Measure theory and stochastic processes
Additional exercises
1. Let $A = \{A_1, A_2, A_3\}$ be non-empty sets that form a partition of a set $\Omega$. Write down all elements of $\sigma(A)$. Let $B_1, B_2$ be two subsets of $\Omega$ such that $B_1 \cap B_2$ and $(B_1 \cup B_2)^c$ are non-empty. Write down all elements of $\sigma(\{B_1, B_2\})$.

2. Let $\Omega$ be a nonempty set and let for each $i$ in some (index) set $I$ $F_i$ be a $\sigma$-algebra on $\Omega$. Let $C$ be some collection of subsets of $\Omega$. In alternative wordings compared to Section A.2, but in content the same, we define $\sigma(C)$ to be the smallest $\sigma$-algebra that contains $C$, i.e. the intersection of all $\sigma$-algebras that contain $C$.

(a) Show that $\bigcap_{i \in I} F_i$ (the intersection of all $\sigma$-algebras $F_i$) is a $\sigma$-algebra.

(b) Why is there is at least one $\sigma$-algebra that contains $C$?

(c) Here we take $\Omega = \mathbb{R}$. Argue that $B(\mathbb{R})$ is equal to $\sigma(C)$, where $C = \{(-\infty, a], a \in \mathbb{R}\}$.

(d) Consider a function $X : \Omega \to \mathbb{R}$. Let $C$ be a collection of subsets of $\mathbb{R}$ that is such that $\sigma(C) = B(\mathbb{R})$. Suppose that all sets $\{X \in C\}$ (for $C \in C$) belong to a $\sigma$-algebra $F$ on $\Omega$. Show that $X$ is a random variable (Definition 1.1.5).

(e) Suppose that for all $a \in \mathbb{R}$ the set $\{X = a\}$ belongs to a $\sigma$-algebra $F$ on $\Omega$. Show that $X$ is a random variable.

(f) Suppose that for all $a \in \mathbb{R}$ the set $\{X < a\}$ belongs to a $\sigma$-algebra $F$ on $\Omega$. Is $X$ a random variable?

3. Let $\mu_X$ be the distribution of a random variable $X$, see Definition 1.2.3. Show that $\mu_X$ is a probability measure on the Borel sets of $\mathbb{R}$.

4. Assume that the random variable $X$ takes on the different values $x_0, x_1, \ldots$ in $\mathbb{R}$ and that $\mathbb{E}|X| < \infty$. Show that $\mathbb{E}X = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k)$. Special case: $X$ is such that $\mathbb{P}(X = k) = \frac{e^{-\lambda} \lambda^k}{2 \pi k!}$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\mathbb{P}(X = 0) = e^{-\lambda}$. What is $\mathbb{E}X$?

5. Consider the setting of Theorem 1.6.1. Show that $\tilde{P}$ and $\mathbb{P}$ are equivalent iff $\mathbb{P}(Z > 0) = 1$.

6. Show that $\sigma(X)$ as defined in Definition 2.1.3 is indeed a $\sigma$-algebra and that $\sigma(X) \subset F$ if $X$ is a random variable on $(\Omega, F, \mathbb{P})$ (there is almost nothing to prove).

7. Let $X$ be a nonnegative random variable. Show that $\int X d\mathbb{P} \geq \frac{\mathbb{P}(X > 1/n)}{n}$.

8. Show (use the previous exercise) that the Radon-Nikodym derivative $Z$ of Theorem 1.6.7 satisfies $Z \geq 0$ $\mathbb{P}$-a.s. (integrate over the set $\{Z < 0\}$). Use the equivalence of $\tilde{P}$ and $\mathbb{P}$ to show that even $Z > 0$ $\mathbb{P}$-a.s. Show also that for a possibly different $Z'$ satisfying the assertions of Theorem 1.6.7 one has that $\mathbb{P}(Z > Z') = 0$ and therefore $\mathbb{P}(Z = Z') = 1$. 1
9. Show (use the previous exercise) that the random variable \( Y \) of Theorem B.1 is a.s. nonnegative. Alternative, you can modify the proof of Theorem B.1 with the integrand \( \frac{X+a}{\varepsilon X+a} \) for arbitrary rational \( a > 0 \) instead of \( \frac{X+1}{\varepsilon X+1} \). This yields the existence of \( G \)-measurable random variables \( Y_a \). Show that they are a.s. all the same. So we can define an a.s. limit of them, \( Y \) say. Show that it follows that \( Y \geq 0 \) a.s.

10. Let \( \Pi = \{ A_1, \ldots, A_n \} \) be a partition of \( \Omega \), i.e. the \( A_i \) are non-empty, \( A_i \cap A_j = \emptyset \) for \( i \neq j \) and \( A_1 \cup \cdots \cup A_n = \Omega \). Let \( G = \sigma(\Pi) \) and \( X : \Omega \to \mathbb{R} \). Show that \( X \) is constant on each \( A_i \) iff \( X \) is \( G \)-measurable. If \( X \) is constant on the whole set \( \Omega \), what is \( \sigma(X) \)?

11. Let \( (Z_t)_{t \geq 0} \) be a sequence of independent random variables, also independent of another random variable \( X_0 \). Assume that the following recursion hold for some ‘good’ measurable functions.

\[
X_{t+1} = f(X_t, Z_t), \quad t \geq 0.
\]

Find a filtration to which the sequence \( (X_t) \) is adapted and that \( (X_t) \) is a (discrete time) Markov process (w.r.t. this filtration).

12. Use moment generating functions to show that \( W(u) - W(t) \) and \( W(t) - W(s) \) are independent random variables if \( s < t < u \) (of course \( W \) is a standard Brownian motion). Compute also the conditional MGF \( E[\exp(uW(t))|\mathcal{F}(s)] \) for \( s < t \), where \( \{\mathcal{F}(s)\}_{s \geq 0} \) is a filtration for the Brownian motion. What is the conditional distribution of \( W(t) \) given \( \mathcal{F}(s) \)?