Continuous Time Finance

Black-Scholes

(Ch 6-7)

Tomas Björk

Start of lecture 1, 2025

slides 42-97.

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1.

Introduction

European Call Option

The holder of this paper has the <u>right</u> , not the obligation

to buy

1 ACME INC

on the date

2026

June 30, ~~2017~~

(or any other future date)

at the price

$100

Financial Derivative

- A financial asset which is defined **in terms of** some **underlying** asset.

- Future stochastic claim.

Examples

- European calls and puts
- American options
- Forward rate agreements
- Convertibles
- Futures
- Bond options
- Caps & Floors
- Interest rate swaps
- CDO:s
- CDS:s

Main problems

- What is a "reasonable" price for a derivative?
- How do you hedge yourself against a derivative.

Natural Answers

Consider a random cash payment $\mathcal{Z}$ at time T.

What is a reasonable price $\Pi_0[\mathcal{Z}]$ at time 0?

Natural answers: (possibly incorrect)

1. Price = Discounted present value of future payouts.

$$\Pi_0[\mathcal{Z}] = e^{-rT} E[\mathcal{Z}]$$

interest rate is r !

2. The question is meaningless.

Both answers are incorrect!

- Given some assumptions we **can** really talk about "the correct price" of an option.
- The correct pricing formula is **not** the one on the previous slide.

Philosophy

- The derivative is **defined in terms of** underlying.
- The derivative can be **priced in terms of** underlying price.
- **Consistent** pricing.
- **Relative** pricing.

Before we can go on further we need some simple portfolio theory

2.

Portfolio Theory

Portfolios

We consider a market with N assets.

$$S_t^i = \text{price at } t\text{, of asset No } i. \qquad i = 1, \dots, N$$

A **portfolio** strategy is an adapted vector process

$$h_t = (h_t^1, \cdots, h_t^N)$$

where

$$\begin{aligned} h_t^i &= \text{number of units of asset } i, \\ V_t &= \text{market value of the portfolio} \end{aligned}$$

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

The portfolio is typically of the form

$$h_t = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

(sometimes also on prices from the past)

Self financing portfolios

We want to study self financing portfolio strategies, i.e. portfolios where purchase of a "new" asset must be financed through sale of an "old" asset.

How is this formalized?

Definition:
The strategy h is **self financing** if

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

Interpret!

See Appendix B for details. (p. 95)
and motivation from discrete time

Accept this definition for the time being.

Relative weights

Definition:

ω_t^i = relative portfolio weight on asset No i.

We have

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t}$$

and then $h_t^i = w_t^i \frac{V_t}{S_t^i}$.

Insert this into the self financing condition

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

We obtain

Portfolio dynamics:

equivalent to

$$dV_t = V_t \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}, \quad \frac{dV_t}{V_t} = \sum w_t^i \frac{dS_t^i}{S_t}$$

Interpret!

(also p. 94)

3.

Deriving the Black-Scholes PDE

Back to Financial Derivatives

Consider the Black-Scholes model

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

$B_t = e^{rt} B_0$, usually $B_0 = 1$ (normalization)

~~b~~bank account
r = interest rate

We want to price a European call with strike price K and exercise time T. This is a stochastic claim on the future. The future pay-out (at T) is a stochastic variable, $\mathcal{Z}$, given by

$$\mathcal{Z} = \max[S_T - K, 0],$$

$= (S_T - K)^+$, in different notation.

More general:

$$\mathcal{Z} = \Phi(S_T)$$

for some contract function Φ.

Main problem: What is a "reasonable" price, $\Pi_t[\mathcal{Z}]$, for $\mathcal{Z}$ at t?

Main Idea

- We demand **consistent** pricing between derivative and underlying.
- No **mispricing** between derivative and underlying.
- No **arbitrage possibilities** on the market (B, S, Π)

i.e., a viable market

Arbitrage

The portfolio ω is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T > 0$ with probability one.

(or, weaker, $P(V_T \geq 0) = 1$, and $P(V_T > 0) > 0$)

See p. 172

Moral:

- **Arbitrage = Free Lunch**
- **No arbitrage possibilities in an efficient market.**

arbitrage possibility only in a market with "wrong" prices

Arbitrage test

(fundamental idea)

Suppose that a portfolio ω is self financing whith dynamics

$$dV_t = kV_t dt$$

- No driving Wiener process
- Risk free rate of return.
- "Synthetic bank" with rate of return k.

If the market is free of arbitrage we must have:

$$k = r ,$$

because, otherwise, ...

Remark: k and r may depend on t, but still nonrandom. Then we get $k_t = r_t$.

Main Idea of Black-Scholes

- Since the derivative is defined in terms of the underlying, the derivative price should be highly correlated with the underlying price.
- We should be able to balance dervative against underlying in our portfolio, so as to cancel the randomness.
- Thus we will obtain a riskless rate of return k on our portfolio.
- Absence of arbitrage must imply

$$k = r \quad (\text{or } k_t = r_t)$$

End of lecture 1a

Two Approaches

The program above can be formally carried out in two slightly different ways:

- The way Black-Scholes did it in the original paper. This leads to some logical problems. (?)

- A more conceptually satisfying way, first presented by Merton.

Here we use the Merton method. You will find the original BS method in Appendix C at the end of this lecture. [p. 95]

Formalized program a la Merton (outline)

- Assume that the derivative price is of the form

$$\Pi_t[\mathcal{Z}] = f(t, S_t).$$

self financing

- Form a portfolio based on the underlying S and the derivative f, with portfolio dynamics

$$dV_t = V_t \left\{ \omega_t^S \cdot \frac{dS_t}{S_t} + \omega_t^f \cdot \frac{df}{f} \right\}$$

relative weights

see p.55 for the definition of the general case

- Choose ω^S and ω^f such that the dW-term is wiped out. This gives us

$$dV_t = V_t \cdot k dt$$

- Absence of arbitrage implies

$$k = r$$

- This relation will say something about f.

Back to Black-Scholes

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ \Pi_t[\mathcal{Z}] &= f(t, S_t) \end{aligned}$$

Itô's formula gives us the f dynamics as

← from dS_t

$$\begin{aligned} df &= \left\{ \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\ &+ \sigma S \frac{\partial f}{\partial s} dW \end{aligned}$$

Write this as

$$df = \mu_f \cdot f dt + \sigma_f \cdot f dW$$

where

$$\begin{aligned} \mu_f &= \frac{\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}}{f} \\ \sigma_f &= \frac{\sigma S \frac{\partial f}{\partial s}}{f} \end{aligned}$$

both in terms of partial derivatives of f.

Recall from previous pages:

$$df = \mu_f \cdot f dt + \sigma_f \cdot f dW$$

$$\begin{aligned} dV &= V\left\{\omega^S \cdot \frac{dS}{S} + \omega^f \cdot \frac{df}{f}\right\} \\ &= V\left\{\omega^S(\mu dt + \sigma dW) + \omega^f(\mu_f dt + \sigma_f dW)\right\} \end{aligned}$$

$$dV = V\left\{\omega^S \mu + \omega^f \mu_f\right\} dt + V\left\{\omega^S \sigma + \omega^f \sigma_f\right\} dW$$

Now we kill the dW-term!

Choose (ω^S, ω^f) such that

$$\begin{aligned} \omega^S \sigma + \omega^f \sigma_f &= 0 \\ \omega^S + \omega^f &= 1 \end{aligned}$$

Linear system with solution (if you don't divide by zero!)

$$\omega^S = \frac{\sigma_f}{\sigma_f - \sigma}, \quad \omega^f = \frac{-\sigma}{\sigma_f - \sigma}$$

Plug into dV!

We obtain

$$dV = V\left\{\omega^S \mu + \omega^f \mu_f\right\} dt$$

This is a risk free “synthetic bank” with short rate

$$\left\{\omega^S \mu + \omega^f \mu_F\right\}$$

.

Absence of arbitrage implies

$$\left\{\omega^S \mu + \omega^f \mu_f\right\} = r$$

Plug in the expressions for ω^S, ω^f, μ_f and simplify. This will give us the following result.

that involve partial derivatives!
see pp. 64, 65

you do the computations!

Black-Schole's PDE

The price is given by

$$\Pi_t[\mathcal{Z}] = f(t, S_t)$$

where the pricing function f satisfies the PDE (partial differential equation)

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,s) + rs\dfrac{\partial f}{\partial s}(t,s) + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 f}{\partial s^2}(t,s) - rf(t,s) & = & 0 \\ f(T,s) & = & \Phi(s) \end{cases}$$

Theorem!

There is a unique solution to the PDE so there is a unique arbitrage free price process for the contract.

Black-Scholes' PDE ct'd

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} - rf & = & 0 \\ f(T, s) & = & \Phi(s) \end{cases}$$

- The price of **all** derivative contracts have to satisfy the **same** PDE

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} - rf = 0$$

otherwise there will be an arbitrage opportunity.

- The only difference between different contracts is in the boundary value condition

$$f(T, s) = \Phi(s)$$

Data needed

- The contract function Φ.
- Today's date t.
- Today's stock price S.
- Short rate r.
- Volatility σ.

Note: The pricing formula does **not** involve the mean rate of return μ!

miracle **??**

Black-Scholes Basic Assumptions

Assumptions:

- The stock price is Geometric Brownian Motion
- Continuous trading.
- Frictionless efficient market.
- Short positions are allowed.
- Constant volatility σ.
- Constant short rate r.
- Flat yield curve.

Black-Scholes' Formula
European Call

T=date of expiration,
t=today's date,
K=strike price,
r=short rate,
s=today's stock price,
σ=volatility.

$$f(t,s) = sN\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right].$$

$N[\cdot]$=cdf for $N(0,1)$-distribution.

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

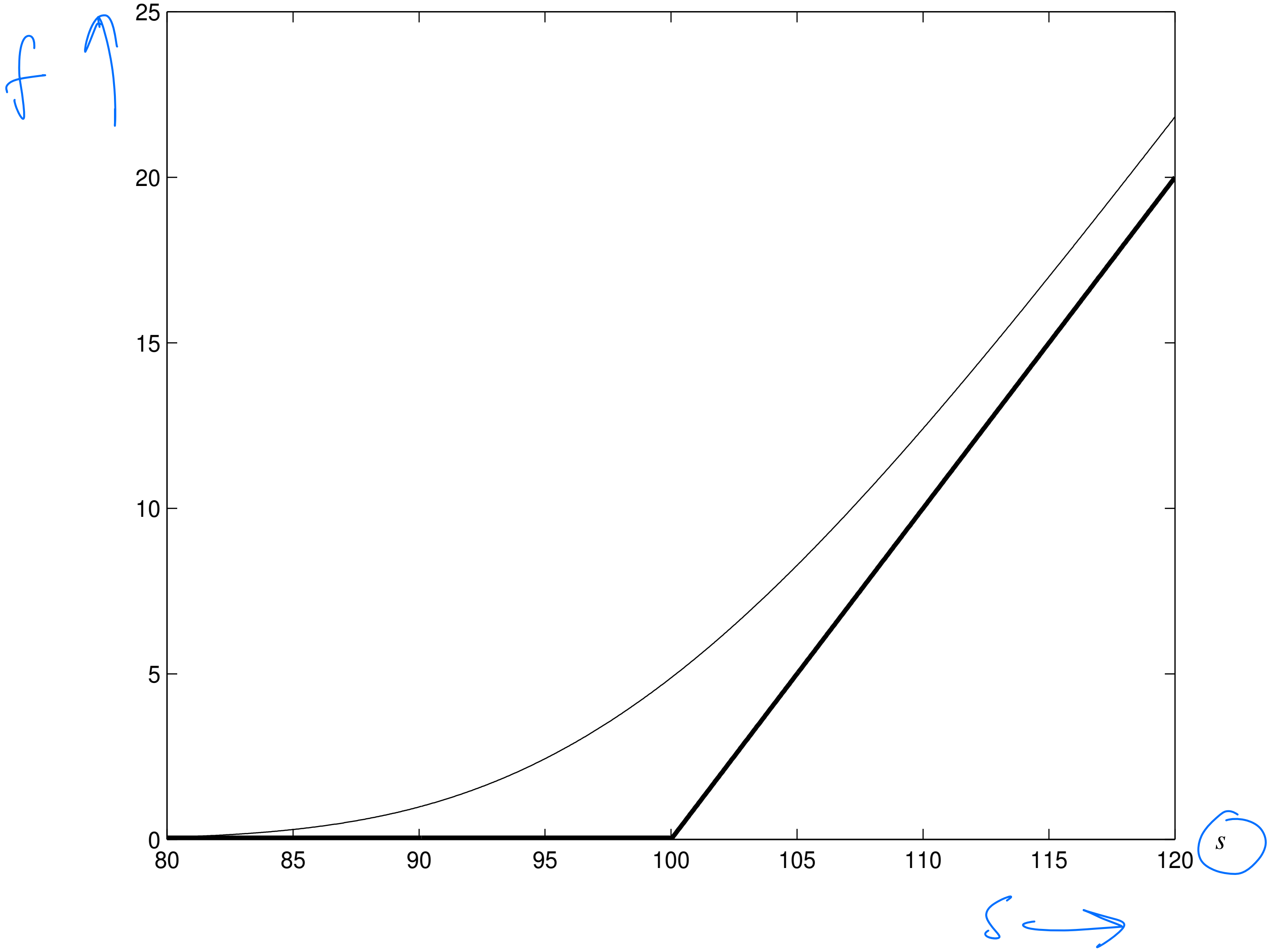
comes out of the blue for the time being; but this f solves the Black-Scholes PDE (check!)

But, see also p. 77

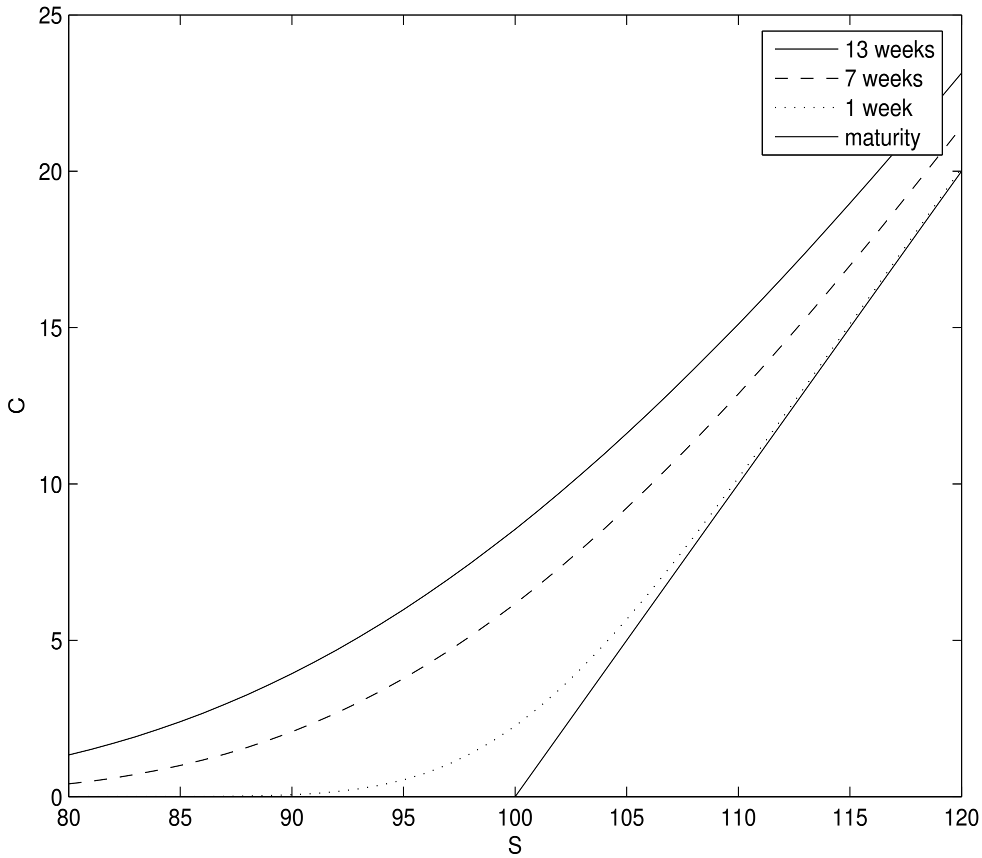
Black-Scholes

European Call,

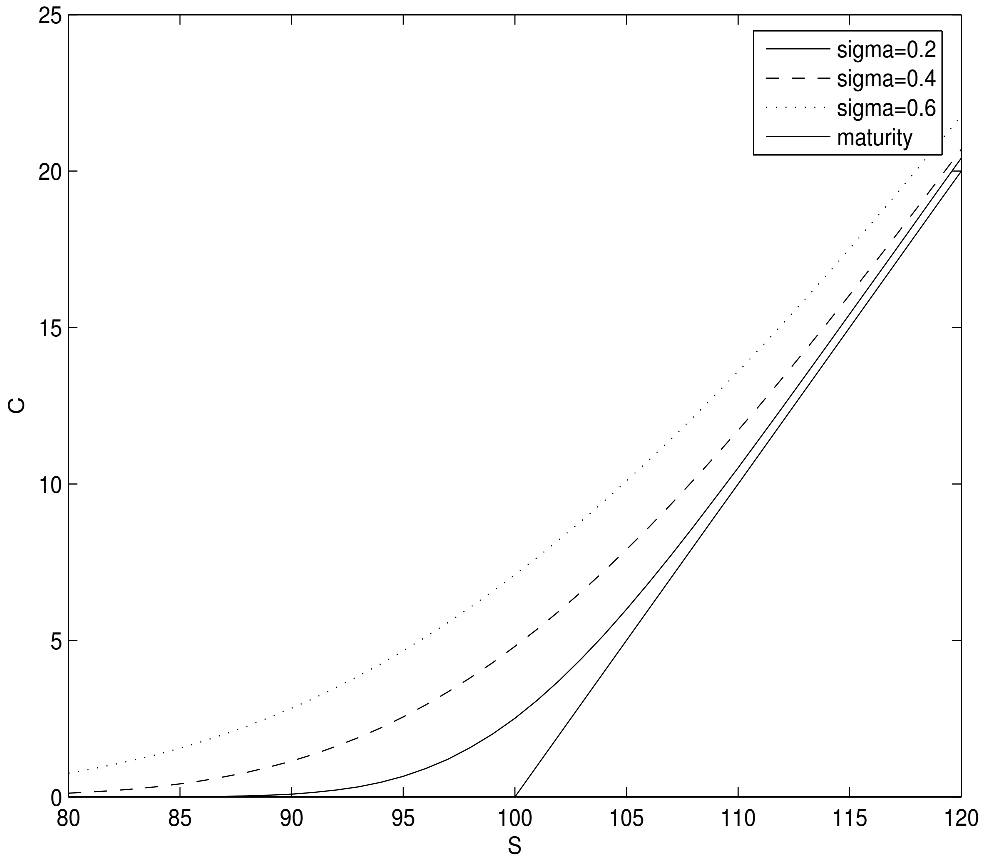
$$K = 100, \quad \sigma = 20\%, \quad r = 7\%, \quad T - t = 1/4$$



Dependence on Time to Maturity



Dependence on Volatility



4.

Risk Neutral Valuation

Risk neutral valuation

Appplying Feynman-Kac to the Black-Scholes PDE we obtain

$$\Pi[t;X] = e^{-r(T-t)}\boxed{E^Q_{t,s}}[X],$$

notation: conditional expectation at time t, with $S_t = s$, under the measure Q.

Q-dynamics:

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dW_t^Q, \\ dB_t = rB_t dt. \end{cases}$$

- Price = Expected discounted value of future payments.
- The expectation shall **not** be taken under the "objective" probability measure P, but under the "risk adjusted" measure ("martingale measure") Q.

Note: $P \sim Q$ (Girsanov), equivalence of the two probability measures on $\mathcal{F}_T$.

[See later]

Concrete formulas

$t = 0$:
$$\Pi\left[0;\Phi\right] = e^{-rT}\int_{-\infty}^{\infty}\Phi(se^{z})f(z)dz$$

$$f(z) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left\{-\frac{\left[z-(r-\frac{1}{2}\sigma^2)T\right]^2}{2\sigma^2 T}\right\}$$

density of $N\left(\left(r-\frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$

variance

Note: $S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^Q\right)$

For the European with strike K we get (see p.57)

$$\Pi(0,\Phi) = e^{-rT}\int_{-\infty}^{\infty}(se^{z}-K)^{+} f(z)dz$$

$$= e^{-rT}\int_{\log\frac{K}{s}}^{\infty} se^{z} f(z)dz + $$

$$- e^{-rT}K\int_{\log\frac{K}{s}}^{\infty} f(z)dz,$$

do the further calculations!

Interpretation of the risk adjusted measure

- **Assume** a risk neutral world.
- Then the following must hold

$$s = S_0 = e^{-rt} E^{Q} [S_t]$$

- In our model this means that

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

- The risk adjusted probabilities can be intrepreted as probabilities in a fictuous risk neutral economy.

Moral

- When we compute prices, we can compute **as if** we live in a risk neutral world.

- This does **not** mean that we live (or think that we live) in a risk neutral world.

- The formulas above hold regardless of the investor's attitude to risk, as long as he/she prefers more to less.

- The valuation formulas are therefore called "preference free valuation formulas".

Quick mentioning of

Properties of Q

More on this in lecture 5

- $P \sim Q$ (Girsanov)

- For the price pricess π of any traded asset, derivative or underlying, the process

$$Z_t = \frac{\pi_t}{B_t}$$

is a Q-martingale. (details later)

- Under Q, the price pricess π of any traded asset, derivative or underlying, has r as its local rate of return:

$$d\pi_t = r\pi_t dt + \sigma_\pi \pi_t dW_t^Q$$

- The volatility of π is the same under Q as under P.

→ end of lecture 16 [or after next slide] ←

A Preview of Martingale Measures

[more on this on pp 179 etc.]

Consider a market, under an objective probability measure P, with underlying assets

$$B, S^1, \ldots, S^N$$

Definition: A probability measure Q is called a **martingale measure** if

- $P \sim Q$
- For every i, the process

$$Z_t^i = \frac{S_t^i}{B_t}$$

is a Q-martingale.

Theorem: The market is arbitrage free **iff** there exists a martingale measure. FTAP 1:

1st fundamental theorem of asset pricing

→ end of lecture 1b ←

→ Start of Lecture 2 (or with p.81)

5.

Appendices

Appendix A: Black-Scholes vs Binomial

(if you know this)

book: Chapter 2

Consider a binomial model for an option with a fixed time to maturity T and a fixed strike price K.

- Build a binomial model with n periods for each $n = 1, 2, \ldots$.

- Use the standard formulas for scaling the jumps:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad \Delta t = T/n, \qquad \sigma > 0, \; d < 1 < u$$

- For a large n, the stock **price** at time T will then be a **product** of a large number of i.i.d. random variables.

