.

Table 2

Quantiles and expected losses beyond sample

q	c_0	c_1	$\hat{c}_1$	Δ	c_2	$\hat{c}_2$	Δ	E_1	$\hat{E}_1$	Δ	E_2	$\hat{E}_2$	Δ
	ML fit, $\mu = \infty$, $\nu = \infty$												
10^{-1}	0.03	0.08	0.07	–0.02	0.15	0.12	–0.18	0.11	0.09	–0.11	0.18	0.14	–0.24
10^{-2}	0.08	0.15	0.14	–0.01	0.23	0.21	–0.09	0.18	0.17	–0.05	0.27	0.23	–0.13
10^{-3}	0.15	0.23	0.23	–0.01	0.32	0.30	–0.05	0.27	0.26	–0.03	0.36	0.33	–0.08
10^{-4}	0.23	0.32	0.32	–0.00	0.41	0.40	–0.03	0.36	0.35	–0.02	0.45	0.43	–0.05
	EVT fit, $\mu = \infty$, $\nu = \infty$												
10^{-1}	0.03	0.08	0.45	5.00	0.15	0.69	3.75	0.11	0.56	4.34	0.18	0.75	3.15
10^{-2}	0.08	0.15	0.48	2.30	0.23	0.71	2.09	0.18	0.58	2.22	0.27	0.77	1.86
10^{-3}	0.15	0.23	0.52	1.27	0.32	0.73	1.28	0.27	0.62	1.29	0.36	0.78	1.18
10^{-4}	0.23	0.32	0.57	0.77	0.41	0.76	0.84	0.36	0.65	0.82	0.45	0.81	0.79
	ML fit, $\mu = 5$, $\nu = 3$												
10^{-1}	0.03	0.06	0.08	0.20	0.24	0.12	–0.48	0.14	0.10	–0.28	0.46	0.14	–0.69
10^{-2}	0.06	0.24	0.27	0.14	0.82	0.43	–0.48	0.46	0.34	–0.26	0.91	0.49	–0.47
10^{-3}	0.24	0.82	0.77	–0.06	0.99	0.93	–0.05	0.91	0.85	–0.07	0.99	0.96	–0.04
10^{-4}	0.82	0.99	0.99	–0.00	1.00	1.00	0.00	0.99	0.99	0.00	1.00	1.00	0.00
	EVT fit, $\mu = 5$, $\nu = 3$												
10^{-1}	0.03	0.06	0.76	10.70	0.24	0.94	2.95	0.14	0.85	5.26	0.46	0.96	1.09
10^{-2}	0.06	0.24	0.76	2.22	0.82	0.94	0.15	0.46	0.85	0.85	0.91	0.96	0.05
10^{-3}	0.24	0.82	0.81	–0.01	0.99	0.95	–0.03	0.91	0.88	–0.04	0.99	0.97	–0.02
10^{-4}	0.82	0.99	0.95	–0.03	1.00	0.99	–0.01	0.99	0.97	–0.02	1.00	0.99	–0.01

Note: Starting from the true quantile c_0 corresponding to a tail probability of q, we use the estimates of Table 1 to approximate the tail using a Weibull with the ML or EVT fit for α. Next, we compute the true $q/10$ quantile c_1 and the $q/100$ quantile c_2 and compare these with their Weibull approximations $\hat{c}_1$ and $\hat{c}_2$, respectively. We do the same for the conditional expected loss beyond c_1 and c_2 for the true distribution (E_1 and E_2, respectively), and beyond $\hat{c}_1$ and $\hat{c}_2$ for the Weibull approximation ($\hat{E}_1$ and $\hat{E}_2$, respectively). The fraction increase of the fitted/approximated value vis-à-vis the true one is given in the Δ columns.

