Financial Stochastics 2002
Exercises

version March 19, 2002

Week 1

1. Prove the strong law of large numbers for Brownian motion: $\frac{W_t}{t} \to 0$ a.s. *Hint:* Use Doob's maximal inequality with $p = 2$ to deduce that $\mathbb{E}\sup_{t_0<t<t_1} \frac{W_t^2}{t^2} \leq 4\frac{t_1}{t_0^2}$ and then $\mathbb{P}(\sup_{2^n<t<2^{n+1}} \frac{|W_t|}{t} > \varepsilon) \leq \frac{8}{\varepsilon^2 2^n}$. Finish the proof by application of the Borel-Cantelli lemma.

2. Show that for Brownian motion (according to definition 2.1) it holds that $W_t - W_s$ is independent of $\mathcal{F}_s^W$ for all $t \geq s$. *Hint:* Let $\mathcal{C}$ be the union of all σ-algebras of the form $\sigma(W_{s_1}, \dots, W_{s_n})$, where $0 \leq s_1 < \dots < s_n \leq s$. Then $\mathcal{C}$ is closed under finite intersections. Let $\mathcal{D}$ be the sets in $\mathcal{F}_s^W$ that are independent of $W_t - W_s$. Then $\mathcal{D}$ is a d-system that contains $\mathcal{C}$.

3. Let T be a stopping time that is a.s. finite. Define for all $k \in \mathbb{N}$ the random variables T_k by $T_k = 2^{-k}[2^k T + 1]$ ($[\cdot]$ means integer part). Show that the T_k are stopping times as well and that $T_k \downarrow T$ a.s.

4. Use the reflection principle to show that $\mathbb{P}(\sup_{0\leq s\leq t} W_s > a) = 2\mathbb{P}(W_t > a)$ for all $a > 0$.

5. Show that for two stopping times S and T the identity $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$ holds.

6. Let T be a finite stopping time and X an adapted cadlag process. Show that X_T is $\mathcal{F}_T$-measurable, if the underlying filtration contains all null sets of $\mathcal{F}$.

Week 2

1. Show that every constant map $T : \Omega \to [0, \infty)$, $T = t_0$ say for some t_0, is a stopping time and that $\mathcal{F}_T = \mathcal{F}_{t_0}$.
   Show also that for a set $F \in \mathcal{F}_t$ the random variable $t \cdot 1_F + \infty \cdot 1_{F^c}$ is a stopping time.

2. Show that a map $T : \Omega \to [0, \infty]$ is a stopping time w.r.t. $\{\mathcal{F}_{t+}\}_{t \geq 0}$ iff $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

3. Prove that for closed sets Γ and entirely continuous adapted processes X the random variables $D_\Gamma = \inf\{t > 0 : X_t \in \Gamma\}$ are stopping times.

4. Show that for a right continuous adapted process X, a stopping time T and a right continuous filtration we have that X_T is $\mathcal{F}_T$-measurable. *Hint:* Do this first for stopping times that take values in a countable set.

5. Show that a nonnegative local martingale M is a supermartingale. *Hint:* Use 'Fatou'.

6. Show that a martingale is a local martingale.

7. If T is s stopping time, then the process $1_{(0,T]}$ is predictable. Show this, and deduce that all simple processes are predictable.

8. Show that for a martingale $M \in \mathcal{M}_0^2$ and a simple process H the stochastic integral $H \bullet M$ belongs to $\mathcal{M}_0^2$ as well and that $||H||_{[M]} = ||H \bullet M||_2$. Prove that the last equality also holds for $H \in \mathcal{L}^2(M)$.

9. Show that $L^2(M)$ is a Hilbert space.

Week 3

1. Let $M \in c\mathcal{M}_0^2$. Show that the map $I_M : L^2(M) \to c\mathcal{M}_0^2$ defined by $I_M(H) = H \bullet M$ is linear.

2. Let T be a stopping time, $M \in c\mathcal{M}_0^2$ and $H \in L^2(M)$. Show that $(H \bullet M)^T = H1_{(0,T]} \bullet M$.

3. Let $M \in c\mathcal{M}_{0,\mathrm{loc}}$ and $H \in \Pi(M)$. Show that the definition of $H \bullet M$ via stopping times is independent of the chosen sequence of stopping times.

4. We proved for a bounded $X \in c\mathcal{S}$ the validity of the formula

$$X_t^2 = X_0^2 + 2\int_0^t X_u \, dX_u + [M]_t.$$

Show that it also holds for arbitrary $X \in c\mathcal{S}$. Deduce now the (stochastic) product rule (integration by parts formula) for $X, Y \in c\mathcal{S}$.