- More precisely

0 $\quad$ $T = n\Delta t$

n divisions of $[0, T]$

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

where n is the number of periods in the binomial model and $Z_i = u, d$. In S_T number of u's and d's matters only, not the order $\Rightarrow$ looks like successes/failures in Binomial models

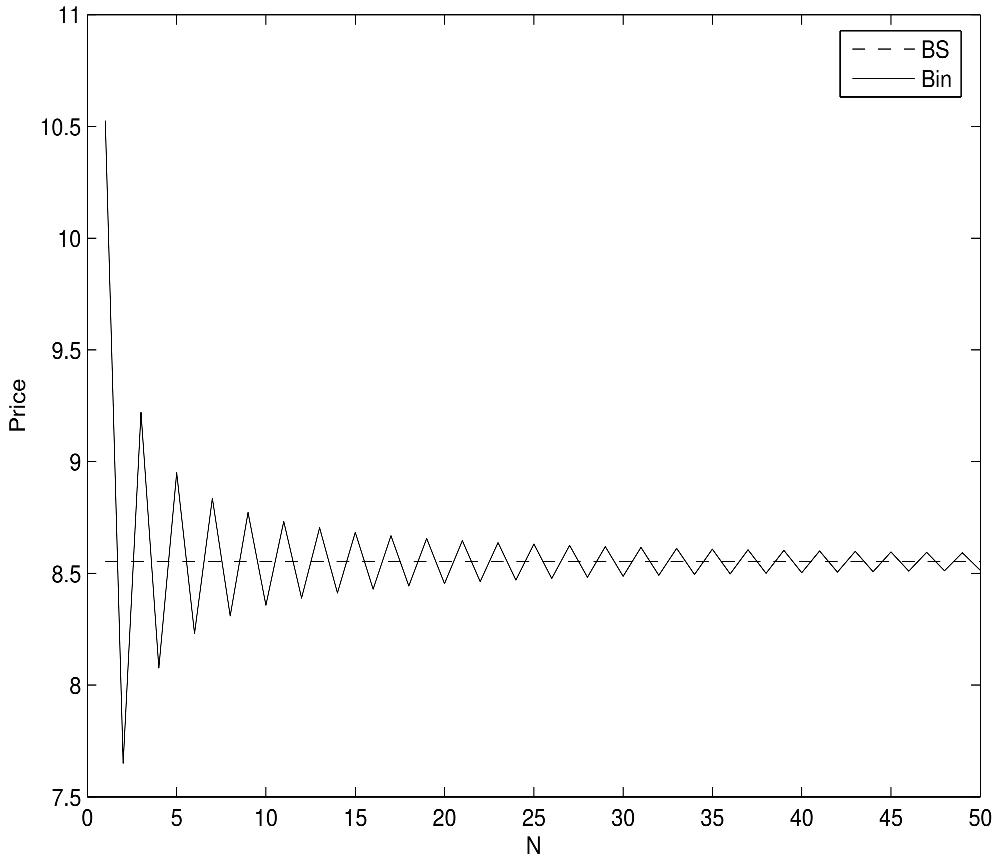
Recall (this is the Cox-Ross-Rubinstein model)

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

- The stock **price** at time T will be a **product** of a large number of i.i.d. random variables.
- The log- **return** will be a large **sum** of i.i.d. variables.
  $\log S_T = \log S_0 + \sum_{i=1}^{n} \log Z_i$
- The Central Limit Theorem will kick in. (details omitted)
- In the limit, **returns** will be **normally** distributed. $\Rightarrow$
- Stock **prices** will be **lognormally** distributed.
- We are in the Black-Scholes model.
- The binomial price will converge to the Black-Scholes price.

Theorem! (no proof/details)

Binomial convergence to Black-Scholes



Binomial ∼ Black-Scholes

The intuition from the Binomial model carries over to Black-Scholes.

- The B-S model is "just" a binomial model where we rebalance the portfolio infinitely often.
- The B-S model is "thus" complete. (notion comes later)
- Completeness explains the unique prices for options in the B-S model.
- The B-S price for a derivative is the limit of the binomial price when the number of periods is very large.

These statements are actually theorems.
Take them for granted.

- Remark: Binomial models have been used in practice (even in Excel)

Appendix B: Portfolio theory

(this is a copy of page 53)

We consider a market with N assets.

$$S_t^i = \text{price at } t\text{, of asset No } i.$$

A **portfolio** strategy is an adapted vector process

$$h_t = (h_t^1, \cdots, h_t^N)$$

where

$$\begin{aligned} h_t^i &= \text{number of units of asset } i, \\ V_t &= \text{market value of the portfolio} \end{aligned}$$

$$V_t = \sum_{i=1}^N h_t^i S_t^i$$

The portfolio is typically of the form

$$h_t = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

Self financing portfolios

We want to study **self financing** portfolio strategies, i.e. portfolios where

- There is now external infusion and/or withdrawal of money to/from the portfolio.
- Purchase of a "new" asset must be financed through sale of an "old" asset.

How is this formalized?

the one on p.54

Problem: Derive an expression for dV_t for a self financing portfolio.

We analyze in discrete time, and then go to the continuous time limit.

Discrete time portfolios

We trade at discrete points in time $t = 0, 1, 2, \ldots$.

Price vector process:

$$S_n = (S_n^1, \cdots, S_n^N), \quad n = 0, 1, 2, \ldots$$

Portfolio process:

$$h_n = (h_n^1, \cdots, h_n^N), \quad n = 0, 1, 2, \ldots$$

Interpretation: At time n we buy the portfolio h_n at the price S_n, and keep it until time $n + 1$.

Value process:

$$V_n = \sum_{i=1}^{N} h_n^i S_n^i = \underbrace{h_n S_n}_{\text{inner}}$$

inner product notation

The self financing condition

- At time $n-1$ we buy the portfolio h_{n-1} at the price S_{n-1}.
- At time n this portfolio is worth $h_{n-1}S_n$.

[prices jump from S_{n-1} to S_n]

- At time n we buy the new portfolio h_n at the price S_n.
- The cost of this new portfolio is h_nS_n.
- The self financing condition is the **budget constraint**

$$h_{n-1}S_n = h_nS_n$$

The self financing condition

Recall:

$$V_n = h_n S_n$$

Definition: For any sequence $x_1, x_2, \ldots$ we define the sequence Δx_n by

$$\Delta x_n = x_n - x_{n-1}$$

Problem: Derive an expression for ΔV_n for a self financing portfolio.

Lemma: For any pair of sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ we have the relation

$$\Delta(xy)_n = x_{n-1}\Delta y_n + y_n \Delta x_n$$

(Abel's summation formula: $x_n y_n - x_0 y_0 = \sum x_{i-1} \Delta y_i + \sum y_i \Delta x_i$)

Proof: Do it yourself.

like integration by parts !

Recall

$$V_n = h_n S_n$$

From the Lemma we have

$$\Delta V_n = \Delta(hS)_n = h_{n-1}\Delta S_n + S_n \Delta h_n$$

Recall the self financing condition

$$h_{n-1}S_n = h_n S_n$$

which we can write as

$$S_n \Delta h_n = 0$$

Inserting this into the expression for ΔV_n gives us.

Proposition: The dynamics of a self financing portfolio are given by

$$\Delta V_n = h_{n-1}\Delta S_n$$

Note the forward increments!

Portfolios in continuous time

Price process:

$$S_t^i = \text{price at } t, \text{ of asset No } i.$$

Portfolio:

$$h_t = (h_t^1, \cdots, h_t^N)$$

Value process

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

From the self financing condition in discrete time

$$\Delta V_n = h_{n-1} \Delta S_n$$

we are led to the following definition. (by analogy!)

Definition: The portfolio h is self financing if and only if

the one on page 54

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

need that the S^i are "semimartingales" (or diffusions)

[copy of page 55]

Relative weights

Definition:

ω_t^i = relative portfolio weight on asset No i.

We have

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t}$$

Insert this into the self financing condition

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

We obtain

Portfolio dynamics:

$$dV_t = V_t \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$$

Interpret!

recall model: $dS_t = \mu S_t dt + \sigma S_t dW_t$

Appendix C: The original Black-Scholes PDE argument

Consider the following portfolio.

- "borrow"
  Short one unit of the derivative, with pricing function $f(t, s)$: you have -1 as a quantity
- Hold x units of the underlying S. (or x_t at time t)
  [later we find a "good" x]

The portfolio value is given by

$$V = -f(t, S_T) + x_t S_t$$

($x_t = x$, short hand notation)

The object is to choose x_t such that the portfolio is risk free for an infinitesimal interval of length dt.

self financing!

We have $dV = -df + x_t dS$ and from Itô we obtain

for df, use dS_t

$$\begin{aligned} dV &= -\left\{\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}\right\} dt \\ &- \sigma S \frac{\partial f}{\partial s} dW + \underbrace{x\mu S dt + x\sigma S dW}_{x_t dS_t} \end{aligned}$$

Rearrange:

$$dV = \left\{ x\mu S - \frac{\partial f}{\partial t} - \mu S \frac{\partial f}{\partial s} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt + \sigma S \left\{ x - \frac{\partial f}{\partial s} \right\} dW$$

We obtain a risk free portfolio if we choose x as

$$x = \frac{\partial f}{\partial s}$$ (the "good" x)

and then we have, after simplification, (inserting this x in $dV = \ldots$)

$$dV = \left\{ -\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt$$

Using $V = -f + xS$ and x as above, the return dV/V is thus given by

$$\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}} dt$$

Remark: not clear what the "logical problems" of page 62 are.

We had (previous page)

$$\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}}dt$$

This portfolio is risk free, so absence of arbitrage implies that

$$\frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}} \overset{!}{=} r$$

(see p.60)

Simplifying this expression gives us the Black-Scholes PDE.

$$\begin{aligned}
\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2} - rf &= 0, \\
f(T,s) &= \Phi(s).
\end{aligned}$$

→ end of lecture 2a ←

Continuous Time Finance

Completeness and Hedging

(Ch 8-9)

Tomas Björk

Problems around Standard Black-Scholes

"over the counter"

- We **assumed** that the derivative was traded. How do we price OTC products?

- Why is the option price independent of the expected rate of return α of the underlying stock?

previously, we used μ instead of α as notation

- Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with **completeness**.

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$$

[The contents of this page will return on p. 191.]

Definition:
We say that a T-claim X can be **replicated**, alternatively that it is **reachable** or **hedgeable**, if there exists a self financing portfolio h such that

$$V_T^h = X, \quad P - a.s.$$

$V_t^h = V_t = h_t S_t , t \leq T$

In this case we say that h is a **hedge** against X. Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**

Basic Idea: If X can be replicated by a portfolio h then the arbitrage free price for X is given by

$$\Pi_t [X] = V_t^h.$$

(law of one price for reachable claim)

[If $\Pi_t(X) < V_t^h$, you sell the portfolio, buy the claim and put the surplus aside. At time T you sell the claim and buy the portfolio back: net cost is zero.]

Similar argument for $t=0$

Consider the following

Trading Strategy

Consider a replicable claim X which we want to sell at $t = 0$..

- Compute the price $\Pi_0[X]$ and sell X at a slightly (well) higher price. [Suppose you are able to do that]
- Buy the hedging portfolio and invest the surplus in the bank.
- Wait until expiration date T.
- The liabilities stemming from X is exactly matched by V_T^h, and we have our surplus in the bank.

Completeness of Black-Scholes

Theorem: The Black-Scholes model is complete.

"arbitrary" function

Proof. Fix a claim $X = \Phi(S_T)$. We want to find processes V, u^B and u^S such that

relative portfolios!

Recall p.55: self financing condition:

$$
\begin{aligned}
dV_t &= V_t \left\{ u_t^B \frac{dB_t}{B_t} + u_t^S \frac{dS_t}{S_t} \right\} \\
V_T &= \Phi(S_T).
\end{aligned}
$$

i.e. (recall $dB_t = rB_t dt$, $dS_t = \alpha S_t dt + \sigma S_t dW_t$)

$$
\begin{aligned}
dV_t &= V_t \left\{ u_t^B r + u_t^S \alpha \right\} dt + V_t u_t^S \sigma dW_t, \\
V_T &= \Phi(S_T).
\end{aligned}
$$

$V_t = h_t^B B_t + h_t^S S_t \Rightarrow$

$u_t^B = \dfrac{h_t^B B_t}{V_t}$

Heuristics:

Let us **assume** that X is replicated by $h = (u^B, u^S)$ with value process V.

Ansatz: (reasonable, based on $X = \Phi(S_T)$ and X is Markov)

$$V_t = F(t, S_t)$$

for F to be found

Ito gives us

$$dV = \left\{ F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss} \right\} dt + \sigma S F_s dW,$$

Write this as

$$dV = V \left\{ \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{S F_s}{V} \sigma dW. \qquad (*)$$

Compare with

$$dV = V \left\{ u^B r + u^S \alpha \right\} dt + V u^S \sigma dW$$

[dW, and dt, terms should coincide]

Define u^S by (time index t and S_t explicitly written)

$$u_t^S = \frac{S_t F_s(t, S_t)}{F(t, S_t)},$$

This gives us the eqn, from (*) on p.103,

$$dV = V\left\{\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^S \alpha\right\} dt + V u^S \sigma dW.$$

Again Compare with

$$dV = V\left\{u^B r + u^S \alpha\right\} dt + V u^S \sigma dW$$

Natural choice for u^B is given by (match the dt terms)

$$u^B = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF},$$

with u_B and u_S of p.104

The relation $u^B + u^S = 1$ gives us the Black-Scholes PDE

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0.$$

The condition

$$V_T = \Phi\left(S_T\right)$$

gives us the boundary condition

$$F(T, s) = \Phi(s)$$

even

Moral: The model is complete and we have explicit formulas for the replicating portfolio.

the u^B and u^S of p. 104

Main Result

Theorem: Define F as the solution to the boundary value problem

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

$= \Phi(S_T)$

Then X can be replicated by the relative portfolio

use u^B on p. 104 and the PDE

$$\begin{aligned} u_t^B &= \frac{F(t,S_t) - S_t F_s(t,S_t)}{F(t,S_t)}, \\ u_t^S &= \frac{S_t F_s(t,S_t)}{F(t,S_t)}. \end{aligned}$$

The corresponding absolute portfolio is given by

$h_t^B = \frac{u_t^B V_t}{B_t}$, $V_t = F(t,S_t)$

$$\begin{aligned} h_t^B &= \frac{F(t,S_t) - S_t F_s(t,S_t)}{B_t}, \\ h_t^S &= F_s(t,S_t), \end{aligned}$$

and the value process V^h is given by

$$V_t^h = F(t,S_t).$$

(see also book Lemma 8.4).

Notes

use the PDE

- Completeness explains unique price - the claim is superfluous! nothing "new" compared to S_t and B_t in the market

- Replicating the claim $P - a.s. \Longleftrightarrow$ Replicating the claim $Q - a.s.$ for any $Q \sim P$. Thus the price only depends on the support of P.

determined by P

- Thus (Girsanov) it will not depend on the drift α of the state equation.

- The completeness theorem is a nice theoretical result, but the replicating portfolio is **continuously rebalanced**. Thus we are facing very high transaction costs.

- Proof only given for claims of the type $\Phi(S_T)$ and under the Ansatz $V_t = F(t, S_t)$. More general result still true: "Any" claim can be hedged

Completeness vs No Arbitrage

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$R = \text{number of random sources}$$

$$N = \text{number of risky assets}$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

for instance to annihilate (the) random sources and find arbitrage opportunities

Meta-Theorem

(Compare to solve $Ax=b$, with $A \in \mathbb{R}^{m\times n}$ etc.
when (unique) solution? $m \le n$ (if you
ignore "rank"
conditions)

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model. R=N=1. Arbitrage free and complete.

→ End of lecture 2b ←

→ Start of lecture 3 ←

Parity Relations

Let Φ and Ψ be contract functions for the T-claims $Z = \Phi(S_T)$ and $Y = \Psi(S_T)$. Then for any real numbers α and β we have the following price relation.

$$\Pi_t[\alpha\Phi + \beta\Psi] = \alpha\Pi_t[\Phi] + \beta\Pi_t[\Psi].$$

see Feynman-Kac, p.76

or use a no-arbitrage argument

Proof. Linearity of mathematical $\boxed{\text{expectation}}$.

Consider the following "basic" contract functions.

$$\begin{aligned}
\Phi_S(x) &= x, \\
\Phi_B(x) &\equiv 1, \\
\Phi_{C,K}(x) &= \max[x - K, 0].
\end{aligned}$$

(think of x as a value of S_T)

Prices:

$$\begin{aligned}
\Pi_t[\Phi_S] &= S_t, \\
\Pi_t[\Phi_B] &= e^{-r(T-t)}, \\
\Pi_t[\Phi_{C,K}] &= c(t, S_t; K, T).
\end{aligned}$$

think a bit!

just notation to express that the price of the call depends on t, S_t, K, T (and more)

If we have for more options, with strike K_i

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i},$$

then

$$\Pi_t[\Phi] = \alpha\Pi_t[\Phi_S] + \beta\Pi_t[\Phi_B] + \sum_{i=1}^{n} \gamma_i \Pi_t[\Phi_{C,K_i}]$$

We may replicate the claim Φ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a **buy-and hold** portfolio:

the α, β, γ_i

- α shares of the underlying stock,
- β zero coupon T-bonds with face value \$1,;
- γ_i European call options with strike price K_i, all maturing at T.

pay off at time T is \$1;
value at time $t<T$: $e^{-r(T-t)}$ (in \$)

Put-Call Parity

Consider a European put contract

$$\Phi_{P,K}(s) = \max\left[K - s, 0\right]$$

It is easy to see (draw a figure) that
(or simple algebra)

$$\begin{aligned}\Phi_{P,K}(x) &= \Phi_{C,K}(x) - s + K \\ &= \Phi_{\cancel{P}C,K}(x) - \Phi_S(x) + \Phi_B(x)K\end{aligned}$$

We immediately get

Put-call parity:

$e^{-r(T-t)}$

$$p(t, s; K, T) = c(t, s; K, T) - s + Ke^{-r(T-t)}$$

Thus you can construct a synthetic put option, using a buy-and-hold portfolio. (with a call option)

(see Prop. 9.3 in the book)

Delta Hedging

name has to do with the "Greeks", see further down

Consider a fixed claim

$$X = \Phi(S_T)$$

with pricing function

$$F(t, s).$$ (or $F(t, S_t)$ with $S_t = s$)

Setup:
We are at time t, and have a short (interpret!) position in the contract.
("debt" in the contract)

Goal:
Offset the risk in the derivative by buying (or selling) the (highly correlated) underlying; find how much to buy/sell

Definition:
A position in the underlying (S_t) is a **delta hedge** against the derivative if the portfolio (underlying + derivative) is immune against small changes in the underlying price.
calls for differentiation, derivatives in the sense of calculus

Formal Analysis

minus because of the short position

$$
\begin{array}{rcl}
-1 & = & \text{number of units of the derivative product} \\
x & = & \text{number of units of the underlying} \\
s & = & \text{today's stock price} \\
t & = & \text{today's date}
\end{array}
$$

, $S_t = s$

Value of the portfolio:

$$V = -1 \cdot F(t,s) + x \cdot s$$

A delta hedge is characterized by the property that

$$\frac{\partial V}{\partial s} = 0.$$

(insensitive to changes in s)

We obtain

$$-\frac{\partial F}{\partial s} + x = 0$$

Solve for x!

see also lecture 5 on p.106

Result:
We should have

$$\hat{x} = \frac{\partial F}{\partial s}$$

shares of the underlying in the delta hedged portfolio.

Definition:
For any contract, its "delta" is defined by

$$\Delta = \frac{\partial F}{\partial s}.$$

Recall: (F = pricing function)

Result:
We should have

$$\hat{x} = \Delta$$

shares of the underlying in the delta hedged portfolio.

Warning:
The delta hedge must be rebalanced over time. (why?)

$\left(\hat{x}_t = \Delta_t = \frac{\partial F}{\partial s}(t, S_t), \text{ time dependent} \right)$

Black Scholes

For a European Call in the Black-Scholes model we have

$$\Delta = N[d_1] = P(N(0,1) \leq d_1)$$

NB This is **not** a trivial result! But see p. 71, Black-Scholes case.

From put call parity it follows (how?) that Δ for a European Put is given by

$$\Delta = N[d_1] - 1$$

$$= -P(N(0,1) > d_1) < 0$$

Check signs and interpret!

Rebalanced Delta Hedge

- Sell one call option at time $t = 0$ at the B-S price F.
- Compute Δ and by Δ shares. (Use the income from the sale of the option, and borrow money if necessary.)
- Wait one day (week, minute, second..). The stock price has now changed.
- Compute the new value of Δ, and borrow money in order to adjust your stock holdings.
- Repeat this procedure until $t = T$. Then the value of your portfolio (B+S) will match the value of the option almost exactly.

and better if you rebalance more frequently

- Lack of perfection comes from discrete, instead of continuous, trading.

- You have created a "synthetic" option. (Replicating portfolio).

Formal result:
The relative weights in the replicating portfolio are

$$u_S = \frac{S \cdot \Delta}{F},$$

$$u_B = \frac{F - S \cdot \Delta}{F}$$

(See p.106, with $\Delta = F_s(t, S_t)$)

Portfolio Delta

Assume that you have a portfolio consisting of derivatives

$$\Phi_i(S_{T_i}), \quad i = 1, \cdots, n$$

all **written on the same underlying** stock S.

$$\begin{aligned} F_i(t,s) &= \text{pricing function for i:th derivative} \quad (S_t = s) \\ \Delta_i &= \frac{\partial F_i}{\partial s} \\ h_i &= \text{units of i:th derivative} \end{aligned}$$

Portfolio value:

$$\Pi = \sum_{i=1}^{n} h_i F_i$$

Portfolio delta:

$$\Delta_\Pi = \sum_{i=1}^{n} h_i \Delta_i$$

Gamma

A problem with discrete delta-hedging is.

- As time goes by, S will change.
- This will cause $\Delta = \frac{\partial F}{\partial s}$ to change, see page 115
- Thus you are sitting with the "wrong" value of delta.
  (at a later time instant)

Moral:

- If delta is sensitive to changes in S, then you have to rebalance often.
- If delta is insensitive to changes in S you do not need to rebalance so often, or perhaps not at all

Definition:
Let Π be the value of a derivative (or portfolio). **Gamma** (Γ) is defined as

$$\Gamma = \frac{\partial \Delta}{\partial s}$$

i.e.

$$\Gamma = \frac{\partial^2 \Pi}{\partial s^2},$$ Π is the pricing function, often $F(t,s)$

Gamma is a measure of the sensitivity of Δ to changes in S.

Result: For a European Call in a Black-Scholes model, Γ can be calculated as

$$\Gamma = \frac{N'[d_1]}{S\sigma\sqrt{T-t}}$$ (Exercise!)

Important fact:
For a position in the underlying stock itself we have

$$\Gamma = 0$$ (trivial!)

Gamma Neutrality

A portfolio Π is said to be **gamma neutral** if its gamma equals zero, i.e.

$$\Gamma_\Pi = 0$$

- Since $\Gamma = 0$ for a stock you can not gamma-hedge using only stocks. Typically you use some derivative to obtain gamma neutrality.

→ End of lecture 3a ←

General procedure

Given a portfolio Π with underlying S. Consider two derivatives with pricing functions F and G.

$$
\begin{aligned}
x_F &= \text{number of units of } F \\
x_G &= \text{number of units of } G
\end{aligned}
$$

Problem:
Choose x_F and x_G such that the entire portfolio is delta- and gamma-neutral.

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

repeat:

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

We get the equations

$$\frac{\partial V}{\partial s} = 0, \quad \text{(delta neutral)}$$

$$\frac{\partial^2 V}{\partial s^2} = 0. \quad \text{(gamma neutral)}$$

i.e.

$$\Delta_\Pi + x_F \Delta_F + x_G \Delta_G = 0,$$

$$\Gamma_\Pi + x_F \Gamma_F + x_G \Gamma_G = 0$$

Solve for x_F and x_G! (linear system, has a unique solution?)

in general yes, if G is sufficiently different from F

Particular Case

- In many practical cases the original portfolio Π is already delta neutral.
- Then it is natural to use a derivative to obtain gamma-neutrality.
- This will destroy the delta-neutrality. for the new portfolio
- Therefore we use the underlying stock (with zero gamma!) to delta hedge in the end: next page

Formally:

additional derivative

$$V = \Pi + x_F \cdot F + x_S \cdot S$$

stock needed for Δ-neutrality

$$\begin{aligned}
\Delta_\Pi + x_F \Delta_F + x_S \Delta_S &= 0, \\
\Gamma_\Pi + x_F \Gamma_F + x_S \Gamma_S &= 0
\end{aligned}$$

We have

$$\begin{aligned}
\Delta_\Pi &= 0, \\
\Delta_S &= 1 \\
\Gamma_S &= 0.
\end{aligned}$$

(assumption was Π is Δ neutral)

i.e.

$$\begin{aligned}
\Delta_\Pi + x_F \Delta_F + x_S &= 0, \\
\Gamma_\Pi + x_F \Gamma_F &= 0
\end{aligned}$$

Solution is

$$\left\{\begin{aligned}
x_F &= -\frac{\Gamma_\Pi}{\Gamma_F} \\
x_S &= \frac{\Delta_F \Gamma_\Pi}{\Gamma_F} - \Delta_\Pi
\end{aligned}\right.$$

Further Greeks

$$\Theta = \frac{\partial \Pi}{\partial t},$$

$$V = \frac{\partial \Pi}{\partial \sigma},$$

$$\rho = \frac{\partial \Pi}{\partial r}$$

V is pronounced "Vega".

NB!

- A delta hedge is a hedge against the movements in the underlying stock, given a **fixed model**.
- A Vega-hedge is not a hedge against movements of the underlying asset. It is a hedge against a **change of the model itself**: σ is a model parameter.

Continuous Time Finance

The Martingale Approach

I: Mathematics

(Ch 10-12)

Tomas Björk

a purely theoretical lecture

Introduction

which you probably know already

In order to understand and to apply the martingale approach to derivative pricing and hedging we will need to some basic concepts and results from measure theory. These will be introduced below in an informal manner - for full details see the textbook.