Let us first consider the Gaussian model and the ML fit. If VaRs, or quantiles, slightly out of sample are estimated and the fit is very good (compare $\hat{c}_1$ with c_1), then the true VaR is underestimated by only 1% or 2%. Further out of sample, however, the approximation works less satisfactorily (compare $\hat{c}_2$ with c_2) and approximation errors increase within a range of 3% to as high as 18%. The approximation works better if the α parameter is estimated further out in the tails, ie for lower values of the tail probability q. A similar picture emerges if we consider expected losses rather than VaRs. The lower q, the better the out-of-sample approximation. Moreover, the approximation becomes worse the further we try to apply it out of sample. Also note that percentage mismatches of expected loss are already significant (11%) for $q = 10^{-1}$ and moderately out of sample ($q/10$). This is due to the fact that the expected loss also takes the goodness of fit of the tail approximation beyond the VaR quantile into account. From the quantiles we already noted that the $q/10$ quantile is approximately correct, but the tail approximation beyond that point becomes increasingly worse (see the $q/100$ quantile c_2 and $\hat{c}_2$). In any case, it is clear from all the Δ columns that the standard empirical application of EVT to the Gaussian model generally leads to an underestimation of the risk involved out of sample.

We now turn to the Student-t model and the ML fit. First, we note that the percentage and absolute approximation errors are much larger in general than for the Gaussian model. Moreover, the VaR moderately out of sample ($q/10$) may be under- or overestimated. The expected loss is underestimated. The same underestimation of risk is apparent if we look further out of sample ($q/100$) to either the VaR or the expected loss. Clearly, in the case of fat-tailed systematic and idiosyncratic risk factors, our results suggest that one should be more cautious in straightforwardly applying EVT approximations in the standard way to increase simulation efficiency and approximate risk measures out of (the simulated) sample.

Finally, we turn to the results for the Weibull approximation based on the EVT fit, ie on the exact rather than estimated tail index. The picture confirms the results from Figures 1 and 2, ie that the EVT fit works less well than the ML one. For the Gaussian model, percentage errors for the EVT fit are considerably higher than for the ML fit. Quantiles and expected losses are all more than 75% off mark. The use of the exact extreme value index α in the Weibull approximation leads to much too conservative (or prudent) estimates of risk. The picture is more subtle for fat-tailed risk factors. In particular, if one goes far out into the tails ($q = 10^{-3}$, 10^{-4}) to estimate the tail index α by ML, the EVT and ML fits produce very similar risk measures, which are both accurate to an error of about 5%. If one does not go far into the tails ($q = 0.1$), the ML fit is much better than the EVT fit for extrapolation purposes (at least up to $q/100$). For the intermediate case, $q = 0.01$, the EVT fit is much more useful if extrapolated far out into the tail ($q/100$; see $\hat{c}_2$ and $\hat{E}_2$). For nearer quantiles (see $\hat{c}_1$ and $\hat{E}_1$) the ML fit is considerably better. So the usefulness of Weibull approximations based on exact extreme value indices compared to ML estimates in the credit risk context very much depends on the tail area the ML estimate is based on and the extent of extrapolation beyond the sample envisaged for the EVT fit. If the tail area considered for ML estimation is large (high q) and one does not need to extrapolate further than $q/100$, then the exact EVT indices are of limited use. Note, however, that the approximations of quantiles and expected losses based on EVT fits improve broadly speaking when applied further out of sample ($q/100$ versus $q/10$). This holds for both the Gaussian and the Student-t models and corresponds to what one would expect. Though better, the approximation may, however, still be too prudent for empirical use.

5. Concluding remarks

The statistical theory of extreme values has been gaining in popularity within the financial research area for quite some time now. Researchers increasingly use tail index and quantile estimators (value-at-risk) in order to assess the tails of return distributions, both for single positions and for fully fledged portfolios. These statistical techniques can also be applied to calculate extreme credit loss quantiles. We investigated in this paper whether the application of extreme value theory (EVT) to the tails of portfolio credit losses is useful for the credit risk manager, ie are estimated EVT quantiles acceptably accurate or is the estimation error too large?