5. Let $M \in c\mathcal{M}_{0,\mathrm{loc}}$ and $Z = Z_0 \exp(M - \frac{1}{2}[M])$, for some $Z_0 \in \mathcal{F}_0$. Show the representation

$$Z_t = Z_0 + \int_0^t Z_s \, dM_s. \tag{1}$$

Let $Y = \frac{1}{Z}$. Show that $dY = -Y\,dM + Y\,d[M]$. Let $\tilde{Z}$ be any process that can be represented as in equation (1). Use the product rule to show that $\tilde{Z}Y = 1$, and hence that Z is the only 'solution' to (1).

6. (a) A process X is called progressive if for all $t > 0$ the maps $X^t : [0,t] \times \Omega \to \mathbb{R}$ defined by $X^t(s,\omega) = X(s,\omega)$ are $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable. Show that a progressive process is adapted.

   (b) Let X be an entirely right continuous adapted process and let $t_k^n = kt2^{-n}$. Define the processes $X^{(n)}$ by

$$X^{(n)}(s,\omega) = \sum_k X(t_k^n, \omega) 1_{(t_{k-1}^n, t_k^n]}(s).$$

   Show that $X^{(n)}(s,\omega) \to X(s,\omega)$ for all $(s,\omega) \in [0,t] \times \Omega$ and that $(s,\omega) \mapsto X^{(n)}(s,\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable for all n. Deduce that X is progressive.

7. Let X be a progressive process and T be a stopping time. Show (use that a composition of measurable maps is measurable itself) that the stopped process X^T is also progressive. Deduce that X_T is $\mathcal{F}_T$-measurable. We follow the notation and conventions of HK.

Week 7

In all exercises of this week the basic setup is on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that is assumed to support a d-dimensional Brownian motion which, together with the $\mathbb{P}$-null sets of $\mathcal{F}_0$, generates the filtration $\mathbb{F}$. All semimartingales are supposed to be (at least) defined on this space.

1. Let Z be a positive continuous semimartingale with $\mathbb{E}_{\mathbb{P}}|Z_t| < \infty$ for all t and define the pre-term structure by

   $$D_{tT} = \frac{1}{Z_t}\mathbb{E}_{\mathbb{P}}[Z_T|\mathcal{F}_t].$$

   Show that with $\tilde{Z}_t = \mathbb{E}_{\mathbb{P}}[Z_t|\mathcal{F}_t^A]$ we get a *pricing kernel* term structure model for the collection of D_{tT}.

2. Show that the *unit-rolling numeraire* defined by

   $$N_t^u = \frac{D_{t\lfloor t+1 \rfloor}}{\prod_{i=1}^{\lfloor t+1 \rfloor} D_{i-1,i}}$$

   is indeed a numeraire. Find the corresponding self-financing portfolio and show that it is predictable.

3. Suppose that $\mathbb{Q}$ is a probability measure that is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ $(T > 0)$ with density process z, a positive $(\mathbb{F}, \mathbb{P})$ martingale of the form $z = \mathcal{E}(\mu)$, where μ is some $(\mathbb{F}, \mathbb{P})$-local martingale. We know that $\tilde{W} := W - [W, \mu]$ is Brownian motion under $\mathbb{Q}$. Let M be an $(\mathbb{F}, \mathbb{Q})$ martingale with $M_0 = 0$. Show that the martingale representation holds for M, i.e. there exists a predictable process H such that for all $t \leq T$

   $$M_t = \int_0^t H_s \, d\tilde{W}_s.$$

   Hint: Write $M_t = \frac{\mathbb{E}_{\mathbb{P}}[M_T z_T|\mathcal{F}_t]}{z_t}$. Apply the Itô-rule and the Martingale representation theorem for $(\mathbb{F}, \mathbb{P})$ (local) martingales.
   Note: This result is used in example 8.11.

4. Consider the setting of theorem 8.18. Show that this generates a *short rate* model. So you have to show that $r_t = -\lim_{h \downarrow 0} \frac{1}{h} \log D_{t,t+h}$ (a.s.) and that $D_{tT} = \mathbb{E}_{\mathbb{Z}}[\exp(-\int_t^T r_s \, ds)|\mathcal{F}_t]$.
   Let for each T the process $m_{\cdot T}$ be the martingale (under $\mathbb{Z}$) defined by $m_{tT} = \mathbb{E}_{\mathbb{Z}}[\zeta_T|\mathcal{F}_t]$ $(t \leq T)$. Show that $D_{\cdot T}$ is a submartingale that satisfies

   $$D_{tT} = D_{0T} + \int_0^t r_s D_{sT} \, ds + \int_0^t \frac{1}{\zeta_s} \, dm_{sT}.$$

   Why do we call $\mathbb{Z}$ the risk-neutral measure?