Many propositions below will be proved but we will also present a couple of central results without proofs, and these must then be considered as dogmatic truths. You are of course not expected to know the proofs of such results (this is outside the scope of this course) but you are supposed to be able to **use** the results in an operational manner.

They are working knowledge, part of your toolkit.

Contents

1. Events and sigma-algebras
2. Conditional expectations
3. Changing measures
4. The Martingale Representation Theorem
5. The Girsanov Theorem

1.

Events and sigma-algebras

Events and sigma-algebras

(book: Section A.2)

contains all "relevant" outcomes

Consider a probability measure P on a sample space Ω. An **event** is simply a subset $A \subseteq \Omega$ and $P(A)$ is the probability that the event A occurs.

For technical reasons, a probability measure can only be defined for a certain "nice" class $\mathcal{F}$ of events, so for $A \in \mathcal{F}$ we are allowed to write $P(A)$ as the probability for the event A.

For technical reasons the class $\mathcal{F}$ must be a **sigma-algebra**, which means that $\mathcal{F}$ is closed under the usual set theoretic operations like complements, countable intersections and countable unions. !

Interpretation: We can view a σ-algebra $\mathcal{F}$ as formalizing the idea of information. More precisely: A σ-algebra $\mathcal{F}$ is a collection of events, and if we assume that we have access to the information contained in $\mathcal{F}$, this means that for every $A \in \mathcal{F}$ we know exactly if A has occured or not.

Probability space is the triple $(\Omega, \mathcal{F}, P)$

Borel sets

Definition: The **Borel algebra** $\mathcal{B}$ is the smallest sigma-algebra on R which contains all intervals. A set B in $\mathcal{B}$ is called a **Borel set**.

Remark: There is no constructive definition of $\mathcal{B}$, but almost all subsets of R that you will ever see will in fact be Borel sets, so the reader can without danger think about a Borel set as "an arbitrary subset of R".

alternatively, contains all open sets, or
contains all closed sets
(these are topological concepts)

Random variables

Section B.1

An $\mathcal{F}$-**measurable random variable** X is a a mapping

$$X : \Omega \rightarrow R$$

such that $\{X \in B\} = \{\omega \in \Omega : \; X(\omega) \in B\}$ belongs to $\mathcal{F}$ for all Borel sets B. This guarantees that we are allowed to write $P(X \in B)$. Instad of writing "X is $\mathcal{F}$-measurable" we will often write $X \in \mathcal{F}$.

This means that if $X \in \mathcal{F}$ then the value of X is completely determined by the information contained in $\mathcal{F}$.

If we have another σ-algebra $\mathcal{G}$ with $\mathcal{G} \subseteq \mathcal{F}$ then we interpret this as "$\mathcal{G}$ contains less information than $\mathcal{F}$".

→ End of lecture 3b ←

→ Start of lecture 4a ←

2.

Conditional Expectation (Section B.5)

formal definition:

Let $\mathcal{G} \subset \mathcal{F}$, $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$: $E|X| < \infty$, $X \in \mathcal{F}$

Thm: "$\exists!$" $\hat{X} \in \mathcal{L}^1(\Omega, \mathcal{G}, P)$ s.t. !

$E\hat{X} 1_G = E X 1_G$, $\forall G \in \mathcal{G}$ $\left[1_G(\omega) = \begin{cases} 1 & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases} \right]$

uniqueness: another $\hat{X}'$ satisfies $P(\hat{X} = \hat{X}') = 1$.

Such an $\hat{X}$ is a version of the conditional expectation of X given $\mathcal{G}$, we often write $E[X|\mathcal{G}]$ instead of $\hat{X}$

Conditional Expectation

If $X \in \mathcal{F}$ and if $\mathcal{G} \subseteq \mathcal{F}$ then we write $E[X|\mathcal{G}]$ for the conditional expectation of X given the information contained in $\mathcal{G}$. Sometimes we use the notation $E_{\mathcal{G}}[X]$.

p.137

The following proposition contains everything that we will need to know about conditional expectations within this course. Also:

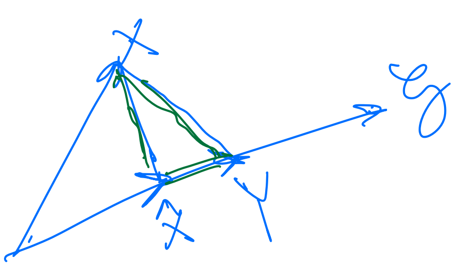
If EX^2, then for all $Y \in \mathcal{L}^2(\Omega, \mathcal{G}, P)$ one has

$$E(X-Y)^2 = E(X-\hat{X})^2 + E(\hat{X}-Y)^2$$

[Think of Pythagoras and ordinary projections, you "see" in the picture

$$X-\hat{X} \perp \hat{X}-Y$$

]



Main Results

Proposition 1: Assume that $X \in \mathcal{F}$, and that $\mathcal{G} \subseteq \mathcal{F}$. Then the following hold.

- The random variable $E[X|\mathcal{G}]$ is completely determined by the information in $\mathcal{G}$ so we have

$$E[X|\mathcal{G}] \in \mathcal{G}$$

(by definition)

- If we have $Y \in \mathcal{G}$ then Y is completely determined by $\mathcal{G}$ so we have

$$E[XY|\mathcal{G}] = Y E[X|\mathcal{G}]$$

  In particular we have

$$E[Y|\mathcal{G}] = Y$$

- If $\mathcal{H} \subseteq \mathcal{G}$ then we have the "law of iterated expectations"

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$$

(analogy with iterated projections)

- In particular we have

$$E[X] = E[E[X|\mathcal{G}]]$$

3.

Changing Measures

Changing Measures

Section B.6

Consider a probability measure P on $(\Omega, \mathcal{F})$, and assume that $L \in \mathcal{F}$ is a random variable with the properties that

$$L \geq 0$$

and

$$E^P[L] = 1.$$

For every event $A \in \mathcal{F}$ we now define the real number $Q(A)$ by the prescription

$$Q(A) = E^P[L \cdot I_A]$$

where the random variable I_A is the indicator for A, i.e.

$$I_A = \begin{cases} 1 & \text{if} \quad A \text{ occurs} \\ 0 & \text{if} \quad A^c \text{ occurs} \end{cases}$$

[I wrote before 1_A instead of I_A.]

Recall that

$$Q(A) = E^P[L \cdot I_A]$$

We now see that $Q(A) \geq 0$ for all A, and that

$$Q(\Omega) = E^P[L \cdot I_\Omega] = E^P[L \cdot 1] = 1$$

We also see that if $A \cap B = \emptyset$ then

$$\begin{aligned} Q(A \cup B) &= E^P[L \cdot I_{A \cup B}] = E^P[L \cdot (I_A + I_B)] \\ &= E^P[L \cdot I_A] + E^P[L \cdot I_B] \\ &= Q(A) + Q(B) \end{aligned}$$

(extends to finite disjoint unions)

Furthermore we see that

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0$$

We have thus more or less proved the following

Proposition 2: If $L \in \mathcal{F}$ is a nonnegative random variable with $E^P[L] = 1$ and Q is defined by

$$Q(A) = E^P[L \cdot I_A]$$

then Q will be a probability measure on $\mathcal{F}$ with the property that

for which you need countable additivity (true!)

$$\boxed{P(A) = 0 \quad \Rightarrow \quad Q(A) = 0.}$$

I turns out that the property above is a very important one, so we give it a name.

Absolute Continuity

Definition: Given two probability measures P and Q on $\mathcal{F}$ we say that Q is **absolutely continuous** w.r.t. P on $\mathcal{F}$ if, for all $A \in \mathcal{F}$, we have

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0$$

We write this as

$$Q << P.$$

If $Q << P$ and $P << Q$ then we say that P and Q are **equivalent** and write

$$Q \sim P$$

(this does <u>NOT</u> mean $Q = P$!)

Equivalent measures

It is easy to see that P and Q are equivalent if and only if

$$P(A) = 0 \quad \Leftrightarrow \quad Q(A) = 0$$

Other equivalent statements are
$P(A) > 0 \Leftrightarrow Q(A) > 0$,

$P(A) < 1 \Leftrightarrow Q(A) < 1$

or, equivalently,

$$P(A) = 1 \quad \Leftrightarrow \quad Q(A) = 1$$

(look at complements)

Two equivalent measures thus agree on all certain events and on all impossible events, but can disagree on all other events.

Simple examples:

- All non degenerate Gaussian distributions on R are equivalent.

- If P is Gaussian on R and Q is exponential then $Q << P$ but not the other way around. (why?)

Absolute Continuity ct'd

We have seen that if we are given P and **define** Q by

$$Q(A) = E^P[L \cdot I_A] \quad (*)$$

for $L \geq 0$ with $E^P[L] = 1$, then Q is a probability measure and $Q << P$. .

A natural question is now if **all** measures $Q << P$ are obtained in this way. The answer is yes, and the precise (quite deep) result is as follows. The proof is difficult and therefore omitted.

that is, by
formula $(*)$
for some L.

→ End of lecture 4a ←

The Radon Nikodym Theorem

Consider two probability measures P and Q on $(\Omega, \mathcal{F})$, and assume that $Q << P$ on $\mathcal{F}$. Then there exists a unique random variable L with the following properties

unique means that for another L' satisfying 1-4, one has $P(L = L') = 1$.

1. $Q(A) = E^P[L \cdot I_A], \quad \forall A \in \mathcal{F}$
2. $L \geq 0, \quad P - a.s.$
3. $E^P[L] = 1,$
4. $L \in \mathcal{F}$

The random variable L is denoted as

$$L = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}$$

and it is called the **Radon-Nikodym derivative** of Q w.r.t. P on $\mathcal{F}$, or the **likelihood ratio** between Q and P on $\mathcal{F}$.

A simple example

The Radon-Nikodym derivative L is intuitively the local scale factor between P and Q. If the sample space Ω is finite so $\Omega = \{\omega_1, \ldots, \omega_n\}$ then P is determined by the probabilities $p_1, \ldots, p_n$ where

$$p_i = P(\omega_i) \quad i = 1, \ldots, n$$

Now consider a measure Q with probabilities

$$q_i = Q(\omega_i) \quad i = 1, \ldots, n$$

If $Q << P$ this simply says that

$$p_i = 0 \quad \Rightarrow \quad q_i = 0$$

and it is easy to see that the Radon-Nikodym derivative $L = dQ/dP$ is given by

$$L(\omega_i) = \frac{q_i}{p_i} \quad i = 1, \ldots, n \qquad (\text{if } p_i > 0),$$

If $p_i = 0$ then we also have $q_i = 0$ and we can define the ratio q_i/p_i arbitrarily.

If $p_1, \ldots, p_n$ as well as $q_1, \ldots, q_n$ are all positive, then we see that $Q \sim P$ and in fact

$$\frac{dP}{dQ} = \frac{1}{L} = \left(\frac{dQ}{dP}\right)^{-1}$$

as could be expected.

Note on notation: $E_P X$ is ~~often~~ written as the Lebesgue integral $\int X \, dP$

Then $E_P(L I_A) = \int L I_A \, dP$ (1)

But $Q(A) \stackrel{!}{=} E_Q[I_A] = \int I_A \, dQ$ (2)

Now divide formally by dP (and multiply):

$$dQ = \frac{dQ}{dP} dP = L \, dP$$

and we understand (1) and (2) are equal.

mathematical nonsense

Computing expected values

A main use of Radon-Nikodym derivatives is for the computation of expected values.

Suppose therefore that $Q << P$ on $\mathcal{F}$ and that X is a random variable with $X \in \mathcal{F}$. With $L = dQ/dP$ on $\mathcal{F}$ then have the following result.

Proposition 3: With notation as above we have

$$E^Q[X] = E^P[L \cdot X],$$

$\int X\,dQ = \int LX\,dP$ (write $L = \frac{dQ}{dP}$ etc....)

Proof: We only give a proof for the simple example above where $\Omega = \{\omega_1, \ldots, \omega_n\}$. We then have

$$\begin{aligned} E^Q[X] &= \sum_{i=1}^{n} X(\omega_i) q_i = \sum_{i=1}^{n} X(\omega_i) \frac{q_i}{p_i} p_i \\ &= \sum_{i=1}^{n} X(\omega_i) L(\omega_i) p_i = E^P[X \cdot L] \end{aligned}$$

The Abstract Bayes' Formula

We can also use Radon-Nikodym derivatives in order to compute conditional expectations. The result, known as the abstract **Bayes' Formula**, is as follows.

Theorem 4: Consider two measures P and Q with $Q << P$ on $\mathcal{F}$ and with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Assume that $\mathcal{G} \subseteq \mathcal{F}$ and let X be a random variable with $X \in \mathcal{F}$. Then the following holds

$$E^Q[X|\mathcal{G}] = \frac{E^P\left[L^{\mathcal{F}}X \middle| \mathcal{G}\right]}{E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]}$$

note the denominator,

different from $E^Q X = E^P[LX]$ ----

(see book Proposition B.41)

Dependence of the σ-algebra

Suppose that we have $Q << P$ on $\mathcal{F}$ with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Now consider smaller σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. Our problem is to find the R-N derivative

$$L^{\mathcal{G}} = \frac{dQ}{dP} \quad \text{on } \mathcal{G}$$

Note that also $Q \ll P$ on $\mathcal{G}$!

We recall that $L^{\mathcal{G}}$ is characterized by the following properties

1. $Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$

2. $L^{\mathcal{G}} \geq 0$

3. $E^P\left[L^{\mathcal{G}}\right] = 1$

4. $L^{\mathcal{G}} \in \mathcal{G}$

crucial!

A natural guess would perhaps be that $L^{\mathcal{G}} = L^{\mathcal{F}}$, so let us check if $L^{\mathcal{F}}$ satisfies points 1-4 above.

By assumption we have

$$Q(A) = E^P \left[L^{\mathcal{F}} \cdot I_A \right] \quad \forall A \in \mathcal{F}$$

Since $\mathcal{G} \subseteq \mathcal{F}$ we then have

$$Q(A) = E^P \left[L^{\mathcal{F}} \cdot I_A \right] \quad \forall A \in \mathcal{G}$$

so point 1 above is certainly satisfied by $L^{\mathcal{F}}$. It is also clear that $L^{\mathcal{F}}$ satisfies points 2 and 3. It thus seems that $L^{\mathcal{F}}$ is also a natural candidate for the R-N derivative $L^{\mathcal{G}}$, but the problem is that we do not in general have $L^{\mathcal{F}} \in \mathcal{G}$. So $L^{\mathcal{G}} \neq L^{\mathcal{F}}$ in general.

This problem can, however, be fixed. By iterated expectations we have, for all $A \in \mathcal{G}$,

$$Q(A) = E^P \left[L^{\mathcal{F}} \cdot I_A \right] = E^P \Big[\underbrace{E^P \left[L^{\mathcal{F}} \cdot I_A \middle| \mathcal{G} \right]}_{\text{Consider this}} \Big]$$

Then previous formula becomes

$Q(A) = E^P\left[E^P[L^{\mathcal{F}} | \mathcal{G}] I_A \right]$

Since $A \in \mathcal{G}$ we have

$$E^P\left[L^{\mathcal{F}} \cdot I_A \middle| \mathcal{G}\right] = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right] I_A$$

Let us now define $L^{\mathcal{G}}$ by

$$L^{\mathcal{G}} = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]$$

We then obviously ! have $L^{\mathcal{G}} \in \mathcal{G}$ and

by a defining property of conditional expectation given $\mathcal{G}$

$$Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$$

It is easy to see that also points 2-3 are satisfied so we have proved the following result.

A formula for $L^{\mathcal{G}}$

Proposition 5: If $Q << P$ on $\mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$ then, with notation as above, we have

$$L^{\mathcal{G}} = E^{P}\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]$$

!

Here, the point was that we wanted

$L^{\mathcal{G}}$ to be $\mathcal{G}$-measurable

Another example is that P and Q are two possible distributions of a random variable with densities f_P and f_Q. Assume $f_P > 0$ and $f_Q > 0$ on $\mathbb{R}$.

Then $P \sim Q$ (seen as probability measures on $\mathcal{F} = \mathcal{B}$, Borel sets, and

$$\frac{dQ}{dP}(x) = \frac{f_Q(x)}{f_P(x)}, \quad \frac{dP}{dQ}(x) = \frac{f_P(x)}{f_Q(x)}.$$

Think of two normal densities f_P, f_Q.

→ End of lecture 4b ←

→ Start of lecture 5 ←

The likelihood process on a filtered space

We now consider the case when we have a probability measure P on some space Ω and that instead of just one σ-algebra $\mathcal{F}$ we have a **filtration**, i.e. an increasing family of σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$.

The interpretation is as usual that $\mathcal{F}_t$ is the information available to us at time t, and that we have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Now assume that we also have another measure Q, and that for some fixed T, we have $Q << P$ on $\mathcal{F}_T$. We define the random variable L_T by

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

as on p.150

Since $Q << P$ on $\mathcal{F}_T$ we also have $Q << P$ on $\mathcal{F}_t$ for all $t \leq T$ and we define

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \leq t \leq T$$

part of the R-N theorem, p.154

For every t we have $\boxed{L_t \in \mathcal{F}_t}$, so L is an adapted process, known as the **likelihood process**.

A process X is adapted (to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$) if for every $t \geq 0$ X_t is $\mathcal{F}_t$-measurable.

The L process is a P martingale

We recall that

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \le t \le T$$

p. 153

Since $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$ we can use Proposition 5 and deduce that

$$L_s = E^P \left[L_t \middle| \mathcal{F}_s \right] \quad s \le t \le T$$

and we have thus proved the following result.

Proposition: Given the assumptions above, the likelihood process L is a P-martingale.

A process X is a $(P, \mathcal{F}_t)$-martingale if it

(i) it is adapted (to a filtration $\{\mathcal{F}_t\}_{t \ge 0}$),

(ii) $E|X_t| < \infty$, $\forall t \ge 0$,

(iii) $E[X_t | \mathcal{F}_s] = X_s$, $\forall t \ge s \ge 0$ ← martingale property

Here $E = E^P$, expectation using P.

Where are we heading?

(and why do we have to know this?)

We are now going to perform measure transformations on Wiener spaces, where P will correspond to the objective measure and Q will be the risk neutral measure.

For this we need define the proper likelihood process L and, since L is a P-martingale, we have the following natural questions.

- What does a martingale look like in a Wiener driven framework? (like Black. Scholes setting)
- Suppose that we have a P-Wiener process W and then change measure from P to Q. What are the properties of W under the new measure Q?

These questions are handled by the Martingale Representation Theorem, and the Girsanov Theorem respectively.

Recall BS framework, with $dS = \mu S dt + \sigma S dW$ (under P)

and $dS = r S dt + \sigma S dW^Q$ (under Q)

We will see that $P \sim Q$ on $\mathcal{F}_T$ (and then also on $\mathcal{F}_t$, $t \le T$)

4.

The Martingale Representation Theorem

(Section 11.1)

Intuition

from general Itô theory

Suppose that we have a Wiener process W under the measure P. We recall that if h is adapted (and integrable enough) and if the process X is defined by

$E\int_0^T h_s^2 ds < \infty$

$$X_t = x_0 + \int_0^t h_s dW_s$$

then X is a a martingale. We now have the following natural question:

Question: Assume that X is an arbitrary martingale. Does it then follow that X has the form

!

$$X_t = x_0 + \int_0^t h_s dW_s$$

for some adapted process h?

In other words: Are **all** martingales stochastic integrals w.r.t. W?

Answer : No, but .....

It is immediately clear that all martingales can **not** be written as stochastic integrals w.r.t. W. Consider for example the process X defined by

$$X_t = \begin{cases} 0 & \text{for} \quad 0 \leq t < 1 \\ Z & \text{for} \quad t \geq 1 \end{cases}$$

where Z is an random variable, independent of W, with $E[Z] = 0$.

You(!) compute $E[X_t | \mathcal{F}_s]$ for different $t > s$

X is then a martingale (why?) but it is clear (how?) that it cannot be written as

?

which filtration? X adapted ⇒ Z should be $\in \mathcal{F}_t$, for $t \geq 1$.

$$X_t = x_0 + \int_0^t h_s dW_s$$

for any process h. See next page

Simplest case: $\mathcal{F}_t = \begin{cases} \{\emptyset, \Omega\}, & t \leq 1 \\ \sigma(Z), & t > 1. \end{cases}$

Intuition

The intuitive reason why we cannot write

$$X_t = x_0 + \int_0^t h_s dW_s$$

in the example above is of course that the random variable Z "has nothing to do with" the Wiener process W. In order to exclude examples like this, we thus need an assumption which guarantees that our probability space only contains the Wiener process W and nothing else.

This idea is formalized by assuming that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ **is the one generated by the Wiener process** W,

$$\mathcal{F}_t = \sigma(W_s,\ s \leq t).$$

not the case with the example on p. 159

Note that the X of p. 159 is NOT adapted to this filtration.

The Martingale Representation Theorem

Theorem. Let W be a P-Wiener process and assume that the filtation is the **internal** one i.e.

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\left\{W_s;\ 0 \leq s \leq t\right\}$$

Then, for every $(P, \mathcal{F}_t)$-martingale X, there exists a real number x and an adapted process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

i.e.

$$dX_t = h_t dW_t.$$

Proof: Hard. This is very deep result.

Crucial is that X is adapted to this special filtration

Note

For a given martingale X, the Representation Theorem above guarantees the existence of a process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

The Theorem does **not**, however, tell us how to find or construct the process h.

5.

The Girsanov Theorem

Sections 11.2, 11.3

Setup

Let W be a P-Wiener process and fix a time horizon T. Suppose that we want to change measure from P to Q on $\mathcal{F}_T$. For this we need a P-martingale L with $L_0 = 1$ to use as a likelihood process, and a natural way of constructing this is to choose a process g and then define L by

see p.155

$$\begin{cases} dL_t &= g_t dW_t \\ L_0 &= 1 \end{cases}$$

(to get a martingale)

This definition does not guarantee that $L \geq 0$, so we make a small adjustment. We choose a process φ and define L by

take $g_t = L_t \varphi_t$ for some φ_t

$$(*) \quad \begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

under integrability conditions!

The process L will again be a martingale and we easily obtain

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

Apply the Itô formula to L_t to see that $(*)$ holds:

$L_t = f(X_t)$, with $f(x) = e^x$, $X_t = \int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds$,

and $(dX_t)^2 = \varphi_t^2 dt$

Thus we are guaranteed that $L \geq 0$. We now change measure form P to Q by setting

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \leq t \leq T$$

"The" main problem is to find out what the properties of W are, under the new measure Q. This problem is resolved by the **Girsanov Theorem**.

Recall : $dQ = L_T dP$ and

$$L_t = E^P[L_T | \mathcal{F}_t]$$

The Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem: Choose an adapted process φ, and define the process L by

OK if L is a P-martingale

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases} \qquad (*)$$

*)

Assume that $E^P[L_T] = 1$, and define a new mesure Q on $\mathcal{F}_T$ by

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

Then $Q << P$ and the process W^Q, defined by

$$W_t^Q = W_t - \int_0^t \varphi_s ds$$

is Q-Wiener. We can also write this as

$$dW_t = \varphi_t dt + dW_t^Q$$

*) In fact, with (*), L is a martingale ($t \le T$) iff $E^P[L_T] = 1$

Changing the drift in an SDE

(Section 11.5)

The single most common use of the Girsanov Theorem is as follows. (related to BS like models)

Suppose that we have a process X with P dynamics

$$(*) \qquad dX_t = \mu_t dt + \sigma_t dW_t$$

where μ and σ are adapted and W is P-Wiener.

We now do a Girsanov Transformation as above, and the question is what the Q-dynamics look like.

From the Girsanov Theorem we have

$$dW_t = \varphi_t dt + dW_t^Q \qquad \text{(page 166)}$$

and substituting this into the P-dynamics $(*)$ we obtain the Q dynamics as

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t dW_t^Q$$

Moral: The drift changes but the diffusion is unaffected, meaning that we keep on having the same σ_t in front of the new Brownian motion W^Q

The Converse Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem. Assume that:

- $Q << P$ on $\mathcal{F}_T$, with likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t \; 0, \leq t \leq T$$

- The filtation is the **internal** one .i.e.

$$\mathcal{F}_t = \sigma \{W_s; \; 0 \leq s \leq t\}$$

Then there exists a process φ such that

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

note (p.155) that
L is a P-martingale.