We started the analysis by calculating extreme quantile probabilities using the exact analytic expression of the portfolio credit loss distribution. We derived the loss distribution if the number of portfolio exposures grows large within the traditional *CreditMetrics* framework, ie portfolio exposures default either because of idiosyncratic shocks (ε_j) or because of systematic shocks (f). The analytic expression for the portfolio credit loss distribution for a large number of exposures exists upon knowledge of the distributional parameterisations for these factors. We therefore calculated the analytic credit loss quantiles conditional upon two different parametric choices for f and ε_j: Gaussian and Student-t distributed factors. The analytic portfolio credit loss distribution is heavy-tailed under either of the distributional choices for the underlying factors triggering defaults. As a consequence, we know from EVT that credit loss tail probabilities $P(C > c)$ can be factorised into a Pareto tail $(1 - c)^{\alpha}$ and a slowly varying function. We then considered a linear approximation for this factorisation and calculated extreme value probabilities, conditional upon both true values of the tail index and estimated values.

Upon comparing the analytic tail probabilities with their extreme value counterparts, we found that the extreme value probabilities come close to their true values provided one goes very far into the credit loss tail. Using higher-order expansions, we showed that very far out in the tail may mean, for empirical reasons, moving unrealistically far into the tails for higher-order terms to become negligible. It is doubtful whether credit risk managers would ever be interested in these remote tail areas. We conclude that standard use of EVT methods as applied in, for example, the market risk context is inappropriate in the credit risk context. More care should be taken when using EVT for credit risk management, and possibly a different method of tail approximation might be called for altogether.

Appendix
Proof of (9)

From (8), we obtain

$$h(c) = K_h \frac{\left[1 + \frac{(s^* - \sqrt{1-\rho^2}F^{-1}(c))}{3\rho^2}\right]^{-3}}{\left[1 + (F^{-1}(c))^2\right]^{-2}} = \tag{A1}$$

$$K_h\left[1+t^2\right]^2\left[1 + \frac{(s^* - \sqrt{1-\rho^2}t)^2}{3\rho^2}\right]^{-3} \tag{A2}$$

with $t = (1.1/(1-c))^{1/3}$. Define $y = 1/t$, and use the definitions in (10) to (16), then from (A2)

$$h(c) = K_h\left[1+t^2\right]^2\left[1+(a_0 + a_1 t)^2\right]^{-3} \tag{A3}$$

$$= K_h y^2\left[1+y^2\right]^2\left[\tilde{a}_2 y^2 + \tilde{a}_1 y + \tilde{a}_0\right]^{-3}. \tag{A4}$$

Note that for $y \approx 0$ we have

$$(a+y)^{-3} = a^{-3}\sum_{k=0}^{\infty}(-a)^{-k}(k+2)_k\frac{y^k}{k!} = a^{-3}\sum_{k=0}^{\infty}\binom{k+2}{k}(-a)^{-k}y^k,$$

where $a_n = a\cdot(a-1)\cdots(a-n+1)$ is the Pochammer symbol. Using this result, rewrite

$$h(c) = K_h\frac{y^2}{\tilde{a}_0^3}(1+2y^2+y^4)\sum_{k=0}^{\infty}\binom{k+2}{k}(-\tilde{a}_0)^{-k}(\tilde{a}_2 y^2 + \tilde{a}_1 y)^k \tag{A5}$$

$$= K_h\frac{y^2}{\tilde{a}_0^3}(1+2y^2+y^4)\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{\tilde{a}_1}{-\tilde{a}_0}\right)^k y^k\sum_{j=0}^{k}\binom{k}{j}\left(\frac{\tilde{a}_2}{\tilde{a}_1}\right)^j y^j,$$

or

$$h(c) = K_h\frac{y^2}{\tilde{a}_0^3}(1+2y^2+y^4)\sum_{k=0}^{\infty}d_k y^k, \tag{A6}$$

with

$$d_k = \left(\frac{\tilde{a}_2}{\tilde{a}_1}\right)^k \cdot \sum_{j=\lceil k/2\rceil}^{k}\binom{j+2}{j}\binom{j}{k-j}\left(\frac{\tilde{a}_1^2}{-\tilde{a}_0\tilde{a}_2}\right)^j.$$