Week 8

In all exercises of this week the basic setup is on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that is assumed to support a d-dimensional Brownian motion which, together with the $\mathbb{P}$-null sets of $\mathcal{F}_0$, generates the filtration $\mathbb{F}$. All semimartingales are supposed to be (at least) defined on this space.

1. Show that, in the context of theorem 8.21, the forward rates satisfy

$$df_{tT} = -\sigma_{tT}\Sigma_{tT}\, dt + \sigma_{tT}\, dW_t.$$

2. Let a and b be nonnegative measurable functions with $\int_0^\infty a_s\, ds < \infty$ and $\int_0^\infty b_s\, ds < \infty$. Consider the Flesaker-Hughston term structure given by

$$D_{tT} = \frac{A_T + B_T M_t}{A_t + B_t M_t}$$

with $A_t = \int_t^\infty a_s\, ds$, $B_t = \int_t^\infty b_s\, ds$ and M a nonnegative positive martingale under some measure $\mathbb{Z}$ (locally equivalent to $\mathbb{P}$) with $M_0 = 1$.

   (a) Show that the above defines a pricing kernel term structure.

   (b) Investigate whether it is possible (by a change of measure) to get a Finite Variation Kernel term structure.

   (c) Investigate (add assumptions where necessary) whether we can cast this model as a Heath-Jarrow-Morton term structure.

3. Consider the Vasicek-Hull-White model with constant coefficients, so for the short rate r we have

$$dr_t = (\theta - ar_t)\, dt + \sigma\, dW_t,$$

where W is a Brownian motion under the risk neutral measure $\mathbb{Q}$. Compute the (non-random) constants A_{tT} and B_{tT} such that

$$D_{tT} = A_{tT} e^{-B_{tT} r_t}.$$

4. Consider the Vasicek-Hull-White model:

$$dr_t = (\theta - ar_t)\, dt + \sigma\, dW_t,$$

where W is Brownian motion (under the risk neutral measure). Show the equality

$$D_{tT} = \frac{D_{0T}}{D_{0t}} \exp(B_{tT} f_{0t} - \frac{\sigma^2}{4a} B_{tT}^2 (1 - e^{-2at}) - B_{tT} r_t,$$

where the B_{tT} are as in exercise 3 and where f_{0t} is the forward rate over the interval $[0, t]$. (Notice that this expression doesn't explicitly depend on θ, so that we may replace the contant θ in the SDE for r by a function without changing the resulting formula for D_{tT}).

5. Consider the Cox-Ingersoll-Ross model with constant coefficients, so for the short rate r we have

$$dr_t = (\theta - ar_t)\,dt + \sigma\sqrt{r_t}\,dW_t,$$

where W is a Brownian motion under the risk neutral measure $\mathbb{Q}$. Show that there exist (non-random) constants A_{tT} and B_{tT} such that

$$D_{tT} = A_{tT}e^{-B_{tT}r_t}.$$

(You may want to determine these constants, in which case you have to perform some tedious computations).

6. Consider the Vasicek-Hull-White model with constant coefficients, so for the short rate r we have

$$dr_t = (\theta - ar_t)\,dt + \sigma\,dW_t,$$

where W is a Brownian motion under the risk neutral measure $\mathbb{Q}$. Compute the coefficients in the Heath-Jarrow-Morton description of the associated forward rates.

7. (*A version of the Stochastic Fubini theorem*) Let $T > 0$ and for each $s \in [0,T]$ the process $H_{\cdot s}$ be adapted and assume that $(s,t) \mapsto H_{ts}(\omega)$ is continuous for all ω. Assume also that $\mathbb{E}\int_0^T\int_0^T H_{ts}\,ds\,dt < \infty$. Put for $s,t \in [0,T]$

$$X_s = \int_0^T H_{ts}\,dW_t$$

and

$$Y_t = \int_0^T H_{ts}\,ds.$$

   (a) Show that X_s and Y_t are well defined.

   (b) Let $0 = t_0 < \cdots < t_n = T$ and put $H^n_{ts} = \sum_{i=1}^n 1_{(t_{i-1},t_i]}(t)H_{t_{i-1}s}$ Show that

$$\int_0^T X^n_s\,ds = \int_0^T Y^n_t\,dW_t$$

   with $X^n_s = \int_0^T H^n_{ts}\,dW_t$ and $Y^n_t = \int_0^T H^n_{ts}\,ds$.

   (c) Show that $\int_0^T X_s\,ds = \int_0^T Y_t\,dW_t$.