

→ Start of lecture 10a ←

Continuous Time Finance

Incomplete Markets

Ch 15

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Recall: In a complete market, every claim can be hedged (and thus has a unique price).

Metatheorem: $N < R$ (not sufficiently many risky assets) in incomplete markets.



What else?
Typical examples?

Derivatives on Non Financial Underlying

Recall: The Black-Scholes theory assumes that the market for the underlying asset has (among other things) the following properties.

- The underlying is a liquidly traded asset.
- Shortselling allowed.
- Portfolios can be carried forward in time.

There exists a large market for derivatives, where the underlying does not satisfy these assumptions.

Examples: *(see next page)*

- Weather derivatives.
- Derivatives on electric energy.
- CAT-bonds.

Typical Contracts

Weather derivatives:

“Heating degree days”. Payoff at maturity T is given by

$$\mathcal{Z} = \max \{X_T - 30, 0\}$$

where X_T is the (mean) temperature at some place.

Electricity option:

The right (but not the obligation) to buy, at time T , at a predetermined price K , a constant flow of energy over a predetermined time interval.

CAT bond:

A bond for which the payment of coupons and nominal value is contingent on some (well specified) natural disaster to take place.

Problems

Weather derivatives:

The temperature is not the price of a traded asset.

Electricity derivatives:

Electric energy cannot easily be stored.

CAT-bonds:

Natural disasters are not traded assets.

We will treat all these problems within a **factor model**.

*for the
underlying*

Typical Factor Model Setup

Given:

- An underlying factor process X , which is **not** the price process of a traded asset, with dynamics under the objective probability measure P as

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t.$$

- A risk free asset with dynamics

$$dB_t = rB_t dt,$$

Problem:

Find arbitrage free price $\Pi_t[\mathcal{Z}]$ of a derivative of the form

$$\mathcal{Z} = \Phi(X_T)$$

Note similarity AND difference with earlier set up.

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293

X is NOT the price of a traded asset

Concrete Examples

Assume that X_t is the temperature at time t at the village of Peniche (Portugal).

Heating degree days:

$$\Phi(X_T) = 100 \cdot \max\{X_T - 30, 0\}$$

Holiday Insurance:

$$\Phi(X_T) = \begin{cases} 1000, & \text{if } X_T < 20 \\ 0, & \text{if } X_T \geq 20 \end{cases}$$

Question

$$\pi_t(\Phi)$$

Is the price $\Pi_t[\Phi]$ uniquely determined by the P -dynamics of X , and the requirement of an arbitrage free derivatives market?

[again a question that
should make you suspicious]

NO!!

WHY?

find the difference(s)
between current and
previous set ups

Stock Price Model ^{vs} ~ Factor Model

Black-Scholes:

$$\begin{aligned}dS_t &= \mu S_t dt + \sigma S_t dW_t, \\dB_t &= r B_t dt.\end{aligned}$$

Factor Model: (similar equation)

$$\begin{aligned}dX_t &= \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \\dB_t &= r B_t dt.\end{aligned}$$

(r can be time dependent too!)

What is the difference?

Answer

- X is not the price of a traded asset!
- We can not form a portfolio based on X , *hedging this way is impossible.*

1. Rule of thumb:

$$\begin{aligned} N &= 0, && \text{(no risky asset)} \\ R &= 1, && \text{(one source of randomness, } W) \end{aligned}$$

We have $N < R$. The exogenously given market, consisting only of B , is incomplete.

2. Replicating portfolios:

We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price X -derivatives.

- There is **not** a unique price for a **particular** derivative. *typical for incomplete markets*
- In order to avoid arbitrage, **different** derivatives have to satisfy internal consistency relations.
- If we take **one** “benchmark” derivative as given, then all other derivatives can be priced **in terms of** the market price of the benchmark. *“replaces the stock”*

We consider two given claims $\Phi(X_T)$ and $\Gamma(X_T)$. We assume they are traded with prices

$$\Pi_t[\Phi] = f(t, X_t)$$

$$\Pi_t[\Gamma] = g(t, X_t)$$

same underlying factor

Program:

- Form portfolio based on Φ and Γ . Use Itô on f and g to get portfolio dynamics.

self-financing condition:

$$dV = V \left\{ u^f \frac{df}{f} + u^g \frac{dg}{g} \right\}$$

u^f, u^g
relative weights

Usual steps:

- Choose portfolio weights such that the dW -term vanishes. Then we have

$$dV = V \cdot k dt,$$

(“synthetic bank” with k as the short rate)

- Absence of arbitrage implies

$$k = r$$

- Read off the relation $k = r$!

carrying out the program:

Recall $\pi_t[\Phi] = f(t, X_t)$, assumption

From Itô:

$$df = f\mu_f dt + f\sigma_f dW,$$

where

$$\begin{cases} \mu_f &= \frac{f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}}{f}, \\ \sigma_f &= \frac{\sigma f_x}{f}. \end{cases}$$

(see equation for dX_t)