→ end of lecture 5a ←

→ Start of lecture 5b ←

Continuous Time Finance

The Martingale Approach

II: Pricing and Hedging

(Ch 10-12)

Tomas Björk

Financial Markets (a recap)

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N \right]$$

Example: (Black-Scholes, $S^0 := B$, $S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N \right]$$

$h_t^i =$ number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = \overbrace{h_t}^{\text{row vector}} \underbrace{S_t}_{\text{column vector}}$$

or interpret as an inner product

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight: (from general theory):
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**. (needs assumptions)

NB! This simple observation is in fact the basis of the following theory.

Itô theory!

discrete time: $\Delta Y_t = h_t \Delta X_t$ predictable:

$E[\Delta Y_t | \mathcal{F}_{t-1}] = E[h_t \Delta X_t | \mathcal{F}_{t-1}]$
$= h_t \, E[\Delta X_t | \mathcal{F}_{t-1}]$
$h_t \in \mathcal{F}_{t-1}$

Arbitrage

The portfolio ~~u~~ h is an **arbitrage** portfolio if (with $V = V^h$)

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T \geq 0, \ P - a.s.$ → $P(V_T \geq 0) = 1$ (weaker than the earlier definition on p. 59)
- $P(V_T > 0) > 0$

This implies $E_P[V_T] > 0$.

Main Question: When is the market free of arbitrage?

Remark: If the market is free of arbitrage, and h is a SF financing portfolio with $P(V_T \geq 0) = 1$, then $P(V_T > 0) = 0$, equivalently $P(V_T = 0) = 1$

First Attempt

not realistic

Proposition: If $S_t^0, \cdots, S_t^N$ are P-martingales, then the market is free of arbitrage.

Proof:

Assume that $\cancel{V}$ h is an arbitrage strategy. Since

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i,$$

V is a P-martingale, so (because then expectations remain the same, $E^P V_T = E^P V_0$)

$$V_0 = E^P [V_T] > 0.$$

see previous page

This contradicts $V_0 = 0$.

True, but useless: next page

(but, as we'll see, there is a point in the argument)

Example: (Black-Scholes)

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$
$$dB_t = r B_t dt.$$

not realistic

(We would have to assume that $\boxed{\alpha = r = 0}$)

We now try to improve on this result.

for S and B martingales

Choose S_0 as numeraire

(look at "normalized" prices)

Definition:
The **normalized price vector** Z is given by

$$Z_t = \frac{S_t}{S_t^0} = \left[1, Z_t^1, ..., Z_t^N\right]$$

$Z_t^0 = 1, \forall t \leq T$: certainly a martingale

The **normalized value process** V^Z is given by

$$V_t^Z = \sum_0^N h_t^i Z_t^i.$$

Idea:
The arbitrage and self financing concepts should be independent of the accounting unit.

Invariance of numeraire

Proposition: One can show (see the book) that

- S-arbitrage $\Longleftrightarrow$ Z-arbitrage.
- S-self-financing $\Longleftrightarrow$ Z-self-financing:
so we can just talk of self-financing

$dV_t = h_t dS_t \Leftrightarrow$

$dV_t^Z = h_t dZ_t$

Insight:

- If h self-financing then

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

(note that we don't need $i=0$)

- Thus, if the **normalized** price process Z is a P-martingale, then V^Z is a martingale, argument as before.

Second Attempt

the normalized processes

Proposition: If $Z_t^0, \cdots, Z_t^N$ are P-martingales, then the market is free of arbitrage.

True, but still fairly useless.

by argument as on p. 173

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Use the quotient rule for differentiation of $Z_t^1 = \frac{S_t}{B_t}$

to get
$$
\begin{aligned}
dZ_t^1 &= (\alpha - r) Z_t^1 dt + \sigma Z_t^1 dW_t, \\
dZ_t^0 &= 0dt.
\end{aligned}
$$

We would have to assume "risk-neutrality", i.e. that $\alpha = r$. to have Z^1 is a P-martingale

But, in principle, $\alpha \neq r$. Later we'll see what happens under Q.

Arbitrage

Recall that h is an arbitrage if
p. 172

- h is self financing
- $V_0 = 0.$
- $V_T \geq 0, \ P - a.s.$
- $P(V_T > 0) > 0$

$\begin{cases} V_T \geq 0, \ Q\text{-a.s.} \\ Q(V_T > 0) > 0 \end{cases}$

Major insight

This concept is $\boxed{\text{invariant}}$ under an **equivalent change of measure!**

$P \sim Q$ iff $\begin{cases} P(A) = 0 \iff Q(A) = 0 \\ P(A) = 1 \iff Q(A) = 1 \\ P(A) > 0 \iff Q(A) > 0 \end{cases}$, as noted on p. 143

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e.

(usually defined on $\mathcal{F}_T$)

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \ldots, N$$

  are **Q-martingales**.

We now state the main result of arbitrage theory.

First Fundamental Theorem
of Asset Pricing (FTAP 1)

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

This theorem was already announced on p.81.

Comments

- It is very easy to prove that existence of EMM imples no arbitrage (see below).

- The other implication is technically very hard.

- For discrete time and finite sample space Ω the hard part follows "easily" from the separation theorem for convex sets.

- For discrete time and more general sample space we need the Hahn-Banach Theorem. (formulated as an infinite dimensional version of the separation theorem

→ End of lecture 5 ←

→ Start of lecture 6a ←

Go back to p. 180 ("NA ⇔ EMM")

Proof that EMM implies no arbitrage

Assume that there exists an EMM denoted by Q. Assume that (existence of an arbitrage opportunity,) $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$. *)

Note: $E^Q(V_T) > 0$!

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

This is a proof by contradiction.

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale (Itô theory)

$$\Downarrow$$

$$V_0 \overset{B_0=1}{=} V_0^Z = E^Q\left[V_T^Z\right] > 0$$

, contradicts $V_0 = 0$ on p. 172

$$\Downarrow$$

No arbitrage

*) All these statements also true for V_T^Z instead of V_T

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

(generalizes $B_t = e^{rt}$, $dB_t = rB_t dt$, which we assume in most examples.)

Example: The Black-Scholes Model

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Look for martingale measure. We set $Z_t = S_t/B_t = S_t e^{-rt}$

Standard calculus gives, differentiate a product:

no Itô is needed as B_t contains no Brownian term

$$
dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t, \qquad (6a)
$$

Girsanov transformation on $[0, T]$:

$$
\begin{cases}
dL_t = L_t \varphi_t dW_t, \\
L_0 = 1.
\end{cases}
$$

$$
dQ = L_T dP, \quad \text{on } \mathcal{F}_T
$$

Girsanov: (see p.166)

$$
dW_t = \varphi_t dt + dW_t^Q, \qquad (6b)
$$

where W^Q is a Q-Wiener process. (whereas W is P-Wiener)

Insert (6b) into (6a).

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

!

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma},$$ then $dZ_t = Z_t \sigma dW_t^Q$: martingale!

Q-dynamics of S: insert (6b) into equation for dS_t:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage, as follows from p.182 since we have now shown that Q is an EMM: Z is a martingale under Q.

Pricing

We consider a market $B_t, S_t^1, \ldots, S_t^N$, (many risky assets).

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent (T)-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

New Approach: use the change of measure framework.

Def: $\Pi_t(X)$ is an arbitrage free price process if the extended market is arbitrage free

Solution

extra asset

The $\boxed{\text{extended}}$ market

$$B_t, S_t^1, \dots, S_t^N, \Pi_t[X]$$

FTAP 1.

must be arbitrage free, so there must exist a martingale measure Q for $(S_t, \Pi_t[X])$. In particular

$$\frac{\Pi_t[X]}{B_t}$$

must be a Q-martingale, i.e. it has the martingale property,

$$\frac{\Pi_t[X]}{B_t} = E^Q\left[\left.\frac{\Pi_T[X]}{B_T}\right| \mathcal{F}_t\right] \qquad (6c)$$

$$\Leftrightarrow \quad \Pi_t(X) = B_t E^Q\left[\left.\frac{\Pi_T(X)}{B_T}\right| \mathcal{F}_t\right] = E^Q\left[\left.\frac{B_t}{B_T} X \right| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T[X] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula , rewrite (6c),

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right],$$

if $dB_t = r_t B_t dt$,

NB: if $r_t \equiv r$, then

$$\Pi_t(X) = e^{-r(T-t)} E^Q\left[X \middle| \mathcal{F}_t \right],$$

which we have encountered on p. 76.

Note: here have <u>not</u> used the Feynman-Kac formula to arrive at the same $\Pi_t(X)$, but used a martingale argument.

Example: The Black-Scholes Model

Q-dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q. \qquad (*)$$

NB: S is a Markov process under Q!

Simple claim:

$$X = \Phi(S_T),$$

Markov property gives

$$\Pi_t[X] = e^{-r(T-t)} E^Q[\Phi(S_T) | \mathcal{F}_t] \qquad (p.188)$$

$$= e^{-r(T-t)} E^Q[\Phi(S_T) | S_t]$$

Kolmogorov $\Rightarrow$

$\Pi_t(X)$ is function of (t and) S_t:

$$\Pi_t[X] = F(t, S_t)$$

where $F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF & = & 0, \\ \\ F(T, s) & = & \Phi(s). \end{cases}$$

use Feynman-Kac and the model (*).

Problem

Recall the valuation formula
(p.188)

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right], \text{ depends on } Q,$$

What if there are several different martingale measures Q?

This is connected with the **completeness** of the market.

Hedging (recall p. 100)

Def: A portfolio h is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X, \quad P - a.s.$: $P(V_T = X) = 1$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural (arbitrage free) way of pricing X is

$$\Pi_t[X] = V_t^h$$

(see p. 101 for a justification)

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

$$\Downarrow$$

"martingale representation theorem"

Fix T-claim X.

If h is a hedge for X then $V_T = X$ and

- $V_T^Z = \frac{X}{B_T}$; recall the normalized prices $Z_t^i = \frac{S_t^i}{B_t}$, which are Q-martingales;
- h is self financing, i.e.

$$dV_t^Z = \sum_1^K h_t^i dZ_t^i \quad , \text{ see p. 176.}$$

Thus V^Z is a Q-**martingale**, $V_t^Z = E^Q\left[V_T^Z \middle| \mathcal{F}_t\right]$:

$$V_t^Z = E^Q\left[\left.\frac{X}{B_T}\right|\mathcal{F}_t\right],$$

if X can be hedged by h.

→ End of lecture 6a ←

→ Start of lecture 66 ←

We reverse the previous argument, which led to p.193.

Lemma:
Fix T-claim X. Define martingale M by

$$M_t = E^Q\left[\left.\frac{X}{B_T}\right|\mathcal{F}_t\right]$$ *)

Suppose that there exist predictable processes $h^1, \cdots, h^N$ such that

$$M_t = x + \sum_{i=1}^{N}\int_0^t h_s^i dZ_s^i,$$

Then X can be replicated.

*) Note that then $M_T B_T = X$ (take $t=T$)

Proof

We guess that (for replication)

$$M_t = V_t^Z = h_t^B \cdot 1 + \sum_{i=1}^{N} h_t^i Z_t^i \quad *)$$

normalized bank account

Define: h^B by

$$h_t^B = M_t - \sum_{i=1}^{N} h_t^i Z_t^i. \quad \text{(the } h^i \text{ are given)}$$

We have $M_t = V_t^Z$, and we get , by assumption,

$$dV_t^Z = dM_t = \sum_{i=1}^{N} h_t^i dZ_t^i, \quad \text{by assumption on p. 194,}$$

so the portfolio is self financing. Furthermore:

$$V_T^Z = M_T \underset{\text{def of } M_T}{=} E^Q\left[\left.\frac{X}{B_T}\right| \mathcal{F}_T\right] = \frac{X}{B_T}.$$

Hedge, as now $V_T = X$!

*) Note that then $X = V_T^Z B_T = h_T B_T + \sum_{i=1}^{N} h_T^i S_T^i$

Second Fundamental Theorem

[FTAP2]

The second most important result in arbitrage theory is the following.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

?

Proof: It is obvious (why?) that if the market is complete, then Q must be unique. The other implication is very hard to prove. It basically relies on duality arguments from functional analysis.

For all $A \in \mathcal{F}_T$, $1_A B_T$ can be hedged and hence has a unique price: for any Q:

$$\Pi_t(1_A B_T) = \frac{E^Q[1_A | \mathcal{F}_t]}{B_t}.$$

For $t=0$: unique

$$\Pi_0(1_A B_T) = E^Q[1_A | \mathcal{F}_0] = E^Q[1_A] = Q(A).$$

Here we used $\mathcal{F}_0$ is the trivial σ-algebra, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $E[X|\mathcal{F}_0] = EX$ for any X.

Black-Scholes Model

Q-dynamics (recall $Z_t = \frac{S_t}{B_t}$)

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\
dZ_t &= Z_t \sigma dW_t^Q
\end{aligned}
$$

, see p. 185.

Consider the martingale (!)

$$
M_t = E^Q\left[e^{-rT}X \middle| \mathcal{F}_t\right],
$$

Here X is an arbitrary claim

Representation theorem for Wiener processes

$$\Downarrow$$

there exists g such that

(if we know that the $\mathcal{F}_t$ are generated by W_t^Q)

$$
M_t = M(0) + \int_0^t g_s dW_s^Q.
$$

same

Thus

$$
M_t = M_0 + \int_0^t h_s^1 dZ_s,
$$

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result: from lemma on p. 194, 195:

X can be replicated using the portfolio defined by

$$
\begin{aligned}
h_t^1 &= g_t/\sigma Z_t, \\
h_t^B &= M_t - h_t^1 Z_t.
\end{aligned}
$$

Moral: The Black Scholes model is complete.

Here we didn't need (as on p. 102) that X is of the form $X = \Phi(S_T)$, but see next page(s).

Special Case: Simple Claims

Assume X is of the form $X = \Phi(S_T)$, and

martingale! $\rightarrow$ $$M_t = E^Q\left[e^{-rT}\Phi(S_T)\middle|\,\mathcal{F}_t\right],$$ normalized price!

Kolmogorov backward equation $\Rightarrow$ $M_t = f(t, S_t)$ $\Leftarrow$ (S is Q-Markov)

(as on p. 189)

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} & = & 0, \\ f(T, s) & = & e^{-rT}\Phi(s). \end{cases}$$

Homework!

Itô $\Rightarrow$ $dM_t = df(t,S_t) = f_t\,dt + f_s\,dS + \frac{1}{2} f_{ss}(dS)^2$; use $\boxed{\text{PDE}}$ to get

$$dM_t = \sigma S_t \frac{\partial f}{\partial s} dW_t^Q,$$

compare to pp 103-106,
$F(t,s) = e^{rt} f(t,s)$
$= B_t f(t,s)$
see also next page

so we know the "abstract" g_t:

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s},$$

Replicating portfolio h:

$$h_t^B = f - S_t \frac{\partial f}{\partial s},$$

$$h_t^1 = \frac{g_t}{\sigma Z_t} = \frac{S_t \frac{\partial f}{\partial s}}{Z_t} = h_t^1 = B_t \frac{\partial f}{\partial s}.$$

Interpretation: $f(t, S_t) = V_t^Z$, normalized price process of X unnormalized, nominal pricing function

Define $F(t,s)$ by

$$F(t,s) = e^{rt} f(t,s) = B_t f(t,s)$$

so $F(t, S_t) = V_t$. Then, from previous page and $\frac{\partial F}{\partial s} = B_t \frac{\partial f}{\partial s}$,

$$\begin{cases} h_t^B = \frac{F(t,S_t) - S_t \frac{\partial F}{\partial s}(t,S_t)}{B_t}, \\ h_t^1 = \frac{\partial F}{\partial s}(t, S_t) \end{cases}$$

where F solves the **Black-Scholes equation**

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T,s) = \Phi(s). \end{cases}$$

Use PDE on p. 199 and

$\frac{\partial F}{\partial t} = rB_t f + B_t \frac{\partial f}{\partial t}$, $\frac{\partial F}{\partial s} = B_t \frac{\partial f}{\partial s}$ and PDE for f

Homework!

Summary: Main Results

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q (FTAP 1)
- The market is complete $\Leftrightarrow$ Q is unique. (FTAP 2)
- Every X must be priced by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right], \text{ complete or not}$$

for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X, if X is not hedgeable
- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t[X] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

because $\Pi_t(X)$ = the value of a portfolio!

Completeness vs No Arbitrage
Rule of Thumb

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$R = \text{number of random sources}$$
$$N = \text{number of risky assets}$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim. (as on p. 108)

Rule of thumb

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

For B-S we have $N = R = 1$. Thus the Black-Scholes model is arbitrage free and complete.

Stochastic Discount Factors/

pricing formula under P

Given a model under P. For every EMM Q we define the corresponding **Stochastic Discount Factor**, or **SDF**, by

$$D_t = e^{-\int_0^t r_s ds} L_t, \quad = L_t / B_t$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

they use L_X

There is thus a one-to-one correspondence between EMMs and SDFs.

The risk neutral valuation formula for a T-claim X can now be expressed under P instead of under Q.

Proposition: With notation as above we have a pricing formula under the measure P (but note that Q is "hidden" in D_t):

$$\Pi_t[X] = \frac{1}{D_t} E^P[D_T X | \mathcal{F}_t]$$

Abstract

Proof: Bayes' formula:

Start from $\frac{\Pi_t(X)}{B_t} = E^Q\left[\frac{X}{B_T}\middle|\mathcal{F}_t\right]$

$$= \frac{E^P[X L_T / B_T | \mathcal{F}_t]}{L_t}$$ (see p. 149)

$$= E^P[X D_T | \mathcal{F}_t] / L_t$$

etc.

Martingale Property of $S \cdot D$

Proposition: If S is an arbitrary price process, then the process

$$S_t D_t$$

is a P-martingale.

Proof: Bayes' formula **again:**

Same trick: we know $E^Q\left[\frac{S_T}{B_T}\middle|\mathcal{F}_t\right]=\frac{S_t}{B_t}$

$$\frac{E^P\left[\frac{S_T}{B_T}L_T\middle|\mathcal{F}_t\right]}{L_t}=\frac{S_t}{B_t}$$

D_T

→ End of lecture 6b ←

→ start of lecture 7a ←

Continuous Time Finance

Dividends,

Forwards, Futures, and Futures Options

Ch 16 & 26

Tomas Björk

Contents

1. Dividends
2. Forward and futures contracts
3. Futures options

1. Dividends

Dividends

Black-Scholes model:

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

New feature:
The underlying stock pays **dividends**.

don't confuse with notation for discount factor on p. 204

$$
D_t = \text{The cumulative dividends over the interval } [0, t]
$$

(will depend on S_u, $u \le t$)

Interpretation:
Over the interval $[t, t+dt]$ you obtain the amount dD_t

Two cases

- Discrete dividends (realistic but messy): we skip!
- Continuous dividends (unrealistic but easy to handle).

in fact differentiable, as we shall see

Portfolios and Dividends

Consider a market with N assets.

$$
\begin{aligned}
S_t^i &= \text{price at } t\text{, of asset No } i \\
D_t^i &= \underline{\text{cumulative}} \text{ dividends for } S^i \text{ over} \\
&\quad \text{the interval } [0,t]\text{,} \quad D_0^i = 0 \\
h_t^i &= \text{number of units of asset } i \\
V_t &= \text{market value of the portfolio } h \text{ at } t
\end{aligned}
$$

Assumption: We assume that D has continuous trajctories.

differentiable

Definition: The **value process** V is defined by

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

(as before)

Self financing portfolios
in presence of dividends

Recall:

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

New **Definition:** The strategy h is **self financing** if

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$

where the **gain** process G^i is defined by

if $D_t^i \equiv 0$, we are back in old case

$$dG_t^i = dS_t^i + dD_t^i$$

extra "income", compared to S_t^i

Interpret!

Note: The definitions above rely on the assumption that D is continuous. In the case of a discontinuous D, the definitions are more complicated.

Relative weights

u_t^i = the relative share of the portfolio value, which is invested in asset No i.

$$u_t^i = \frac{h_t^i S_t^i}{V_t}$$ (as before)

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$ (previous page)

Substitute!

$$dV_t = V_t \sum_{i=1}^{N} u_t^i \frac{dG_t^i}{S_t^i}$$

[without dividends we had
$$dV_t = V_t \sum_{i=1}^{N} u_t^i \frac{dS_t^i}{S_t^i}$$]

Continuous Dividend Yield

Definition: The stock S pays a differentiable **continuous dividend yield** of q, if D has the form (with $q \geq 0$)

$$dD_t = qS_t dt$$

, here the dividend growth qS_t is proportional to S_t, with rate q.

Problem:
How does the dividend affect the price of a European Call? (compared to a non-dividend stock).

Answer:
The price is lower. (why?) you can guess....

Black-Scholes with Cont. Dividend Yield

$$
\begin{aligned}
\text{as before} \rightarrow dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
\text{new} \rightarrow dD_t &= q S_t dt
\end{aligned}
\Bigg\} \Rightarrow
$$

Gain process:

$$dG_t = (\alpha + q)S_t dt + \sigma S_t dW_t$$

Consider a fixed claim

$$X = \Phi(S_T)$$

and assume that

$$\Pi_t [X] = F(t, S_t),$$

justified by some Markov property as before.

Standard Procedure , familiar by now

- Assume that the derivative price is of the form

$$\Pi_t[X] = F(t, S_t).$$

- Form a portfolio based on underlying S and derivative F, with portfolio dynamics with SF property:

$$dV_t = V_t \left\{ u_t^S \cdot \boxed{\frac{dG_t}{S_t}} + u_t^F \cdot \frac{dF}{F} \right\}$$

new

(compare p. 212)

- Choose u^S and u^F such that the dW-term is wiped out. This gives us eventually, after computations,

$$dV_t = V_t \cdot k_t dt$$

- Absence of arbitrage implies

$$k_t = r$$

- This relation will say something about F, as before.

Value dynamics (repeat):

$$dV = V \cdot \left\{ u^S \frac{dG}{S} + u^F \frac{dF}{F} \right\},$$

applied to $F(t, S_t)$

$$dG = S(\alpha + q)dt + \sigma S dW.$$ (previous page)

From Itô we obtain

$$dF = \alpha_F F dt + \sigma_F F dW,$$

where

we use the shorthand notation

$$\begin{aligned} \alpha_F &= \frac{1}{F} \left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\}, \\ \sigma_F &= \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}. \end{aligned}$$

Collecting terms gives us

$$\begin{aligned} dV &= V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt \\ &+ V \cdot \left\{ u^S \sigma + u^F \sigma_F \right\} dW, \end{aligned}$$

Define u^S and u^F by the system

$$\begin{aligned} u^S \sigma + u^F \sigma_F &= 0, \\ u^S + u^F &= 1. \end{aligned}$$

to wipe out the Wiener term

Solution (if $\sigma_F \neq \sigma$)

$$
\begin{aligned}
u^S &= \frac{\sigma_F}{\sigma_F - \sigma}, \\
u^F &= \frac{-\sigma}{\sigma_F - \sigma},
\end{aligned}
$$

Value dynamics (dW term wiped out in previous equation):

$$
dV = V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt.
$$

Absence of arbitrage implies [usual argument]

$$
u^S(\alpha + q) + u^F \alpha_F = r,
$$

We get, using α_F and σ_F of p.216 in u^S and u^F,

$$
\frac{\partial F}{\partial t} + (r - q)S\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.
$$

(verify!)

Pricing PDE

Proposition: The pricing function F is given as the solution to the PDE

$$
\begin{cases}
\dfrac{\partial F}{\partial t} + (r-q)s\dfrac{\partial F}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 F}{\partial s^2} - rF & = & 0, \\
F(T,s) & = & \Phi(s).
\end{cases}
$$

q only here

We can now apply Feynman-Kac to the PDE in order to obtain a risk neutral valuation formula.

under a risk-neutral measure Q:

new: $dS_t = (r-q)S_t\,dt + \sigma S_t\,dW_t^Q$

$dB_t = rB_t\,dt$

$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\,\Phi(S_T)$

differences with previous results are due to q.

If $q=0$ we are back in the old situation.

→ End of lecture 7a ←

→ Start of lecture 7b ←

Risk Neutral Valuation

The pricing function has the representation, still discounting with bank account

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right],$$

where the Q-dynamics of S are given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW^Q_t.$$

Question: Which object is a martingale under the measure Q? Is it $\frac{S_t}{B_t}$ as before?