Combining all this, we obtain

$$h(c) = K_h\frac{y^2}{\tilde{a}_0^3}\sum_{k=0}^{\infty}(d_k + 2d_{k-2} + d_{k-4})y^k = K_h\frac{y^2}{\tilde{a}_0^3}\sum_{k=0}^{\infty}c_k y^k. \tag{A7}$$

References

Abramowitz, M and I Stegun (1970): *Handbook of Mathematical Functions*, New York: Dover.

Altman, E (1983): "Corporate financial distress. A complete guide to predicting, avoiding, and dealing with bankruptcy", New York: Wiley.

Basel Committee on Banking Supervision (2001): *The New Basel Capital Accord*, technical report, Bank for International Settlements, January.

Caouette, J, E Altman and P Narayanan (1998): "Managing credit risk. The next great financial challenge", New York: Wiley.

Credit Suisse (1997): *CreditRisk+*, downloadable: http://www.csfb.com/creditrisk.

Daníelsson, J and C de Vries (2000): "Value-at-Risk and extreme returns", *Annale d'Economie et de Statistique* 60, 239.

Diebold, F, T Schuermann and J Stroughair (1998): "Pitfalls and opportunities in the use of extreme value theory in risk management", in A-P N Refenes, A Burgess and J Moody (eds), *Decision technologies for computational finance*, 3-12, Deventer: Kluwer.

Embrechts, P (2000): "Extremes and integrated risk management", London: Risk Publications.

Embrechts, P, C Klüppelberg and T Mikosch (1997): "Modelling extremal events", vol 33 of *Applications of mathematics; stochastic modelling and applied probability*, Heidelberg: Springer Verlag.

Erdélyi, A (1953): "Higher transcendental functions", vol 1-2, New York: McGraw-Hill.

Finger, C (1999): "Conditional approaches for CreditMetrics portfolio distributions", *CreditMetrics Monitor* 1, 14-33, April.

Gordy, M (2000): "A comparative anatomy of credit risk models", *Journal of Banking and Finance* 24 (1-2), 119-49.

Gupton, G, C Finger and M Bhatia (1997): *CreditMetrics - Technical Document* (1st edition), http://www.riskmetrics.com.

Hill, B (1975): "A simple general approach to inference about the tail of a distribution", *Annals of Statistics* 3, 1163-73.

Journal of Banking and Finance (2001): vol 25 (1), pp 1-270, January.

Kealhofer, S (1995): "Managing default risk in derivative portfolios", in *Derivative credit risk: advances in measurement and management*, London: Risk Publications.

Koyluoglu, H and A Hickman (1998): "Reconcilable differences", *Risk*, 56-62, October.

Longin, F (2000): "From value at risk to stress testing: the extreme value approach", *Journal of Banking and Finance* 24, 1097-130.

Longin, F and B Solnik (2001): "Extreme correlation of international equity markets", *Journal of Finance* 56, 649-76.

Lucas, A, P Klaassen, P Spreij and S Straetmans (2001a): "Tail behaviour of credit loss distributions for general latent factor models", TI Discussion Paper, 01-023/2, Tinbergen Institute.

——— (2001b): "An analytic approach to credit risk of large corporate bond and loan portfolios", *Journal of Banking and Finance* 25, 1635-64.

Mandelbrot, B (1952): "The variation of certain speculative prices", *Journal of Business* 36, 394-440.

Wilson, T (1997a): "Portfolio credit risk, part I", *Risk*, 111-17, September.

——— (1997b): "Portfolio credit risk, part II", *Risk*, 56-61, October.