Answer should be no, as there is "q" in the equation for S_t.

Martingale Property

Proposition: Under the martingale measure Q the **normalized gain process**

$$G_t^Z = e^{-rt}S_t + \underbrace{\int_0^t e^{-ru}dD_u}_{\text{extra term compared to previous case}}$$

is a Q-martingale.

Proof: Exercise: Show $dG_t^Z = e^{-rt}\sigma S_t dW_t^Q$; no dt term

Note: The result above holds in great generality, see p. 228.

Interpretation:
In a risk neutral world, today's stock price should be the expected value of all future discounted earnings which arise from holding the stock, these include dividends,

$$S_0 = E^Q\left[\int_0^t e^{-ru}dD_u + e^{-rt}S_t\right], \geq e^{-rt}E^Q S_t \text{ ! the "old" price}$$

from Proposition upon noticing $G_0^Z = S_0$

Pricing formula

Find

Pricing formula for claims of the type

$$\mathcal{Z} = \Phi(S_T).$$

We are standing at time t, with dividend yield q. Today's stock price is s.

- Suppose that you have the pricing function

$$F^0(t,s) = \Pi_t(\mathcal{Z}), \text{ when } S_t = s.$$

  for a non dividend stock.

- Denote the pricing function for the dividend paying stock by

$$F^q(t,s)$$

Proposition: With notation as above we have

$$F^q(t,s) = F^0\left(t, se^{-q(T-t)}\right)$$

This is Exercise 16.5.

Moral

Use your old formulas, but replace today's stock price s with $se^{-q(T-t)}$.

$\Rightarrow$ explicit expressions in the Black-Scholes case

NB: $se^{-q(T-t)} \leq s$.

If $F^0(t,s)$ is increasing in s, then $F^q(t,s) \leq F^0(t,s)$; this happens for a European call.

For a European put, $F^0(t,s)$ is decreasing in s.

European Call on Dividend-Paying-Stock

$$F^q(t,s) = se^{-q(T-t)}N\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)\right\} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Compare to p. 71, and observe the role of q.

Martingale Analysis

Basic task: We have a general model for stock price S and cumulative dividends D, under P. How do we find a martingale measure Q, and exactly which objects will be martingales under Q?

needed to define a martingale measure

Main Idea: We attack this situation by reducing it to the well known case of a market without dividends. Then we apply standard techniques.

The Reduction Technique

- Consider the self financing portfolio where you keep 1 unit of the stock and invest all dividends in the bank. Denote the portfolio value by V.

- This portfolio can be viewed as a traded asset **without dividends**. (as they disappear into the bank account)

- Now apply the First Fundamental Theorem to the market (B, V) instead of the original market (B, S).

- Thus there exists a martingale measure Q such that $\frac{\Pi_t}{B_t}$ is a Q martingale for all traded assets (underlying and derivatives) without dividends.

- In particular the process

$$\frac{V_t}{B_t}$$

is a Q martingale. Next we study V.

The V Process

Let h_t denote the number of units in the bank account, where $h_0 = 0$. V is then characterized by

from previous page

$$\begin{aligned} V_t &= 1 \cdot S_t + h_t B_t && (1) \\ dV_t &= \underbrace{dS_t + dD_t}_{\text{SF condition, p. 211}} + h_t dB_t && (2) \end{aligned}$$

From (1) we obtain

$$dV_t = dS_t + h_t dB_t + B_t dh_t$$

(ordinary product rule, if dh_t makes sense)

Comparing this with (2) gives us

$$B_t dh_t = dD_t \quad \text{and} \quad dh_t = \frac{1}{B_t} dD_t .$$

Integrating this gives us

$$h_t = \int_0^t \frac{1}{B_s} dD_s$$

We thus have

$$V_t = S_t + B_t \int_0^t \frac{1}{B_s} dD_s \tag{3}$$

and the first fundamental theorem gives us the following result.

Proposition: For a market with dividends, the martingale measure Q is characterized by the fact that the **normalized gain process** $G_t^Z = \frac{V_t}{B_t}$ satisfies

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a Q martingale. (as on p. 221, but from a different argument)

Quiz: Could you have guessed the formula (3) for V?

Another "quiz": Who is you?

Continuous Dividend Yield

Model under P

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dD_t &= q S_t dt
\end{aligned}
$$

We recall (p.228, Proposition)

$$
G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s \quad = Z_t + \int_0^t q Z_s \, ds
$$

You do!

Easy calculation gives us (under P)

$$
dG_t^Z = Z_t \left(\alpha - r + q \right) dt + Z_t \sigma dW_t ,
$$

where $Z = S/B$.

Girsanov transformation $dQ = LdP$, where

$$
dL_t = L_t \varphi_t dW_t \quad \text{(for some } \varphi_t \text{)}
$$

We have

$$
dW_t = \varphi_t dt + dW_t^Q \quad \text{(see p.166)}
$$

Insert this into dG^Z

The Q dynamics for G^Z are

$$dG_t^Z = Z_t \left(\alpha - r + q + \sigma\varphi_t\right) dt + Z_t \sigma dW_t^Q$$

Martingale condition

$$\alpha - r + q + \sigma\varphi_t = 0 \qquad , \quad \alpha + \sigma\varphi_t = r - q$$

Q-dynamics of S : $dS = \alpha S_t dt + \sigma S_t \left(dW^Q + \varphi_t dt\right)$

gives

$$dS_t = S_t \left(\alpha + \sigma\varphi\right) dt + S_t \sigma dW_t^Q$$

Using the martingale condition this gives us the Q-dynamics of S as

$$dS_t = S_t \left(r - q\right) dt + S_t \sigma dW_t^Q \text{, already seen on p. 219}$$

(note again the role of the dividend rate q)

Risk Neutral Valuation

Theorem: For a T-claim X, the price process $\Pi_t[X]$ is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q[X | \mathcal{F}_t],$$

where the Q-dynamics of S are given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^Q.$$

note that q appears (only) in the Q-dynamics of S!.

→ end of lecture 7b ←

→ Start of lecture 8a ←

2. Forward and Futures Contracts

Forward Contracts

A **forward contract** on the T-claim X, **contracted at** t, is defined by the following payment scheme.

- The holder of the forward contract receives, at time T, the stochastic amount X from the underwriter.
  $\hookrightarrow \mathcal{F}_T$ - measurable
- The holder of the contract pays, at time T, the **forward price** $f(t; T, X)$ to the underwriter.
- The forward price $f(t; T, X)$ is determined at time t. (will be $\mathcal{F}_t$ - measurable)
- The forward price $f(t; T, X)$ is determined in such a way that the price of the forward contract equals zero, at the time t when the contract is made.
  (compensating cash flows, swap of products that net to zero)

General Risk Neutral Formula

Suppose we have a bank account B with dynamics

$$dB_t = r_t B_t dt, \quad B_0 = 1$$

with a (possibly stochastic) short rate r_t. Then

$$B_t = e^{\int_0^t r_s ds}$$

adapted process

and we have the following risk neutral valuation for a T-claim X

$$\Pi_t [X] = E^Q \left[\overbrace{e^{-\int_t^T r_s ds}}^{B_t / B_T} \cdot X \middle| \mathcal{F}_t \right] \qquad (*)$$

, $B_t \in \mathcal{F}_t$

$$= B_t E^Q \left[e^{-\int^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]$$

Setting $X = 1$ we have the price, at time t, of a zero coupon bond maturing at T as

$$p(t, T) = E^Q \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = B_t E^Q \left[\frac{1}{B(T)} \middle| \mathcal{F}_t \right]$$

$\frac{p(t,T)}{B(t)}$ is Q-martingale

[see also book, Section 29.1]

generalizes $e^{-r(T-t)}$!

Forward Price Formula

Theorem: The forward price of the claim X is given by

$$f(t,T) = \frac{1}{p(t,T)} E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]$$

where $p(t,T)$ denotes the price at time t of a zero coupon bond maturing at time T.

In particular, if the short rate r is deterministic we have

$$f(t,T) = E^Q \left[X \middle| \mathcal{F}_t \right]$$

note the normalization factor,

not B_t !

Proof

definition of forward contract

The net cash flow at maturity is $X - f(t,T)$. If the value of this at time t equals zero we obtain

$$\Pi_t[X] = \Pi_t[f(t,T)]$$

We have (from p. 234)

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t\right]$$

and, since $f(t,T)$ is known at t, we obviously (why?) have (see definition of $p(t,T)$ on p. 234)

$$\left(\Pi_t[X] =\right) \Pi_t[f(t,T)] = p(t,T)f(t,T).$$

This proves the main result. If r is deterministic and constant then $p(t,T) = e^{-r(T-t)}$ which gives us the second formula.

$$\Pi_t[f(t,T)] = \underbrace{f(t,T)}_{\mathcal{F}_t\text{-measurable}} \underbrace{E^Q\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right]}_{p(t,T)}.$$

Futures Contracts

A **futures contract** on the T-claim X, is a financial asset with the following properties.

$\mathcal{F}_t$-measurable

(i) At every point of time t with $0 \leq t \leq T$, there exists in the market a quoted object $F(t; T, X)$, known as the **futures price** for X at t, for delivery at T.

(ii) At the time T of delivery, the holder of the contract pays $F(T; T, X)$ and receives the claim X, both $\mathcal{F}_T$-measurable
So $F(T;T,X) = X$

(iii) During an arbitrary time interval $(s, t]$ the holder of the contract receives the amount $F(t; T, X) - F(s; T, X)$. The cashflow $F(t;T,X)$ looks like a dividend

(iv) The spot price, at any time t prior to delivery, for buying or selling the futures contract, is by definition equal to zero.

this is not the futures price

Futures Price Formula (Section 29.2)

in particular (iii)

From the definition it is clear that a futures contract is a **price-dividend pair** (S, D) with

$$S \equiv 0, \quad dD_t = dF(t,T)$$

(not of the ~~form~~ $qS_t dt$)

$D_t = F(t,T)$

From general theory, the normalized gains process (p. 228)

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a Q-martingale.

Since $S \equiv 0$ and $dD_t = dF(t,T)$ this implies that

$$\frac{1}{B_t} dF(t,T)$$

usually of the type "$h_t\, dW_t^Q$"

is a martingale increment, which implies (why?) "general theory" that $dF(t,T)$ is a martingale increment. Thus F is a Q-martingale and we have

$B_t h_t dW_t^Q =$

(in t)

martingale in t

$$F(t,T) = E^Q\left[F(T,T) \middle| \mathcal{F}_t \right] = E^Q\left[X \middle| \mathcal{F}_t \right]$$

see p. 237 (ii)

This proves:

Theorem: The futures price process is given by

$$F(t,T) = E^Q\left[X \mid \mathcal{F}_t\right].$$

(no discounting!)

Corollary. If the short rate is deterministic, then the futures and forward prices coincide.

see p. 235

3. Futures Options

Futures Options

We denote the futures price process, at time t with delivery time at T by

$$F(t, T).$$

When T is fixed we sometimes suppress it and write F_t, i.e. $F_t = F(t, T)$

not a partial derivative (in spite of same notation)

Definition:
A European futures call option, with strike price K and exercise date T, on a futures contract with delivery date T_1 will, if exercised at T, pay to the holder:

$\boxed{T < T_1}$

- The amount $F(T, T_1) - K$ in **cash**.

  $X = (F(T, T_1) - K)^+$

- A long postition in the underlying futures contract.

NB! The long position above can immediately be closed at no cost, so focus on $[F(T, T_1) - K)^+$

the spot price of a future is zero

Institutional fact:
The exercise date T of the futures option is typcally very close to the date of delivery of the underlying T_1 futures contract.

Why do Futures Options exist?

- On many markets (such as commodity markets) the futures market is much more liquid than the underlying market.
- Futures options are typically settled in **cash**. This relieves you from handling the underlying (tons of copper, hundreds of pigs, etc.). tons of potatoes
- The market place for futures and futures options is often the same. This facilitates hedging etc.

Pricing Futures Options – Black-76

We consider a futures contract with delivery date T_1 (fixed) and use the notation $F_t = F(t, T_1)$. We assume the following dynamics for F.

$$dF_t = \mu F_t dt + \sigma F_t dW_t$$

Now suppose we want to price a derivative with exercise date T with the T_1-futures price F as underlying, i.e. a claim of the form

$T_1 > T$

$$\Phi(F_T)$$

This turns out to be quite easy.

From risk neutral valuation we know that the price process $\Pi_t[\Phi]$ is of the form (by analogy, replace S_t with F_t):

$$\Pi_t[\Phi] = f(t, F_t)$$

where f is given by

$$f(t, F) = e^{-r(T-t)} E^Q_{t,F}[\Phi(F_T)]$$

conditional expectation under Q, with $F_t = F$

so it only remains to find the Q-dynamics for F.

We now recall from p.238

Proposition: The futures price process F_t is a Q-martingale.

! Thus the Q-dynamics of F are given by

$$dF_t = \sigma F_t dW^Q_t \text{ :}$$

No dt term, and

Note that the diffusion coefficient is the same for $Q \sim P$, see previous page.

We thus have

$$f(t, F) = e^{-r(T-t)} E_{t,F}^{Q} [\Phi(F_T)]$$

with Q-dynamics

$$dF_t = \sigma F_t dW_t^Q$$

from p. 230

Now recall the formula for a stock with continuous dividend yield q.

$$f(t, s) = e^{-r(T-t)} E_{t,s}^{Q} [\Phi(S_T)]$$

with Q-dynamics

$$dS_t = (r - q)S_t + \sigma S_t dW_t^Q$$

Note: If we set $q = r$ the formulas are **identical**!
and replace S_t with F_t

Pricing Formulas

Let $f^0(t,s)$ be the pricing function for the contract $\Phi(S_T)$ for the case when S is a stock without dividends. Let $f(t,F)$ be the pricing formula for the claim $\Phi(F_T)$.

Proposition: With notation as above we have
(book Prop 7.13)

$$f(t,F) = f^0(t, Fe^{-r(T-t)})$$

Moral: Reset today's futures price F to $Fe^{-r(T-t)}$ and use your formulas for stock options.

Compare to p. 222 for dividends, similar story, replace q with r.

Black-76 Formula

The price of a futures option with exercise date T and exercise price K is given by

$$c = e^{-r(T-t)} \left\{ FN[d_1] - KN[d_2] \right\}.$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t) \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

(Use the formula on p. 224 with $q = r$, and $s = F$.

→ End of lecture 8a ←

→ start of lecture 8b ←

Continuous Time Finance

Currency Derivatives

Ch 17

Tomas Björk

(Foreign eXchange markets)
(FX)

Pure Currency Contracts

Consider two markets, domestic (England) and foreign (USA).

$$
\begin{aligned}
r^d &= \text{domestic short rate} \\
r^f &= \text{foreign short rate} \\
X &= \text{exchange rate}
\end{aligned}
$$

generally different!

NB! The exchange rate X is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}}$$

If 1 EUR = 1,09 USD, then

$$X = \frac{1}{1{,}09}$$

Simple Model (Garman-Kohlhagen)

for the exchange rate

The P-dynamics are given as:

different bank accounts

$$
\begin{aligned}
dX_t &= X_t \alpha dt + X_t \sigma dW_t, \\
dB_t^d &= r^d B_t^d dt, \\
dB_t^f &= r^f B_t^f dt,
\end{aligned}
$$

in which currency?

Main Problem:
Find arbitrage free price for currency derivative, Z, of the form

$$Z = \Phi(X_T)$$

Typical example: European Call on X.

$$Z = \max\left[X_T - K, 0\right]$$

Naive idea

For the European Call, use the standard Black-Scholes formula, with S replaced by X and r replaced by r^d.

Is this OK?

"suspicious question"

NO!

WHY?

Main Idea

but asset price keeps on fluctuating

- When you buy stock you just keep the asset until you sell it. (no interest on assets)

- When you buy dollars, these are put into a bank account, giving the interest r^f.

Moral:
Buying a currency is like buying a dividend-paying stock with dividend yield $q = r^f$.

many similarities

But exchange rate keeps on fluctuating, this does NOT affect what you have on your bank account in the foreign currency.

Technique

- Transform all objects into **domestically traded** asset prices.

- Use standard techniques on the transformed model.

Transformed Market

1. Investing foreign currency in the foreign bank gives value dynamics **in foreign currency** according to

$$dB_t^f = r^f B_t^f dt.$$

2. $B_{\cancel{f}}^{f}$ units of the foreign currency is worth $X \cdot B_{\cancel{f}}^{f}$ in the domestic currency. $(X_t B_t^f)$

3. Trading in the foreign currency is equivalent to trading in a domestic market with the domestic price process

$$\tilde{B}_t^f = B_t^f \cdot X_t$$

$\leftarrow$ this is the transformation

4. Study the domestic market consisting of

$$\tilde{B}^f, \quad B^d$$

Market dynamics

Summary:

$$
\begin{aligned}
dX_t &= X_t \alpha dt + X_t \sigma dW \\
\tilde{B}_t^f &= B_t^f \cdot X_t
\end{aligned}
$$

, the domestic prices

Using Itô we have domestic market dynamics

$$
\begin{aligned}
d\tilde{B}_t^f &= \tilde{B}_t^f \left(\alpha + r^f\right) dt + \tilde{B}_t^f \sigma dW_t \\
dB_t^d &= r^d B_t^d dt
\end{aligned}
$$

you want $\frac{\tilde{B}^f}{B^d}$ to be Q-martingale

Standard results gives us Q-dynamics for domestically traded asset prices: (write down $\frac{dQ}{dP}$ on $\mathcal{F}_T$)

compare to $dS_t = rS_t dt + \sigma S_t dW_t^Q$

$$
\begin{aligned}
d\tilde{B}_t^f &= \tilde{B}_t^f r^d dt + \tilde{B}_t^f \sigma dW_t^Q \\
dB_t^d &= r^d B_t^d dt
\end{aligned}
$$

domestic interest rate!

Itô gives us Q-dynamics for $X_t = \tilde{B}_t^f / B_t^f$:

$$
dX_t = X_t(r^d - r^f)dt + X_t \sigma dW_t^Q
$$

$d\left(\frac{\tilde{B}^f}{B^f}\right) = \frac{d\tilde{B}^f}{B^f} + \tilde{B}^f d\left(\frac{1}{B^f}\right)$ (no cross terms)

Verify this!

Risk neutral Valuation

of a currency derivative

Theorem: The arbitrage free price $\Pi_t[\Phi]$ is given by $\Pi_t[\Phi] = F(t, X_t)$ where

$$F(t,x) = e^{-r^d(T-t)} E^Q_{t,x}[\Phi(X_T)]$$

The Q-dynamics of X are given by, see page 257,

$$dX_t = X_t(r^d - r^f)dt + X_t \sigma dW_t^Q$$

→ Feynman-Kac representation:

gives relation to PDE

Pricing PDE

Theorem: The pricing function F solves the boundary value problem

from the equation for X under Q

$$
\begin{aligned}
\frac{\partial F}{\partial t} + \underbrace{x(r^d - r^f)}\frac{\partial F}{\partial x} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 F}{\partial x^2} - r^d F &= 0, \\
F(T,x) &= \Phi(x)
\end{aligned}
$$

(analogy with usual B-S framework, also similarity with results for dividends)

Currency vs Equity Derivatives

Proposition: Introduce the notation:

- $F^0(t,x)$ = the pricing function for the claim $\mathcal{Z} = \Phi(X_T)$, where we interpret X as the price of an ordinary stock without dividends.
- $F(t,x)$ = the pricing function of the same claim when X is interpreted as an exchange rate.

Then the following holds

$$F(t,x) = F_0\left(t, xe^{-r^f(T-t)}\right).$$

like dividend case on p. 222
with $F^q(t,x)$ and $F^0(t,x)$
and q replaced with r^f

Currency Option Formula

The price of a European currency call is given by

$$F(t,x) = xe^{-r^f(T-t)} N[d_1] - e^{-r^d(T-t)} K N[d_2],$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{x}{K}\right) + \left(r^d - r^f + \frac{1}{2}\sigma_X^2\right)(T-t) \right\}$$

$$d_2 = d_1(t,x) - \sigma\sqrt{T-t}$$

Upon renaming the constants, this is the same formula as on p.224 for dividends.

→ End of lecture 8b ←

→ Start of lecture 9a ←

Martingale Analysis

$$
\begin{aligned}
Q^d &= \text{domestic martingale measure} \\
Q^f &= \text{foreign martingale measure}
\end{aligned}
$$

$$
L_t = \frac{dQ^f}{dQ^d}, \quad L_t^d = \frac{dQ^d}{dP}, \quad L_t^f = \frac{dQ^f}{dP}
$$

P-dynamics of X

$$
dX_t = X_t \alpha_t dt + X_t \sigma_t dW_t
$$

where α and σ are arbitrary adapted processes and W is P-Wiener.

Problem: How are Q^d and Q^f related?

through L_t of course,
but how exactly?
via L_t^d, L_t^f?

Main Idea

Fix an arbitrary foreign T-claim Z.

- Compute foreign price and change to domestic currency. The price at $t = 0$ will be

$$\Pi_0 [Z] = X_0 E^{Q^f} \left[e^{-\int_0^T r_s^f ds} Z \right]$$

foreign price

This can be written as

domestic price

$$\Pi_0 [Z] = X_0 E^{Q^d} \left[L_T e^{-\int_0^T r_s^f ds} Z \right]$$

$\Rightarrow Z \to Z X_T$

OR

- Change into domestic currency at T and then compute domestic arbitrage free price. This gives us

$$\Pi_0 [Z] = E^{Q^d} \left[e^{-\int_0^T r_s^d ds} X_T \cdot Z \right]$$

- These expressions must be equal for all choices of $Z \in \mathcal{F}_T$.

Proposition: $E[XZ] = E[YZ], \forall Z \Rightarrow X = Y$ a.s.

This returns on p. 278.

We thus obtain

$$E^{Q^d}\left[e^{-\int_0^T r_s^d ds} X_T \cdot Z\right] = X_0 E^{Q^d}\left[L_T e^{-\int_0^T r_s^f ds} Z\right]$$

for all T-claims Z. This implies the following result. (replace T with t)

Theorem: The exchange rate X is given by

$$X_t = X_0 e^{\int_0^t (r_s^d - r_s^f) ds} L_t = X_0 \frac{B_t^d}{B_t^f} \frac{L_t^f}{L_t^d}$$

alternatively by

$$X_t = X_0 \frac{D_t^f}{D_t^d}$$

where D_t^d is the domestic stochastic discount factor etc.

↳ see p.204 : $D_t^f = \frac{L_t^f}{B_t^f}$, $D_t^d = \frac{L_t^d}{B_t^d}$ etc.

Proof: The last part follows from

$$L = \frac{dQ^f}{dQ^d} = \frac{dQ^f}{dP} \bigg/ \frac{dQ^d}{dP} = \frac{L^f}{L^d}$$

and $B_t^f = \exp\left(\int_0^t r_s^f ds\right)$ etc.

Compare to general theory on p.155, Here: Q^d plays the role of P, Q^f that of Q

Q^d-Dynamics of X

positive

In particular, since L is a Q^d-martingale the Q^d dynamics of L are of the form

$$(a) \qquad dL_t = L_t \varphi_t dW_t^d$$

where W^d is Q^d-Wiener. From (Thm on p.264)

$$(b) \qquad X_t = X_0 e^{\int_0^t (r_s^d - r_s^f) ds} L_t$$

the Q^d-dynamics of X follows ~~as~~ from (a), (b) and Itô as

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \boxed{\varphi_t} dW_t^d$$

σ_t (bit of work)

Compare to p.257 (use $W^Q = W^d$) and conclude that

~~so~~ the Girsanov kernel φ equals the exchange rate volatility σ and we have the general Q^d dynamics.

(see p. 257)

Theorem: The Q^d dynamics of X are of the form

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \sigma_t dW_t^d,$$

known from p.257

Market Prices of Risk

Recall

$$D_t^d = e^{-\int_0^t r_s^d ds} L_t^d \quad \left(= \frac{L_t^d}{B_t^d} \right)$$

We also have a representation like

$$dL_t^d = L_t^d \varphi_t^d dW_t \leftarrow \text{P-Wiener process}$$

where $-\varphi_t^d = \boxed{\lambda^d}$ is the domestic market price of risk and similar for φ^f etc. From

new terminology (returns later, see p. 305 and p. 319)

$$X_t = X_0 \frac{D_t^f}{D_t^d}$$

we now easily obtain ? (exercise in Itô-calculus)

$$dX_t = X_t \alpha_t dt + X_t \left(\lambda_t^d - \lambda_t^f \right) dW_t,$$

where we do not care about the exact shape of α. We thus have

we don't specify it

Theorem: The exchange rate volatility is given by

$$\sigma_t = \lambda_t^d - \lambda_t^f,$$

a relation between volatility and market prices of risk.

Siegel's Paradox

Assume that the domestic and the foreign markets are both risk neutral and assume constant short rates. We now have the following surprising (?) argument.

A: Let us consider a T claim of 1 dollar. The arbitrage free dollar value at $t = 0$ is of course

$$e^{-r^f T}$$

so the Euro value at at $t = 0$ is given by

$$X_0 e^{-r^f T}.$$

The 1-dollar claim is, however, identical to a T-claim of X_T euros. Given domestic risk neutrality, the Euro value at $t = 0$ is then

$Q^d = P$ by assumption

$$e^{-r^d T} E^P [X_T].$$

We thus have

$$X_0 e^{-r^f T} = e^{-r^d T} E^P [X_T]$$

Siegel's Paradox ct'd

B: We now consider a T-claim of one Euro and compute the dollar value of this claim. The Euro value at $t = 0$ is of course

$$e^{-r^d T}$$

so the dollar value is (at $t=0$)

$$\frac{1}{X_0} e^{-r^d T}.$$

The 1-Euro claim is identical to a T-claim of X_T^{-1} Euros so, by foreign risk neutrality, we obtain the dollar price as $Q^f = P$

$$e^{-r^f T} E^P \left[\frac{1}{X_T} \right]$$

which gives us

$$\frac{1}{X_0} e^{-r^d T} = e^{-r^f T} E^P \left[\frac{1}{X_T} \right]$$

Siegel's Paradox ct'd

Recall our earlier results [pp 267 and 268]

$$X_0 e^{-r^f T} = e^{-r^d T} E^P [X_T]$$
$$\frac{1}{X_0} e^{-r^d T} = e^{-r^f T} E^P \left[\frac{1}{X_T} \right]$$

[multiply]

Combining these gives us

$$E^P \left[\frac{1}{X_T} \right] = \frac{1}{E^P [X_T]}$$

which, by Jensen's inequality,*) is impossible unless X_T is deterministic. This is sometimes referred to as (one formulation of) "Siegel's paradox."

It thus seems that Americans cannot be risk neutral at the same time as Europeans.

What is going on?

*) If φ is convex, then $E\varphi(X) \geq \varphi(EX)$. Example: $\varphi(x) = \frac{1}{x}$

If φ is strictly convex: strict inequality, if X not degenerate, equality iff X is degenerate: $\exists x_0 : P(X = x_0) = 1$.

Formal analysis of Siegel's Paradox

Question: Can we assume that both the domestic and the foreign markets are risk neutral?

Answer: Generally no, because of Jensen's inequality

Proof: The assumption would be equivalent to assuming the $P = Q^d = Q^f$ i.e.

$$\lambda_t^d = \lambda_t^f = 0$$ (must have $L_t^d \equiv 1 \equiv L_t^f$)

However, we know that (see p. 266)

$$\sigma_t = \lambda_t^d - \lambda_t^f$$

so we would need to have $\sigma_t = 0$ i.e. a non-stochastic exchange rate, which is not realistic.

$dX_t = \alpha_t X_t dt + 0 dt$, from p. 266

Moral / solving the paradox

The previous slide gave us the mathematical result, but the intuitive question remains why Americans cannot be risk neutral at the same time as Europeans.

The solution is roughly as follows.

- Risk neutrality (or risk aversion) is always **defined in terms of a given numeraire**. (B^d_t and B^f_t).
- It is **not** an attitude towards **risk as such**.
  but also refers to a specific market
- You can therefore **not** be risk neutral w.r.t two different numeraires at the same time unless the ratio between them is deterministic.
- In particular we cannot have risk neutrality w.r.t. Dollars and Euros at the same time.

Convincing?

If you are risk neutral in one market, you cannot be so in the other one, due to random fluctuations of the exchange rate.

→ End of lecture gaz

→ Start of lecture 9b ←

Continuous Time Finance

Change of Numeraire

Ch 26

Tomas Björk

Recap of General Theory

Consider a market with asset prices

$$S_t^0, S_t^1, \ldots, S_t^N$$

FTAP 1:

Theorem: The market is arbitrage free

iff

there exists an EMM, i.e. a measure Q such that

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^N}{S_t^0}$$

are Q-martingales.

Recap continued

Recall the normalized market

$$\left(Z_t^0, Z_t^1, \ldots Z_t^N\right) = \left(\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0} \ldots, \frac{S_t^N}{S_t^0}\right)$$

- We obviously have

$$Z_t^0 \equiv 1$$

- Thus Z^0 is a risk free asset in the normalized economy.
- Z^0 is a bank account in the normalized economy.
- In the normalized economy **the short rate is zero:**

If $\left.\begin{array}{l} dZ_t^0 = r_t Z_t^0 dt \\ Z_0^0 \equiv 1 \end{array}\right\}$ then $Z_t^0 = \exp\left(\int_0^t r_s ds\right) = 1, \forall t \geq 0$

$\Rightarrow r_s = 0, \forall s \geq 0.$

Dependence on numeraire

- The EMM Q will obviously depend on the choice of numeraire, so we should really write Q^0 to emphasize that we are using S^0 as numeraire.

- So far we have only considered the case when the numeraire asset is the bank account, i.e. when $S_t^0 = B_t$. In this case, the martingale measure Q^B is referred to as "the risk neutral martingale measure".

- Henceforth the notation Q (without upper case index) will only be used for the risk neutral martingale measure, i.e. $Q = Q^B = Q^0$.

- We will now consider the case of a general numeraire.

General change of numeraire.

- Consider a financial market, including a bank account B.

- Assume that the market is using a fixed risk neutral measure Q as pricing measure. (S_t^i / B_t are Q-martingales) if $Q = Q^B$

Alternative:

- Choose a fixed asset S as numeraire, and denote the corresponding martingale measure by Q^S. :

$\frac{S_t^i}{S_t}$ become martingales under Q^S

Problems:

- Determine Q^S, i.e. determine

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

(or determine $\frac{dQ^S}{dP}$ on $\mathcal{F}_t$)

- Develop pricing formulas for contingent claims using Q^S instead of Q.

Constructing $\mathbf{Q}^S$

Fix a T-claim X. From general theory we know that

$$\Pi_0[X] = E^Q\left[\frac{X}{B_T}\right]$$

$\left(\frac{\Pi_t(x)}{B_t} \text{ is } Q^B\text{-martingale}\right)$

Since Q^S is a martingale measure for the numeraire S, the normalized process

$$\frac{\Pi_t[X]}{S_t}$$

note the different measures and numeraires

is a Q^S-martingale. We thus have with $L_T = \frac{dQ^S}{dQ}$ on $\mathcal{F}_T$

$$\frac{\Pi_0[X]}{S_0} = E^S\left[\frac{\Pi_T[X]}{S_T}\right] = E^S\left[\frac{X}{S_T}\right] = E^Q\left[L_T\frac{X}{S_T}\right]$$

From this we obtain (use S_0 is a constant)

$$\Pi_0[X] = E^Q\left[L_T\frac{X\cdot S_0}{S_T}\right],$$

For all $X \in \mathcal{F}_T$ we thus have

$$E^Q \left[\frac{X}{B_T} \right] = E^Q \left[L_T \frac{X \cdot S_0}{S_T} \right]$$

Recall the following basic result from probability theory.
(see again p. 264)

Proposition: Consider a probability space $(\Omega, \mathcal{F}, P)$ and assume that

$$E[Y \cdot X] = E[Z \cdot X], \quad \text{for all } \cancel{Z}\ X \in \mathcal{F}. \text{ s.t. expectations exist}$$

Then we have

$$Y = Z, \quad P - a.s. \qquad \text{(Prove this!)}$$

From this result we conclude that

$$\frac{1}{B_T} = L_T \frac{S_0}{S_T}$$

Can do the same for t instead of T:

Main result

Proposition: The likelihood process for the change Q to Q^S,

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

is given by

$$L_t = \frac{S_t}{B_t} \cdot \frac{1}{S_0}$$

p. 155:
General theory says L is a Q-martingale

NB Also $L_t = \dfrac{S_t/S_0}{B_t/B_0}$ (as $B_0 = 1$)

Easy exercises

1. Convince yourself that L is a Q-martingale.
   also follows from formula of L_t, and property of Q.

2. Assume that a process A_t has the property that A_t/B_t is a Q martingale. Show that this implies that A_t/S_t is a Q^S-martingale. Interpret the result.

   Prove by Bayes rule (as one possibility)

   There is a general result in the Exercise class, Exercise 3 in the "additional exercises".

Pricing

Theorem: For every T-claim X we have the pricing formula

$$\Pi_t[X] = S_t E^S\left[\left.\frac{X}{S_T}\right| \mathcal{F}_t\right]$$

Proof: Follows directly from the Q^S-martingale property of $\Pi_t[X]/S_t$. ∎ (parallel to the usual property under Q)

Note 1: We observe S_t directly on the market.

Note 2: The pricing formula above is particularly useful when X is of the form

$$X = S_T \cdot Y$$

In this case we obtain

$$\Pi_t[S_T Y] = \Pi_t[X] = S_t E^S[Y|\mathcal{F}_t]$$

only one random variable here,

instead of the ratio $\frac{X}{S_T}$

Important example

Consider a claim of the form

$$X = \Phi\left[S_T^0, S_T^1\right]$$

We assume that Φ is **linearly homogeneous**, i.e.

$$\Phi(\lambda x, \lambda y) = \lambda \Phi(x, y), \quad \text{for all } \lambda > 0$$

Using Q^0 we obtain

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \frac{\Phi\left[S_T^0, S_T^1\right]}{\underbrace{S_T^0}} \right| \mathcal{F}_t \right]$$

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \Phi\left(1, \frac{S_T^1}{S_T^0}\right) \right| \mathcal{F}_t \right]$$

Φ is linearly homogeneous

Important example cnt'd

Proposition: For a claim of the form

$$X = \Phi\left[S_T^0, S_T^1\right],$$

where Φ is homogeneous, we have

$$\Pi_t[X] = S_t^0 E^0\left[\varphi(Z_T)\,|\,\mathcal{F}_t\right]$$

where

$$\varphi(z) = \Phi[1, z], \quad Z_t = \frac{S_t^1}{S_t^0}$$

notation for normalized process, here with S^0 as numéraire

Exchange option

has nothing to do with exchange rates

Consider an exchange option, i.e. a claim X given by

explain the name!

$$X = \max\left[S_T^1 - S_T^0,\ 0\right]$$

Since $\Phi(x, y) = \max[x - y, 0]$ is homogeneous we obtain

$$\Pi_t[X] = S_t^0 E^0\left[\max\left[Z_T - 1,\ 0\right] \middle| \mathcal{F}_t\right]$$

- This is a European Call on Z with strike price $K = 1$
- Zero interest rate. (?, what if $S_t^0 = B_t$?)
- Piece of cake!
- If S^0 and S^1 are both GBM, then so is Z and the price will be given by the Black-Scholes formula.

$Z_t = \frac{S_t^1}{S_t^0}$

dangerous statement: "product or ratio of two lognormals is lognormal again"

[why dangerous?]

Related: Is the sum of two normals again?

Identifying the Girsanov Transformation

Assume the Q-dynamics of S are known as $(S = S^1 !)$

$$dS_t = r_t S_t dt + S_t v_t dW_t^Q$$

p. 279 gives
$$L_t = \frac{S_t}{S_0 B_t}$$
$\left(\frac{dQ^S}{dQ} \text{ on } \mathcal{F}_t \right)$

From this we immediately have $\left(\text{if } dB_t = r_t B_t dt \right)$

$$dL_t = L_t v_t dW_t^Q.$$

and we can summarize.

Theorem: The Girsanov kernel is given by the numeraire volatility v_t, i.e.

S

$$dL_t = L_t v_t dW_t^Q.$$

Application: → or new theory? see also p.234.

Recap on zero coupon bonds

Recall: A zero coupon T-bond is a contract which gives you the claim

$$X \equiv 1$$

at time T.

The price process $\Pi_t[1]$ is denoted by $p(t,T)$, see also p.234

Allowing a stochastic short rate r_t we have

$$dB_t = r_t B_t dt.$$ r_t is adapted, $r_t \in \mathcal{F}_t$.

This gives us

$$B_t = e^{\int_0^t r_s ds}, \quad \in \mathcal{F}_t$$

and using standard risk neutral valuation we have

$$p(t,T) = E^Q\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = B_t E^Q\left[\frac{1}{B_T} \middle| \mathcal{F}_t \right]$$

Note:

$$p(T,T) = 1$$

Special choice of numéraire leads to

The forward measure Q^T !

- Consider a fixed T.
- Choose the bond price process $p(t,T)$ as numeraire.
- The corresponding martingale measure is denoted by Q^T and referred to as "the T-forward measure".

For any T claim X we obtain

$$\Pi_t[X] = p(t,T)E^{Q^T}\left[\left.\frac{\Pi_T[X]}{p(T,T)}\right|\mathcal{F}_t\right]$$

We have

$$\Pi_T[X] = X, \quad p(T,T) = 1$$

Theorem: For any T-claim X we have

only one random variable !

$$\Pi_t[X] = p(t,T)E^{Q^T}[X|\mathcal{F}_t]$$

"better" than $\Pi_t[X] = B_t E^Q\left[\frac{X}{B_T}\middle|\mathcal{F}_t\right]$

but also note the different expectations

and $\Pi_0[X] = p(0,T)E^{Q^T}[X]$

A general option pricing formula, by use of two different numéraires, and Q^S and Q^T

European call on asset S with strike price K and maturity T.

$$X = \max\left[S_T - K,\ 0\right]$$

Write X as and use Note 2 on p.281

$$X = (S_T - K) \cdot I\left\{S_T \geq K\right\} = S_T I\left\{S_T \geq K\right\} - K \underbrace{I\left\{S_T \geq K\right\}}_{p(T,T)\,1\{S_T \geq K\}}$$

the measure of p.276

forward measure

Use Q^S on the first term and Q^T on the second.

(and p.281)

$$\Pi_0\left[X\right] = S_0 \cdot Q^S\left[S_T \geq K\right] - K \cdot p(0,T) \cdot Q^T\left[S_T \geq K\right]$$

Exercise: find similar expression for $\Pi_t[X]$, at time t instead of at time 0.

→ End of lecture 9b ←

→ Start of lecture 10a ←

Continuous Time Finance

Incomplete Markets

Ch 15

Tomas Björk

Recall: In a complete market, every claim can be hedged (and thus has a unique price).

Metatheorem: $N < R$ (not sufficiently many risky assets) in incomplete markets.

What else?
Typical examples?

Derivatives on Non Financial Underlying

Recall: The Black-Scholes theory assumes that the market for the underlying asset has (among other things) the following properties.

- The underlying is a liquidly traded asset.
- Shortselling allowed.
- Portfolios can be carried forward in time.

There exists a large market for derivatives, where the underlying does not satisfy these assumptions.

Examples: (see next page)

- Weather derivatives.
- Derivatives on electric energy.
- CAT-bonds.

Typical Contracts

Weather derivatives:

"Heating degree days". Payoff at maturity T is given by

$$\mathcal{Z} = \max\{X_T - 30, 0\}$$

where X_T is the (mean) temperature at some place.

Electricity option:

The right (but not the obligation) to buy, at time T, at a predetermined price K, a constant flow of energy over a predetermined time interval.

CAT bond:

A bond for which the payment of coupons and nominal value is contingent on some (well specified) natural disaster to take place.

Problems

Weather derivatives:
The temperature is not the price of a traded asset.

Electricity derivatives:
Electric energy cannot easily be stored.

CAT-bonds:
Natural disasters are not traded assets.

We will treat all these problems within a **factor model**.

for the underlying

Typical Factor Model Setup

Given:

- An underlying factor process X, which is **not** the price process of a traded asset, with dynamics under the objective probability measure P as

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t.$$

- A risk free asset with dynamics

$$dB_t = rB_t dt,$$

Problem:

Find arbitrage free price $\Pi_t\left[\mathcal{Z}\right]$ of a derivative of the form

$$\mathcal{Z} = \Phi(X_T)$$

Note similarity AND difference with earlier set up.

X is NOT the price of a traded asset

Concrete Examples

Assume that X_t is the temperature at time t at the village of Peniche (Portugal).

Heating degree days:

$$\Phi(X_T) = 100 \cdot \max\{X_T - 30, 0\}$$

Holiday Insurance:

$$\Phi(X_T) = \begin{cases} 1000, & \text{if } X_T < 20 \\ 0, & \text{if } X_T \geq 20 \end{cases}$$

Question

$\Pi_t(\cancel{Z})$
$\|$

Is the price $\Pi_t[\Phi]$ uniquely determined by the P-dynamics of X, and the requirement of an arbitrage free derivatives market?

[again a question that should make you suspicious]

NO!!

WHY?

find the difference(s)
between current and
previous set ups

Stock Price Model $\overset{\text{vs}}{\sim}$ Factor Model

Black-Scholes:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$
$$dB_t = r B_t dt.$$

Factor Model: (similar equation)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$
$$dB_t = r B_t dt.$$ (r can be time dependent too!)

What is the difference?

Answer

- X is **not the price of a traded asset!**
- We can not form a portfolio based on X, hedging this way is impossible.

1. **Rule of thumb:**

$$\begin{aligned} N &= 0, \quad \text{(no risky asset)} \\ R &= 1, \quad \text{(one source of randomness, } W\text{)} \end{aligned}$$

   We have $N < R$. The exogenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price X-derivatives.

- There is **not** a unique price for a **particular** derivative. typical for incomplete markets

- In order to avoid arbitrage, **different** derivatives have to satisfy **internal consistency** relations.

- If we take **one** "benchmark" derivative as given, then all other derivatives can be priced **in terms of** the market price of the benchmark. "replaces the stock"

We consider two given claims $\Phi(X_T)$ and $\Gamma(X_T)$. We assume they are traded with prices

same underlying factor

$$
\begin{aligned}
\Pi_t[\Phi] &= f(t, X_t) \\
\Pi_t[\Gamma] &= g(t, X_t)
\end{aligned}
$$

Program:

self financing

value process V

- Form portfolio based on Φ and Γ. Use Itô on f and g to get portfolio dynamics.

self-financing condition: $$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}$$ u^f, u^g relative weights

Usual steps:

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot kdt,$$

("synthetic bank" with k as the short rate)

- Absence of arbitrage implies

$$k = r$$

- Read off the relation $k = r$!

arrying out the program:

Recall $\Pi_t[\Phi] = f(t, X_t)$, assumption

From Itô:

$$df = f\mu_f dt + f\sigma_f dW,$$

where

$$\begin{cases} \mu_f &= \frac{f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}}{f}, \\ \sigma_f &= \frac{\sigma f_x}{f}. \end{cases}$$

(see equation for dX_t)

Portfolio dynamics

$$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}.$$

Use also $\Pi_t[\Gamma] = g(t, X_t)$:

$dg = g\mu_g dt + g\sigma_g dW$

$\mu_g = \cdots, \sigma_g = \cdots$

Reshuffling terms gives us

$$dV = V \cdot \left\{u^f \mu_f + u^g \mu_g\right\} dt + V \cdot \left\{u^f \sigma_f + u^g \sigma_g\right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^f + u^g &= 1, \\ u^f \sigma_f + u^g \sigma_g &= 0. \end{cases}$$

kill dW contribution

$$\boxed{\begin{aligned} u^f &= -\frac{\sigma_g}{\sigma_f - \sigma_g}, \\ u^g &= \frac{\sigma_f}{\sigma_f - \sigma_g}, \end{aligned}}$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^f \mu_f + u^g \mu_g \right\} dt. \quad + 0 \cdot dW$$

i.e.

$$dV = V \cdot \left\{ \frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} = r$$

which can be written as

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Repeat from previous slide:

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Note!
The quotient does **not** depend upon the particular choice of contract. (same for $f, g, \ldots$)

Consider this as an internal consistency relation (see p. 300)

This relation should also hold when X is the price of a traded asset. True? Think of this!

Result

Assume that the market for X-derivatives is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

holds for all t and for every choice of contract f.

NB: The same λ for all choices of f.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{"Market Price of Risk" (cf. CAPM).} \\ &= \text{Sharpe Ratio} \end{aligned}$$

all different terminology

Slogan:
"On an arbitrage free market all X-derivatives have the same market price of risk."

The relation in short (suppress dependence on t, X_t)

$$\frac{\mu_f - r}{\sigma_f} = \lambda$$

is actually a PDE! : Look at the equations for μ_f and σ_f on p.302

→ End of lecture 10a ←

→ start of lecture 10b ←

Pricing Equation

Note that all these functions depend on t, x:
$\mu = \mu(t,x)$, $f_t = f_t(t,x)$ etc.

ETC.

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{cases}$$

Note also that μ (drift of X under P) is present in this equation, not in the Black-Scholes PDE. Why?

P-**dynamics:**

$$dX = \mu(t,X)dt + \sigma(t,X)dW.$$

Can we solve the PDE?

No!!

Why??

Answer

Recall the PDE (in shorthand notation)

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{cases}$$

- In order to solve the PDE **we need to know λ.**
- λ is not given exogenously.
- λ is not determined endogenously.

looks hopeless, way out?

Question:

Who determines λ?

Answer:

THE MARKET!

Interpreting λ

$(\Pi_t[\Phi] = f(t, X_t))$

Recall that the f dynamics are

$$df = f\mu_f dt + f\sigma_f dW_t$$

and λ is defined as

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

- λ **measures the aggregate risk aversion** in the market.
- If λ is big then the market is highly risk averse.
- If λ is zero then the market is **risk neutral**.
- If you make an assumption about λ, then you implicitly make an assumption about the aggregate risk aversion of the market. and v.v.: you may learn about λ from market (agents) behaviour, or perhaps by analyzing a particular derivative.

Moral

- Since the market is incomplete the requirement of an arbitrage free market will **not** lead to unique prices for X-derivatives.

- Prices on derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free derivative market. **All** pricing functions satisfies the **same** PDE. but with different boundary conditions
  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

depending on λ!

Risk Neutral Valuation

We recall the PDE

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{cases}$$

Using Feynman-Kac we obtain a risk neutral valuation formula.

As always: couples this PDE to an SDE for X.

Risk Neutral Valuation

$$f(t,x) = e^{-r(T-t)} E^Q_{t,x}\left[\Phi(X_T)\right]$$

Q-dynamics: Note: these are not constants, see p.293

$$dX_t = \{\mu - \lambda\sigma\}\,dt + \sigma dW^Q_t$$

- Price $=$ expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q. the?

Think of Girsanov. If L_t ($\frac{dQ}{dP}$ on $\mathcal{F}_t$) satisfies $dL_t = L_t \varphi_t dW_t$, then

$$\begin{cases} dW_t = dW_t^Q + \varphi_t\,dt \\ dX_t = \mu\,dt + \sigma\,dW_t \end{cases} \Rightarrow dX_t = (\mu + \sigma\varphi)\,dt + dW_t^Q$$

So, $\varphi = -\lambda$, as on p. 266

Note that Q depends on λ and v.v.

(see also pp. 318, 319)

Interpretation of the risk adjusted probabilities, i.e. Q

As at the beginning of the course:

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.
- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.
- This does **not** mean that we live in, or think that we live in, a risk neutral world.
- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.

Market price of risk

Diversification argument about λ

product dependent, not on market properties

- If the risk factor is **idiosyncratic** and **diversifiable**, then one can argue that the factor should not be priced by the market. Compare with APT, Arbitrage Pricing Theory, e.g. CAPM
- Mathematically this means that $\lambda = 0$, i.e. $P = Q$, i.e. **the risk neutral distribution coincides with the objective distribution**.

see p. 314

- We thus have the **"actuarial pricing formula"**

$$f(t,x) = e^{-r(T-t)} E^{P}_{t,x}\left[\Phi(X_T)\right] \quad (\text{so, if } P=Q)$$

where we use the objective probabiliy measure P.

Modeling Issues

Temperature:

A standard model is given by (Ornstein-Uhlenbeck process)

$$dX_t = \{m(t) - bX_t\}\,dt + \sigma dW_t,$$

where m is the mean temperature capturing seasonal variations. This often works reasonably well. Here X_t has a normal distribution if X_0 is normal (independent of W)

Electricity:

A (naive) model for the spot electricity price is

$$dS_t = S_t\,\{m(t) - a\ln S_t\}\,dt + \sigma S_t dW_t$$

Use Itô to show: $\log S_t$ is an OU process

This implies lognormal prices (why?). Electricty prices are however very $\boxed{\text{far from lognormal}}$, because of "spikes" in the prices. Complicated. in reality!

CAT bonds:

Here we have to use the theory of point processes and the theory of extremal statistics to model natural disasters. Complicated. natural concept here, like Poisson process; beyond scope of this course

Martingale Analysis

(like in p. 314)

Model: Under P we have

$$
\begin{aligned}
dX_t &= \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\
dB_t &= rB_t dt,
\end{aligned}
$$

We look for martingale measures. Since B is the only traded asset we need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q martingale.

Result: In this model, **every** $Q \sim P$ is a martingale measure.

Girsanov

$$dL_t = L_t \varphi_t dW_t$$

P-dynamics

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \qquad \text{(a)}$$

$$dL_t = L_t \varphi_t dW_t$$

$dQ = L_t dP$ on $\mathcal{F}_t$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q \qquad \text{(b)}$$

Martingale pricing: for appropriate T-claim Z

$$F(t, x) = e^{-r(T-t)} E^Q\left[Z \mid \mathcal{F}_t\right]$$

Q-dynamics of X: (Combine (a) and (b)):

$$dX_t = \{\mu(t, X_t) + \sigma(t, X_t)\varphi_t\}\,dt + \sigma(t, X_t)\,dW_t^Q,$$

Result: We have $\lambda_t = -\varphi_t$, i.e,. the Girsanov kernel φ equals minus the market price of risk.

Several Risk Factors

We recall the dynamics of the f-derivative

$$df = f\mu_f dt + f\sigma_f dW_t$$

also $df = f(\mu_f + \sigma_f \varphi)dt + f\sigma_f dW^Q$

and the Market Price of Risk

$$\frac{\mu_f - r}{\sigma_f} = \lambda, \quad \text{i.e.} \quad \mu_f - r = \lambda\sigma_f.$$

In a multifactor model of the type

$$dX_t = \mu(t, X_t)\, dt + \sum_{i=1}^{n} \sigma_i(t, X_t)\, dW_t^i,$$

for multiple Wiener processes (not treated)

it follows from Girsanov that for every risk factor W^i there will exist a market price of risk $\lambda_i = -\varphi_i$ such that

New equation: $$\mu_f - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Compare with CAPM. (if you know what that is)

→ End of lecture 10b ←

→ Start of lecture 11a ←

Continuous Time Finance

Stochastic Control Theory

Ch 19

Tomas Björk

→ Theory of stochastic control

Financial applications in lecture 12 (optimal investment and consumption)

Contents

1. Dynamic programming. (mathematical background)
2. Investment theory. (economic application, next lecture)

1. Dynamic Programming

- The basic idea.
- Deriving the HJB equation.
- The verification theorem.
- The linear quadratic regulator. (classic example from systems theory)

Problem Formulation

$$\max_u \quad E\left[\int_0^T F(t, X_t, u_t)dt + \Phi(X_T)\right]$$

subject to

Markov process for fixed u, $X_t = X_t^u$

$$\begin{aligned} dX_t &= \mu(t, X_t, u_t)\, dt + \sigma(t, X_t, u_t)\, dW_t \\ X_0 &= x_0, \\ u_t &\in U(t, X_t), \quad \forall t. \end{aligned}$$

often $\equiv U$ (for all t, x)

We will only consider **feedback control laws**, i.e. controls of the form

$$u_t = \mathbf{u}(t, X_t)$$

partly justified by previous Markov property

X^u will still be Markov

Terminology:

X	=	state variable
u	=	control variable
U	=	control constraint

$X_t \in R^n$

$u_t \in R^k$

the set U; $U \subset R^k$

Note: No state space constraints. ($e.g.\ X_t \geq 0$)

Main idea

- Embedd the problem above in a family of problems indexed by starting point in time and space. See p. 328

- Tie all these problems together by a PDE: the Hamilton Jacobi Bellman equation. See p. 338

HJB

- The control problem is reduced to the problem of solving the deterministic HJB equation.

can be a very complicated equation,

but it gives a way to "compute" the solution to the original problem, and in a way, it is an "easier" problem

Some notation

(looks messy, but try to see 'through')

- For any fixed vector $u \in R^k$, the functions μ^u, σ^u and C^u are defined by

$$
\left.\begin{aligned}
\mu^u(t,x) &= \mu(t,x,u), \\
\sigma^u(t,x) &= \sigma(t,x,u), \\
C^u(t,x) &= \sigma(t,x,u)\sigma(t,x,u)'.
\end{aligned}\right\} \text{functions of 3 variables}
$$

i.e. $u_t = \mathbf{u}(t,x)$

- For any control law $\mathbf{u}$, the functions $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$, $C^{\mathbf{u}}(t,x)$ and $F^{\mathbf{u}}(t,x)$ are defined by

($\mathbf{u}$ is a function)

these are functions of two variables: t, x But there is also $\mathbf{u}$

$$
\left\{\begin{aligned}
\mu^{\mathbf{u}}(t,x) &= \mu(t,x,\mathbf{u}(t,x)), \\
\sigma^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x)), \\
C^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x))\sigma(t,x,\mathbf{u}(t,x))', \\
F^{\mathbf{u}}(t,x) &= F(t,x,\mathbf{u}(t,x)).
\end{aligned}\right.
$$

Itô: $df(t,X_t) = f_t(t,X_t)dt + \underbrace{\mathcal{A}^u f(t,X_t)}dt + f_x(t,X_t)\sigma(t,X_t)dW_t$

More notation

(to confuse you more)

- For any fixed vector $u \in R^k$, the partial differential operator $\mathcal{A}^u$ is defined by

$$\mathcal{A}^u = \sum_{i=1}^{n} \mu_i^u(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}^u(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

(Generator of X^u)

- For any control law $\mathbf{u}$, the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^{n} \mu_i^{\mathbf{u}}(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}^{\mathbf{u}}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law $\mathbf{u}$, the process $X^{\mathbf{u}}$ is the solution of the SDE

notation!

$$dX_t^{\mathbf{u}} = \underbrace{\mu\left(t, X_t^{\mathbf{u}}, \mathbf{u}_t\right)}_{\mu^{\mathbf{u}}(t,X_t^{\mathbf{u}})} dt + \sigma\left(t, X_t^{\mathbf{u}}, \mathbf{u}_t\right) dW_t,$$

where

$$\mathbf{u}_t = \mathbf{u}(t, X_t^{\mathbf{u}})$$

So: $dX_t^{\mathbf{u}} = \mu(t, X_t^{\mathbf{u}}, \mathbf{u}(t,X_t^{\mathbf{u}}))dt + \sigma(\cdot\cdot)dW_t$

is of the form

$= \tilde{\mu}(t, X_t^{\mathbf{u}})dt + \sigma(t, X_t^{\mathbf{u}})dW_t \Rightarrow$ Markov solution

Embedding the problem of p.324 into a family of problems $\mathcal{P}_{t,x}$

For every fixed (t, x) the control problem $\mathcal{P}_{t,x}$ is defined as the problem to maximize [from initial time t, and initial value x: $X_t^u = x$]

$$E_{t,x}\left[\int_t^T F(s, X_s^{\mathbf{u}}, u_s)ds + \Phi\left(X_T^{\mathbf{u}}\right)\right],$$

$= E\left[\int_t^T F ds + \Phi \,\middle|\, \mathcal{F}_t\right]_{X_t^u = x} = E\left[\int_t^T F ds + \Phi \,\middle|\, X_t^u = x\right]$

Markov

given the dynamics

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + \sigma\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) dW_s, \\ X_t^{u} &= x, \end{aligned}$$

and the constraints

$$\mathbf{u}(s, y) \in U, \quad \forall (s, y) \in [t, T] \times R^n.$$

The original problem was $\mathcal{P}_{0,x_0}$, as special instance of $\mathcal{P}_{t,x}$ with $t=0$, $x = x_0$

The optimal value function

- The **value function**

$$\mathcal{J} : R_+ \times R^n \times \mathcal{U} \to R$$

is defined by (recall x, the initial value at time t)

$$\mathcal{J}(t, x, \mathbf{u}) = E\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi\left(X_T^{\mathbf{u}}\right)\right]$$

given the dynamics above. (and $X_t^{\mathbf{u}} = x$)

Note: in fact $X_s^{\mathbf{u}}$ also depends on x for all $s \geq t$; should write $X_s^{\mathbf{u},x}$, but more cumbersome

- The **optimal value function**

$$V : R_+ \times R^n \to R$$

is defined by (recall we want to maximize)

$$V(t, x) = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t, x, \mathbf{u}).$$ Note: $V(T,x) = \Phi(x)$

Our aim:

- We want to derive a PDE for V.

If sup is attained, then there is some $\mathbf{u} = \hat{\mathbf{u}} = \hat{\mathbf{u}}(t,x)$

Assumptions

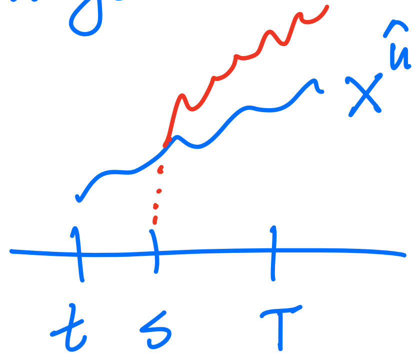
We assume:

- There exists an optimal control law $\hat{\mathbf{u}}$. ($\rightarrow$ $\hat{u}(t,x)$, $\hat{u}(t, X_t)$)
- The optimal value function V is regular in the sense that $V \in C^{1,2}$.
- A number of limiting procedures in the following arguments can be justified. We will make big steps and ignore many mathematical details that would require a finer analysis; beyond the scope and aims of this course.

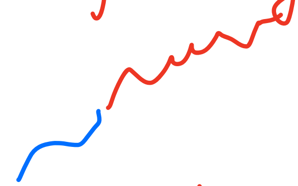
Bellman Optimality Principle

Theorem: If a control law $\hat{\mathbf{u}}$ is optimal for the time interval $[t, T]$ then it is also optimal for all smaller intervals $[s, T]$ where $s \geq t$.

Proof: Exercise. ☺ Use a concatenation argument: other trajectory, that is supposed to have a better performance on $[S, T]$ than $\hat{\mathbf{u}}$.



But then combined trajectory



would have a better performance on $[t, T]$: contradiction

Basic strategy

HJB

To derive the PDE do as follows:

- Fix $(t, x) \in (0, T) \times R^n$.
- Choose a real number h (interpreted as a "small" time increment).
- Choose an arbitrary control law $\mathbf{u}$ on the time inerval $[t, t+h]$.

t

Now define the control law $\mathbf{u}^\star$ by

$$\mathbf{u}^\star(s, y) = \begin{cases} \mathbf{u}(s, y), & (s, y) \in [t, t+h] \times R^n \\ \hat{\mathbf{u}}(s, y), & (s, y) \in (t+h, T] \times R^n. \end{cases}$$

In other words, if we use $\mathbf{u}^\star$ then we use the arbitrary control $\mathbf{u}$ during the time interval $[t, t+h]$, and then we switch to the optimal control law during the rest of the time period.

Note that u^* is worse than $\hat{u}$ on $[t, T]$

Basic idea

The whole idea of DynP boils down to the following procedure.

- Given the point (t, x) above, we consider the following two strategies over the time interval $[t, T]$:

  I: Use the optimal law $\hat{\mathbf{u}}$.

  II: Use the control law $\mathbf{u}^\star$ defined above on p. 332

  [if you can]

- Compute the expected utilities obtained by the respective strategies.

- Using the obvious ! fact that $\hat{\mathbf{u}}$ is at least as good as $\mathbf{u}^\star$, and letting h tend to zero, we obtain our fundamental PDE. (in this step we will reason rather heuristically)

→ End of lecture 11a ←

→ Start of lecture 11b ←

Strategy values

I: **Expected utility for $\hat{\mathbf{u}}$:**

$$\mathcal{J}(t, x, \hat{\mathbf{u}}) = V(t, x)$$

(p. 329, definition of V)

II: **Expected utility for $\mathbf{u}^\star$:** Split the time interval $[t,T]$:

- The expected utility for $[t, t+h)$ is given by

$$E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds\right].$$

- Conditional expected utility over $[t+h, T]$, given (t, x):

$$E_{t,x}\left[V(t+h, X_{t+h}^{\mathbf{u}})\right].$$

starting point at time t+h reached from $X_t = x$ by applying u

- Total expected utility for Strategy II is

$$V_{II} = E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V(t+h, X_{t+h}^{\mathbf{u}})\right]. \quad (*)$$

Comparing strategies

(the math here is a bit sloppy)

We have trivially (V results from optimal $\hat{u}$, strategy I is optimal)

$$(**)\quad V(t,x) \geq E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V(t+h, X_{t+h}^{\mathbf{u}})\right] = V_{\mathrm{II}}$$

Remark (trivial)

We have equality above if and only if the control law $\mathbf{u}$ is the optimal law $\hat{\mathbf{u}}$.

Now use Itô to obtain

$$V(t+h, X_{t+h}^{\mathbf{u}}) = V(t,x)$$

$$+\int_t^{t+h}\left\{\frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}} V(s, X_s^{\mathbf{u}})\right\} ds$$

$$+\underbrace{\int_t^{t+h} \nabla_x V(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s}_{\text{has conditional expectation } E[\cdot\cdot|\mathcal{F}_t]=0},$$

if a true martingale (needs conditions)

and plug into the formula above.

(*) on p.334

We obtain, also using (**),

$$E_{t,x}\left[\int_t^{t+h}\left\{F\left(s,X_s^{\mathbf{u}},\mathbf{u}_s\right)+\frac{\partial V}{\partial t}(s,X_s^{\mathbf{u}})+\mathcal{A}^{\mathbf{u}}V(s,X_s^{\mathbf{u}})\right\}ds\right]\le 0.$$

Going to the limit:
Divide by h, move h within the expectation and let h tend to zero.
We get, with $X_t^u = x$:

$$F(t,x,u)+\frac{\partial V}{\partial t}(t,x)+\mathcal{A}^{u}V(t,x)\le 0,$$

$\left(\frac{1}{h}\int_t^{t+h} g(s)ds \rightarrow g(t) \text{ for continuous } g\right)$

Recall from previous slide:

$$F(t,x,u) + \frac{\partial V}{\partial t}(t,x) + \mathcal{A}^u V(t,x) \leq 0,$$

This holds for all $u = \mathbf{u}(t,x)$, with equality if and only if $\mathbf{u} = \hat{\mathbf{u}}$.

We thus obtain the **HJB equation**

$$\frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0.$$

The HJB equation

Theorem:
Under suitable regularity assumptions the follwing hold:

I: V satisfies the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left\{ F(t,x,u) + \mathcal{A}^u V(t,x) \right\} &= 0, \\ V(T,x) &= \Phi(x), \end{aligned}$$

II: For each $(t,x) \in [0,T] \times R^n$ the supremum in the HJB equation above is attained by $u = \hat{\mathbf{u}}(t,x)$, i.e. by the optimal control.

Logic and problem

Note: We have shown that **if** V is the optimal value function, and **if** V is regular enough, **then** V satisfies the HJB equation. The HJB eqn is thus derived as a **necessary** condition, and requires strong *ad hoc* regularity assumptions, alternatively the use of viscosity solutions techniques.

Problem: Suppose we have solved the HJB equation. Have we then found the optimal value function and the optimal control law? In other words, is HJB a **sufficient** condition for optimality.

Answer: Yes! This follows from the **Verification Theorem**.

The Verification Theorem

Suppose that we have two functions $H(t,x)$ and $g(t,x)$, such that

- H is sufficiently integrable, and solves the HJB equation

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\} & = & 0, \\ H(T,x) & = & \Phi(x), \end{cases}$$

- For each fixed (t,x), the supremum in the expression

$$\sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\}$$ ← static problem for each (t,x)

  is attained by the choice $u = g(t,x)$.

Then the following hold.

1. The optimal value function V to the initial control problem is given by

$$V(t,x) = H(t,x),$$ the function that solves HJB

2. There exists an optimal control law $\hat{\mathbf{u}}$, and in fact

$$\hat{\mathbf{u}}(t,x) = g(t,x)$$

Proof perhaps (see book pp 291 -293)

Handling the HJB equation (Section 19.4)

1. Consider the HJB equation for V.
2. Fix $(t, x) \in [0, T] \times R^n$ and solve, the static optimization problem

(maximizer exists?) $$\max_{u \in U} \quad [F(t, x, u) + \mathcal{A}^u V(t, x)]$$ of slide 340

   Here u is the only variable, whereas t and x are fixed parameters. The functions F, μ, σ and V are considered as given.
3. The optimal $\hat{u}$, will depend on t and x, and on the function V and its partial derivatives. We thus write $\hat{u}$ as

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, x; V). \tag{4}$$

4. The function $\hat{\mathbf{u}}(t, x; V)$ is our candidate for the optimal control law, but since we do not know V this description is incomplete. Therefore we substitute the expression for $\hat{u}$ into the PDE , giving us the highly nonlinear (why?) PDE

$$\begin{aligned} \frac{\partial V}{\partial t}(t, x) + F^{\hat{\mathbf{u}}}(t, x) + \mathcal{A}^{\hat{\mathbf{u}}}(t, x) V(t, x) &= 0, \\ V(T, x) &= \Phi(x). \end{aligned}$$

   if you can!

5. Now we solve the PDE above! Then we put the solution V into expression (4). Using the verification theorem we can identify V as the optimal value function, and $\hat{u}$ as the optimal control law.

Does this work in concrete situations?

Making an Ansatz

- The hard work of dynamic programming consists in solving the highly nonlinear HJB equation
- There are no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed.
- In an actual case one usually tries to **guess** a solution, i.e. we typically make a parameterized **Ansatz** for V then use the PDE in order to identify the parameters.
- **Hint:** V often inherits some structural properties from the boundary function Φ as well as from the instantaneous utility function F. (this is experience)
- Most of the known solved control problems have, to some extent, been "rigged" in order to be analytically solvable.

EXAMPLE

standard, classical problem in systems & contr

The Linear Quadratic Regulator

(section 19.5)

Now: min!

$$\min_{u \in R} \quad E\left[\int_0^T \left\{QX_t^2 + Ru_t^2\right\} dt + HX_T^2\right], \qquad X_t \in R^1$$

with dynamics

$$dX_t = \{AX_t + Bu_t\} dt + CdW_t. \quad \text{multidimensional}$$

for each fixed u_t, this gives Gaussian X_t, OU process, LQG control problem

Example: We want to control a vehicle in such a way that it stays close to the origin (the terms Qx^2 and Hx^2) while at the same time keeping the "energy" Ru^2 small.

Here $X_t \in R$ and $\mathbf{u}_t \in R$, and we impose no control constraints on u.

The real numbers Q, R, H, A, B and C are assumed to be known. We assume that R is strictly positive.

Handling the Problem

The HJB equation becomes (use the generator: $Af = f_x \mu + \frac{1}{2} f_{xx} \sigma^2$ for $dX = \mu\, dt + \sigma\, dW$)

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) & + \quad \inf_{u \in R} \left\{ Qx^2 + Ru^2 + V_x(t,x)\left[Ax + Bu\right] \right\} \\ & + \quad \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t,x) C^2 = 0, \\ V(T,x) & = \quad Hx^2. \end{cases}$$

$\frac{\partial V}{\partial x}$

For each fixed choice of (t,x) we now have to solve the static unconstrained optimization problem to minimize

$$Qx^2 + Ru^2 + V_x(t,x)\left[Ax + Bu\right].$$

The problem was:

$$\min_u \quad Qx^2 + Ru^2 + V_x(t,x)\left[Ax + Bu\right].$$

Since $R > 0$ we set the u-derivative to zero and obtain

$$2Ru = -V_x B,$$

which gives us the optimal u as

$$\hat{u} = -\frac{1}{2}\frac{B}{R}V_x.$$

Note: This is our candidate of optimal control law, but it depends on the unkown function V.

We now make an educated guess about the structure of V.

From the boundary function Hx^2 and the term Qx^2 in the cost function we make the Ansatz

Note $V(T,x) = Hx^2$

$$V(t,x) = P(t)x^2 + q(t),$$

where $P(t)$ and $q(t)$ are deterministic functions. to be found later

With this trial solution we have,

$$\begin{aligned}
\frac{\partial V}{\partial t}(t,x) &= \dot{P}x^2 + \dot{q}, \\
V_x(t,x) &= 2Px, \qquad (P = P(t) \text{ etc.}) \\
V_{xx}(t,x) &= 2P \\
\hat{u} &= -\frac{B}{R}Px. \qquad \text{(see p. 345)}
\end{aligned}$$

Inserting these expressions into the HJB equation we get

$$x^2 \left\{ \dot{P} + Q - \frac{B^2}{R}P^2 + 2AP \right\}$$

$$\cancel{+\dot{q}PC^2 = 0.} \qquad + \dot{q} + PC^2 = 0, \ \forall x!$$

We thus get the following ODE for P

$$\begin{cases} \dot{P} &= \frac{B^2}{R}P^2 - 2AP - Q, \\ P(T) &= H. \end{cases}$$

from $V(T,x) = Hx^2$

and we can integrate directly for q:

$$\begin{cases} \dot{q} &= -C^2 P, \\ q(T) &= 0. \end{cases}$$

The ODE for P is a **Riccati equation**. The equation for q can then be integrated directly, once you have P

Final Result for LQ: (note that P is not given explicitely)

$$V(t,x) = P(t)x^2 + \int_t^T C^2 P(s)ds,$$ ← verify!

$$\hat{\mathbf{u}}(t,x) = -\frac{B}{R}P(t)x,$$ this is a linear feedback law

→ End of lecture 11b ←

→ Start of lecture 12a ←

Back to finance:

2. Investment Theory

(Section 19.6)

- Problem formulation.
- An extension of HJB.
- The simplest consumption-investment problem.
- The Merton fund separation results.

Recap of Basic Facts

We consider a market with n assets.

$$
\begin{aligned}
S_t^i &= \text{price of asset No } i, \\
h_t^i &= \text{units of asset No } i \text{ in portfolio} \\
w_t^i &= \text{portfolio weight on asset No } i \quad \text{(previously } u_t^i\text{)} \\
X_t &= \text{portfolio value} \quad \text{[previously denoted } V_t\text{]} \\
c_t &= \text{consumption rate} \quad \geq 0
\end{aligned}
$$

$\boxed{\text{New concept}} \rightarrow c_t$

We have the relations

$$X_t = \sum_{i=0}^{n} h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=0}^{n} w_t^i = 1.$$

(The 0-asset is the bank account in what follows)

Basic equation: when consumption is present:
Dynamics of self financing portfolio in terms of relative weights

$$dX_t = X_t \sum_{i=0}^{n} w_t^i \frac{dS_t^i}{S_t^i} - c_t dt$$

(opposite to dividends, see pp. 212, 214, now we have a "minus term")

Simplest model (n=1)

Assume a scalar risky asset and a constant short rate.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t \\ dB_t &= r B_t dt \end{aligned}$$

We want to maximize expected utility of consumption over time

$\boxed{\text{note the control variables } w^0, w^1, c}$

utility function $F(t;)$

$$\max_{w^0, w^1, c} \quad E\left[\int_0^T F(t, c_t) dt\right]$$

(also a term $\Phi(T, X_T)$ can be included)

Dynamics

$$dX_t = X_t \left[w_t^0 r + w_t^1 \alpha\right] dt - c_t dt + w_t^1 \sigma X_t dW_t,$$

(verify!)

Constraints

$$\begin{aligned} c_t &\geq 0, \ \forall t \geq 0, \\ w_t^0 + w_t^1 &= 1, \ \forall t \geq 0. \end{aligned}$$

Sensible problem (formulation)?

.... become suspicious ....

Nonsense!

What are the problems?

- We can obtain unlimited utility by simply consuming arbitrary large amounts.

- The wealth will go negative, but there is nothing in the problem formulations which prohibits this.

- We would like to impose a constraint of type $X_t \geq 0$ but this is a **state constraint** and DynP does not allow this. (see p. 324)

  [leading to HJB]

Good News:
DynP can be generalized to handle (some) problems of this kind.

The use of stopping times helps!

Generalized problem

Let D be a nice open subset of $[0,T] \times R^n$ and consider the following problem.

note τ, $\boxed{\text{stopping time}}$

$$\max_{u \in U} \quad E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi\left(\tau, X_\tau^{\mathbf{u}}\right)\right].$$

Dynamics:

$$\begin{aligned} dX_t &= \mu\left(t, X_t, u_t\right) dt + \sigma\left(t, X_t, u_t\right) dW_t, \\ X_0 &= x_0 \in D. \end{aligned}$$

The **stopping time** τ is defined by

$$\tau = \inf\left\{t \geq 0 \,|(t, X_t) \in \partial D\right\} \wedge T. \quad \leq T \quad !$$

a random time!

boundary of D

So, the problem looks as before, but with the difference that the horizon T is random!

Generalized HJB

Theorem: Given enough regularity the follwing hold.

as before

boundary condition on ∂D, not just at $t=T$

1. The optimal value function satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} = 0, & \forall (t,x) \in D \\ V(t,x) = \Phi(t,x), & \boxed{\forall (t,x) \in \partial D.} \end{cases}$$

2. We have an obvious verification theorem.

replace e.g. $H(T,x)$ on p.340 by $H(t,x) = \Phi(t,x)$, $\forall (t,x) \in \partial D$

Reformulated problem

or c_T? now a "premium" on remaining wealth

$$\max_{c \geq 0,\ w \in R} \quad E\left[\int_0^\tau F(t, c_t)dt + \Phi(X_T)\right]$$

open set in $[0,T] \times \mathbb{R}$

The "ruin time" τ is defined by corresponds to $D = [0,T) \times (0,\infty)$

$$\tau = \inf\{t \geq 0 \,|X_t = 0\} \wedge T.$$

$t \leq \tau \Rightarrow X_t \geq 0$!

$\partial D = \{(t,x) \in [0,T] \times [0,\infty) : t = T \text{ or } x = 0\}$

Notation:

$$\begin{aligned} w^1 &= w, \\ w^0 &= 1 - w \end{aligned}$$

Thus no constraint on w, only w and c remain as controls.

Dynamics of simple model on p.350 become

$$dX_t = w_t \left[\alpha - r\right] X_t dt + (rX_t - c_t)\, dt + w\sigma X_t dW_t,$$

now c_t, w_t are the control variables for optimal $\hat{c}, \hat{w}$: $V_t + F(t,\hat{c}) + \hat{w}\, x(\alpha - r)\frac{\partial V}{\partial x}(\cdot\cdot) + \cdots = 0 \quad (*)$

HJB Equation

$$\frac{\partial V}{\partial t} + \sup_{c\geq 0, w\in R}\left\{F(t,c) + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2}\right\} = 0,$$

$\Phi = 0 \Longrightarrow$

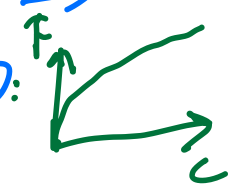
$$\left\{\begin{array}{rcl} V(T,x) & = & 0, \\ V(t,0) & = & 0. \end{array}\right.$$

?

We now specialize (why?) to

$$F(t,c) = e^{-\delta t}c^{\gamma},$$

with $0<\gamma<1 \Longrightarrow F_c(t,0) = +\infty \Longrightarrow$ optimal $c > 0$:



and for simplicity we assume that

$$\Phi = 0,$$

so we have to maximize

$$e^{-\delta t}c^{\gamma} + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

w.r.t. $c \geq 0$ and $w \in R$.

Analysis of the HJB Equation

In the embedded static problem we maximize, over c and w, (repeat from p.356)

$$G(c,w) := e^{-\delta t}c^{\gamma} + wx(\alpha - r)V_x + (rx - c)V_x + \frac{1}{2}x^2w^2\sigma^2V_{xx},$$

First order conditions:

$$
\begin{aligned}
&(1) \quad & \gamma c^{\gamma-1} &= e^{\delta t}V_x, & \left(\text{from } \frac{\partial G}{\partial c} = 0\right)\\
&(2) \quad & w &= \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, & \left(\text{from } \frac{\partial G}{\partial w} = 0\right)
\end{aligned}
$$

Ansatz:

$$V(t,x) = e^{-\delta t}h(t)x^{\gamma}, \quad (\text{like } F(t,c))$$

Because of the boundary conditions, we must demand that $\Phi = 0$

$$h(T) = 0. \qquad (5)$$

alternatively, you could try $V(t,x) = k(t)x^{\gamma}$, should work as well.

Given a V of this form we have (using $\cdot$ to denote the time derivative)

[note: $h = h(t)$]

$$
\begin{aligned}
V_t &= e^{-\delta t}\dot{h}x^{\gamma} - \delta e^{-\delta t}hx^{\gamma}, \\
V_x &= \gamma e^{-\delta t}hx^{\gamma-1}, \\
V_{xx} &= \gamma(\gamma-1)e^{-\delta t}hx^{\gamma-2}.
\end{aligned}
$$

giving us

use p.357, (2): $\widehat{w}(t,x) = \dfrac{\alpha - r}{\sigma^2(1-\gamma)},$ (constant!)

use p.357, (1): $\widehat{c}(t,x) = xh(t)^{-1/(1-\gamma)}.$ (linear in x)

Plug all this into HJB! and try to solve

(*) on ~~top~~ of p. 356, or the one on the bottom with $\hat{w}$ and $\hat{c}$ etc.

tedious work, but you can check!

After rearrangements we obtain

$$x^{\gamma}\left\{\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)}\right\}=0,$$

where the constants A and B are given by

more tedious computations,

$$\begin{aligned} A &= \frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}+r\gamma-\frac{1}{2}\frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}-\delta = \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\gamma-\delta \\ B &= 1-\gamma. \end{aligned}$$

If this equation is to hold for all x and all t, then we see that h must solve the ODE

$$\begin{aligned} \dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)} &= 0, \\ h(T) &= 0. \end{aligned}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly. See Exercises 19.2, 19.3

We are done.

→ End of lecture 12a

→ Start of lecture 12b

Merton's Mutal Fund Theorems

Section 19.7

1. The case with no risk free asset

We consider n risky assets with dynamics

$$dS_i = S_i \alpha_i dt + S_i \sigma_i dW, \quad i = 1, \ldots, n \quad , \sigma_i \in \mathbb{R}^{1 \times k}$$

where W is Wiener in R^k. On vector form:

means every component in Wiener and they are all independent

$$dS = D(S)\alpha dt + D(S)\sigma dW.$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \text{ and } \sigma = \begin{bmatrix} - \sigma_1 - \\ \vdots \\ - \sigma_n - \end{bmatrix} \in \mathbb{R}^{n \times k}$$

$D(S)$ is the diagonal matrix

$$D(S) = diag[S_1, \ldots, S_n]. \in \mathbb{R}^n$$

$$= \begin{bmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_n \end{bmatrix}$$

Formal problem

[Note $\Phi(\cdot\cdot) = 0$]

$$\max_{c,w} \quad E\left[\int_0^{\tau} F(t, c_t)dt\right]$$

given the dynamics (use the ST condition on p.349)

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW,$$

additional to the dS_t^i equations

and constraints

$$\sum_{i=1}^{n} w_t^i = e'w_t = 1, \quad c_t \geq 0.$$

$w_t = (w_t^1, \dots, w_t^n)'$

Assumptions:

- The vector α and the matrix σ are constant and deterministic.
- The volatility matrix σ has full row rank so $\sigma\sigma'$ is positive definite and invertible. $\Rightarrow$ arbitrage free and complete market if $n=k$

Note: S does not (explicitly) turn up in the X-dynamics so V is of the form

$$\underbrace{V(t, x, s)}_{\text{would result from}} = V(t, x)$$

$d\begin{pmatrix} X_t \\ S_t \end{pmatrix} = \dots dt + \dots dW_t$

combined state vector

The HJB equation is

Note that the optimal c,w will depend on time

$$\begin{cases} V_t(t,x) + \sup_{e'w=1,\ c\geq 0} \{F(t,c) + \mathcal{A}^{c,w}V(t,x)\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0. \end{cases}$$

see p. 328 for earlier $\mathcal{A}^u$

boundary conditions as on p. 356

where

$$\mathcal{A}^{c,w}V = xw'\alpha V_x - cV_x + \frac{1}{2}x^2 w'\Sigma w\ V_{xx},$$

property of W (k-dim):
$(dW)(dW)' = I_k\, dt$.
$\Rightarrow dX = \ldots dt + Xw'\sigma dW \Rightarrow$
$(dX)^2 = x^2 w'\sigma (dW)(dW)'\sigma' w$
$= x^2\, w'\sigma\sigma' w\, dt$

The matrix Σ is given by

$$\Sigma = \sigma\sigma'.$$

Symmetric!

The HJB equation ~~is~~ then becomes

$$\begin{cases} V_t + \sup_{w'e=1,\ c\geq 0} \left\{ F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} \right\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0. \end{cases}$$

where $\Sigma = \sigma\sigma'$.

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

standard technique for optimization under linear constraints

$$L = F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\,(1 - w'e)\,.$$

$L(c, w, \lambda)$

Repeat:

$$\begin{aligned} L &= F(t,c) + (xw'\alpha - c)V_x \\ &+ \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\,(1 - w'e)\,. \end{aligned}$$

The first order condition for c is (we have not specified F)

$$F_c = V_x.$$

recall from calculus: $D(w'\Sigma w) = 2w'\Sigma$, w' is a row vector

The first order condition for w is

$$x\alpha' V_x + x^2 V_{xx} w'\Sigma = \lambda e',$$ (row vectors)

$$\Leftrightarrow x\alpha V_x + x^2 V_{xx}\Sigma w = \lambda e$$ (column vectors)

so we can solve for w in order to obtain

$$\hat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}} e - \frac{x V_x}{x^2 V_{xx}}\alpha\right].$$ (column vector)

Using the relation $e'w = 1$ this gives λ as

$$\lambda = \frac{x^2 V_{xx} + x V_x e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

$$1 = e'\hat{w} = \lambda \frac{e'\Sigma^{-1}e}{x^2 V_{xx}} - \frac{x V_x e'\Sigma^{-1}\alpha}{x^2 V_{xx}}$$

in $\hat{w} = \Sigma^{-1}[\dots]$ of previous page

Inserting λ gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \underbrace{\frac{V_x}{xV_{xx}}}\Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right].$$

[the quotient depends on t !]

We can write this as

$$\hat{w}(t) = g + Y(t)h,$$

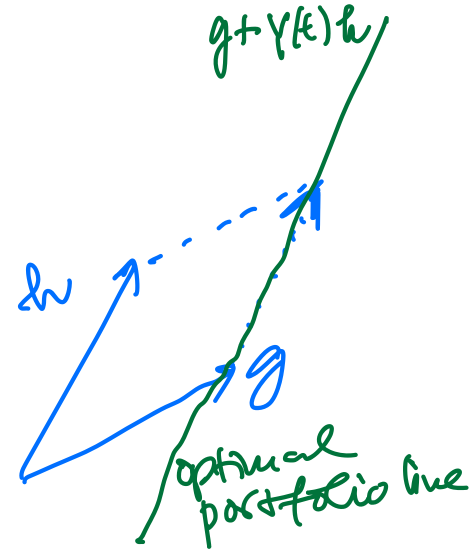
where the fixed vectors g and h are given by

not depending on t

$$\left\{\begin{aligned} g &= \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \\ h &= \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \end{aligned}\right.$$

whereas Y is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}.$$



We had

$$\hat{w}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" Δ, where

$$\Delta = \{w \in R^n \; |e'w = 1\}.$$

If we fix two points on the optimal portfolio line, say $w^a = g + ah$ and $w^b = g + bh$, then any point w on the line can be written as an affine combination of the basis points w^a and w^b. An easy calculation shows that if $w^s = g + sh$ then we can write

do it!

$$w^s = \mu w^a + (1-\mu) w^b,$$

where

$$\mu = \frac{s-b}{a-b}.$$

Summary:

Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed s the portfolio w^s stays fixed over time.

2. For fixed a, b with $a \neq b$ the optimal portfolio $\hat{\mathbf{w}}(t)$ is, obtained by allocating all resources between the fixed funds w^a and w^b, i.e.

$$\hat{w}(t) = \mu^a(t)w^a + \mu^b(t)w^b,$$

$$\mu^a(t) = \frac{Y(t) - b}{b - a}, \quad \mu^b(t) = 1 - \mu^a(t)$$

(note $\mu^a(t) + \mu^b(t) = 1$)

Remark: to obtain the geometric interpretation (see also p.366) it is not necessary to (precisely) know the utility function F and/or the optimal value function V. Surprising!

See pp 360–367 for the case "without"

Merton's mutual funds:

The case with a risk free asset

Again we consider the standard model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t),$$

We also assume the risk free asset B with dynamics

$$dB = rBdt.$$

We denote $B = S_0$ and consider portfolio weights $(w_0, w_1, \ldots, w_n)'$ where $\sum_0^n w_i = 1$. We then eliminate w_0 by the relation

extra term

$$w_0 = 1 - \sum_1^n w_i,$$

(method could also be used in previous case)

and use the letter w to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

Note: $w \in R^n$ without constraints. (no "Laplace" needed, as w is not constrained)

HJB

We obtain (again from the SF condition)

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW,$$

where $e = (1, 1, \ldots, 1)'$. (note: $w'e \neq 1$ in general here, because we removed w_0)

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \{F(t,c) + \mathcal{A}^{c,w}V(t,x)\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}^{c,w}V &= xw'(\alpha - re)V_x(t,x) + (rx - c)V_x(t,x) \\ &+ \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x). \end{aligned}$$

again $\Sigma = \sigma\sigma'$

First order conditions

We maximize

$$F(t,c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx}$$

with $c \geq 0$ and $w \in R^n$.

The first order conditions are (parallel to p.364)

$$\begin{aligned} F_c &= V_x, \\ \hat{w} &= \underbrace{-\frac{V_x}{xV_{xx}}}_{=:\mu^f}\underbrace{\Sigma^{-1}(\alpha - re)}_{=:w^f \in R^n}, \end{aligned}$$

, only risky investments

with geometrically obvious economic interpretation.

like on p.366

see definitions on p.371

Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds w^0 and w^f.

2. The fund w^0 consists only of the risk free asset.

3. The fund w^f consists only of the risky assets, and is given by

$$w^f = \Sigma^{-1}(\alpha - re).$$

and relative allocations of wealth are $u^f = -\frac{V_x}{xV_{xx}}$ (everything depending on $t, X(t)$)

$u_0 = 1 - u^f$

Remark: These results do not follow from the case without riskfree asset (see the book).

→ end of lecture 12b ←

→ Start of Lecture 13a ←

More (alternative) theory

Continuous Time Finance

The Martingale Approach to Optimal Investment Theory

Ch 20

Tomas Björk

essential ingredient is completeness of the market

Contents

- Decoupling the wealth profile from the portfolio choice.
- Lagrange relaxation. (seen before, but will be explained again)
- Solving the general wealth problem.
- Example: Log utility.
- Example: The numeraire portfolio.

Problem Formulation

Standard model with internal filtration (see pp. 360, 368)

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$
$$dB_t = rB_t dt.$$

Assumptions:

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is **complete**. ("N = R")
- We have a given initial wealth x_0

Problem:

$$\max_{h \in \mathcal{H}} \quad E^P\left[\Phi(X_T)\right]$$

(only terminal wealth)

where

$$\mathcal{H} = \{\text{self financing portfolios}\}$$

given the initial wealth $X_0 = x_0$.

Some observations

- In a complete market, there is a unique martingale measure Q.

- Every claim Z satisfying the budget constraint

$$e^{-rT}E^{Q}[Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

self-financing

$h_T' S_T = Z$ and

$x_0 = h_0' S_0 = e^{-rT} E^Q h_T' S_T$ ($=: V_T^h$)

and the value process V^h of a self-financing portfolio process is a Q-martingale.

- We can thus write our problem as

Note: E^P

$$\max_{Z} \quad E^{P}[\Phi(Z)]$$

subject to the constraint

Note: E^Q

$$e^{-rT}E^{Q}[Z] = x_0.$$

- We can forget the wealth dynamics! (for the time being, see step 2 below)

Basic Ideas

Our problem was

$$\max_{Z} \quad E^P \left[\Phi(Z) \right]$$

subject to $$e^{-rT} E^Q \left[Z \right] = x_0.$$

Idea I:

We can **decouple** the optimal portfolio problem into:

1. Finding the optimal wealth profile $\hat{Z}$.
2. Given $\hat{Z}$, find the replicating portfolio. (Here the dynamics come in!)

Idea II:

- Rewrite the constraint under the measure P (instead of Q).
- Use Lagrangian techniques to relax the constraint.

Lagrange formulation

Recall

Problem:

$$\max_{Z} \quad E^{P}\left[\Phi(Z)\right] \qquad Z \in \mathcal{F}_T$$

subject to

$$e^{-rT}E^{P}\left[L_T Z\right] = x_0.$$

Now: constraint in terms of measure P!

Here L is the likelihood process, i.e.

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t, \quad 0 \le t \le T \text{ , and}$$

recall $E^Q Z = E^P[L_T Z]$

The Lagrangian of the problem is

$$\mathcal{L} = E^{P}\left[\Phi(Z)\right] + \lambda\left\{x_0 - e^{-rT}E^{P}\left[L_T Z\right]\right\}$$

i.e.

$$\mathcal{L} = E^{P}\left[\Phi(Z) - \lambda e^{-rT}L_T Z\right] + \lambda x_0$$

note: both expectations under P!

The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable λ, to find the optimal Z by maximizing

$$\mathcal{L} = E^P \left[\Phi(Z) - \lambda e^{-rT} L_T Z \right] + \lambda x_0$$

over unconstrained Z, i.e. to maximize the Lebesgue integral

$$\int_\Omega \left\{ \Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega) \right\} dP(\omega)$$

This is a trivial problem! (if you look at it the right way)

We can simply maximize $Z(\omega)$ for each ω separately.

$$\max_z \quad \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\},$$ with $L_T = L_T(\omega)$, $z = Z(\omega)$

under the integral

The optimal wealth profile

Our problem: (repeat from previous slide)

$$\max_z \quad \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\}$$

First order condition

$$\Phi'(z) = \lambda e^{-rT} L_T$$

The optimal Z is thus given by

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right)$$

$\hat{Z}$ depends on λ!

where

$$G(y) = \left[\Phi'\right]^{-1}(y).$$

[if Φ' has inverse with y in its domain]

The dual variable λ is determined by the constraint

implicitly

$$e^{-rT} E^P \left[L_T \hat{Z} \right] = x_0.$$

you have to solve λ from this equation, $\hat{Z}$ depends on λ, and hope/prove that a unique solution exists

→ end of lecture 13a ←

– start of lecture 13b

Example – log utility

Assume that

$$\Phi(x) = \ln(x) \quad , \Phi'(x) = \frac{1}{x} \Rightarrow$$

Then inverse of Φ' is

$$g(y) = \frac{1}{y} \quad , \text{for all } y>0$$

Thus

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right) = \frac{1}{\lambda} e^{rT} L_T^{-1}$$

Finally λ is determined by

$$e^{-rT} E^P \left[L_T \hat{Z} \right] = x_0.$$

i.e.

$$e^{-rT} E^P \left[L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0.$$

so $\lambda = x_0^{-1}$ and

$$\hat{Z} = x_0 e^{rT} L_T^{-1},$$

to be interpreted as optimal wealth at time T, given the budget constraint.

The optimal wealth process

- We have computed the optimal **terminal** wealth profile

$$\widehat{Z} = \widehat{X}_T = x_0 e^{rT} L_T^{-1} \qquad (1)$$

- What does the optimal wealth **process** $\widehat{X}_t$ look like?

We have (why?) (discounted traded assets are Q-martingales)

$$\widehat{X}_t = e^{-r(T-t)} E^Q \left[\widehat{X}_T \middle| \mathcal{F}_t \right] \qquad (2)$$

so we obtain from (1) and (2):

Q is now the base measure

$$\widehat{X}_t = x_0 e^{rt} E^Q \left[L_T^{-1} \middle| \mathcal{F}_t \right]$$

abstract theory $\to$ $L_T^{-1} = dP/dQ$ on $\mathcal{F}_T$

But L^{-1} is a Q-martingale (why?) so we obtain

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

The Optimal Portfolio

- We have computed the optimal wealth process: $\hat{X}_t$,
  once we know L_t.
- How do we compute the optimal portfolio?

Assume for simplicity that we have a standard Black-Scholes model (complete model!)

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt
\end{aligned}
$$

Recall that

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

L_t^{-1} satisfies (Girsanov theory) an equation like $dL_t^{-1} = L_t^{-1} \psi_t dW_t^Q$ for some ψ_t

Q-Wiener

$\boxed{W \text{ is } P\text{-Wiener}}$

$= L_t^{-1} (\dots) dt + L_t^{-1} \psi_t \, dW_t$, as we know $dW^Q = (\dots) dt + dW_t$.

Basic Program

$z_t = z_0 e^{rt} L_t^{-1}$, you "know" L_t

1. Use Ito and the formula for $\widehat{X}_t$ to compute $d\widehat{X}_t$ like

$$d\widehat{X}_t = \widehat{X}_t(\quad)dt + \widehat{X}_t\beta_t dW_t$$

(find β_t later)

where we do not care about $(\quad)$.

2. Recall that (for some $\hat{u}_t$, portfolio weight)

involves dW_t

see p.382

$$d\widehat{X}_t = \widehat{X}_t\left\{(1-\hat{u}_t)\frac{dB_t}{B_t} + \hat{u}_t\frac{dS_t}{S_t}\right\}$$

which we write as

$$d\widehat{X}_t = \widehat{X}_t\{\quad\}\,dt + \widehat{X}_t\hat{u}_t\sigma dW_t$$

3. We can identify $\hat{u}$ as

$$\hat{u}_t = \frac{\beta_t}{\sigma}$$

(if we know β_t, see further down)

ψ_t on p. 382 in term $\psi_t = -\varphi_t$

We recall

(1) $$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

We also recall that

(L_t is $\frac{dQ}{dP}$ on $\mathcal{F}_t$) $$dL_t = L_t \varphi dW_t,$$

where

$$\varphi = \frac{r - \mu}{\sigma}$$ (many lectures ago)

From this we have (Itô for L_t^{-1})

(2) $$dL_t^{-1} = \varphi^2 L_t^{-1} dt - L_t^{-1} \varphi dW_t = -\varphi L_t^{-1} dW_t^{Q}$$

and we obtain from (1) and (2), do the calculus!,

$$d\widehat{X}_t = \widehat{X}_t \left\{ \quad \right\} dt - \widehat{X}_t \varphi dW_t \rightarrow \beta_t = -\varphi$$

Result: The optimal portfolio is given by $\hat{u}_t = \frac{\beta_t}{\sigma}$

$$\Rightarrow \quad \hat{u}_t = \frac{\mu - r}{\sigma^2}$$ (which we have seen as Market price of risk)

Note that $\hat{u}$ is a "myopic" portfolio in the sense that it does not depend on the time horizon T.

(Another example)

A Digression: The Numeraire Portfolio

Standard approach:

- Choose a fixed numeraire (portfolio) N.
- Find the corresponding martingale measure, i.e. find Q^N s.t.

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are Q^N -martingales.

Alternative approach: (swap the two steps above)

- Choose a fixed measure $Q \sim P$.
- Find numeraire N such that $Q = Q^N$:

$\frac{X_t}{N_t}$ is Q^N-martingale if X_t is value of traded asset

Special case:

- Set $Q = P$, our choice
- Find numeraire N such that $Q^N = P$ i.e. such that

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are $\overset{P}{\overset{\shortparallel}{Q^N}}$-martingales under the **objective** measure P.
- This N is called the **numeraire portfolio**.

Specialize further:

Log utility and the numeraire portfolio

Definition:
The **growth optimal portfolio** (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date T.

wealth process is $\hat{X}_t = x_0 e^{rt} L_t^{-1}$ (p. 381)

Theorem:
Assume that X is GOP. Then X is the numeraire portfolio.

Proof:
see p. 385

We have to show that the process

$$Y_t = \frac{S_t}{X_t}$$

is a P martingale. (and likewise is $\frac{B_t}{X_t} = x_0^{-1} L_t$)

We have (see p. 381)

$$\frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t$$

likelihood ratio $\frac{dQ}{dP}$ on $\mathcal{F}_t$

which is a P martingale, since $x_0^{-1} e^{-rt} S_t$ is a Q martingale. use "Bayes" (Additional exercise 3 = exercise C.9 in the book)

→ End of lecture 13b ←