Portfolio dynamics

$$dV = V \left\{ u^f \frac{df}{f} + u^g \frac{dg}{g} \right\}.$$

Use also $\pi_t[\Gamma] = g(t, X_t)$:

$$dg = g\mu_g dt + g\sigma_g dW$$

$\mu_g = \dots, \sigma_g = \dots$

Reshuffling terms gives us

$$dV = V \cdot \{u^f \mu_f + u^g \mu_g\} dt + V \cdot \{u^f \sigma_f + u^g \sigma_g\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^f + u^g &= 1, \\ u^f \sigma_f + u^g \sigma_g &= 0. \end{cases}$$

← kill dW contribution

$$u^f = -\frac{\sigma_g}{\sigma_f - \sigma_g},$$

$$u^g = \frac{\sigma_f}{\sigma_f - \sigma_g},$$

Portfolio dynamics

$$dV \equiv V \cdot \{u^f \mu_f + u^g \mu_g\} dt. \quad + 0, dW$$

i.e.

$$dV \equiv V \cdot \left\{ \frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} = r$$

which can be written as

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Repeat from previous slide:

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Note!

The quotient does **not** depend upon the particular choice of contract. (same for $f_1, \dots$)

Consider this as an internal consistency relation (see p.300)

This relation should also hold when X is the price of a traded asset.
True? Think of this!

Result

Assume that the market for X -derivatives is free of arbitrage. Then there exists a universal process λ , such that

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

holds for all t and for every choice of contract f .

NB: The same λ for all choices of f .

- λ = Risk premium per unit of volatility
- = "Market Price of Risk" (cf. CAPM).
- = Sharpe Ratio

all different terminology

Slogan:

"On an arbitrage free market all X -derivatives have the same market price of risk."

The relation *in short (suppress dependence on t, X_t)*

$$\frac{\mu_f - r}{\sigma_f} = \lambda$$

is actually a PDE! *: look at the equations for μ_f and σ_f on p.302*

→ End of lecture 10A ←

→ start of lecture 106 ←

Pricing Equation

Note that all these functions depend on t, x :
 $\mu = \mu(t, x), f_t = f_t(t, x)$ etc.

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf = 0 \\ f(T, x) = \Phi(x), \end{cases}$$

ETC.

Note also that μ (drift of X under P) is present in this equation, not in the Black-Scholes PDE. Why?

P -dynamics:

$$dX = \mu(t, X)dt + \sigma(t, X)dW.$$

Can we solve the PDE?

No!!

Why??

Answer

Recall the PDE *(in short hand notation)*

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf = 0 \\ f(T, x) = \Phi(x), \end{cases}$$

- In order to solve the PDE **we need to know** λ .
- λ is not given exogenously.
- λ is not determined endogenously.

looks hopeless, way out?

Question:

Who determines λ ?

Answer:

THE MARKET!

Interpreting λ

Recall that the f dynamics are

$$(\pi_t[\Phi] = f(t, X_t))$$

$$df = f\mu_f dt + f\sigma_f dW_t$$

and λ is defined as

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

- λ measures the aggregate risk aversion in the market.
- If λ is big then the market is highly risk averse.
- If λ is zero then the market is **risk neutral**.
- If you make an assumption about λ , then you implicitly make an assumption about the aggregate risk aversion of the market. *and v.v.: you may learn about λ from market (agents) behaviour, or perhaps by analysing a particular derivative.*

Moral

- Since the market is incomplete the requirement of an arbitrage free market will **not** lead to unique prices for X -derivatives.
- Prices on derivatives are determined by two main factors.
 1. **Partly** by the requirement of an arbitrage free derivative market. **All** pricing functions satisfies the **same PDE**. *but with different boundary conditions*
 2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

depending on λ !

Risk Neutral Valuation

We recall the PDE

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf = 0 \\ f(T, x) = \Phi(x), \end{cases}$$

Using Feynman-Kac we obtain a risk neutral valuation formula.



As always: couples this PDE to an SDE for X .

Risk Neutral Valuation

$$f(t, x) = e^{-r(T-t)} E_{t,x}^Q [\Phi(X_T)]$$

Q-dynamics: Note: these are not constants, see p. 293

$$dX_t = \{\mu - \lambda\sigma\} dt + \sigma dW_t^Q$$

- Price = expected value of future payments
- The expectation should **not** be taken under the “objective” probabilities P , but under the “risk adjusted” probabilities Q . *the?*

Think of Girsanov. If $L_t \left(\frac{dQ}{dP} \text{ on } \mathcal{F}_t \right)$ satisfies $dL_t = L_t \varphi_t dW_t$, then

$$\begin{cases} dW_t = dW_t^Q + \varphi_t dt \\ dX_t = \mu dt + \sigma dW_t \end{cases} \Rightarrow dX_t = (\mu + \sigma\varphi) dt + \sigma dW_t^Q$$

So, $\varphi = -\lambda$,
as on p. 266

Note that Q depends on λ and v.v.

(see also pp. 318, 319)

Interpretation of the risk adjusted probabilities, i.e. Q

As at the beginning of the course:

- The risk adjusted probabilities can be interpreted as probabilities in a (fictitious) risk neutral world.
- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.
- This does **not** mean that we live in, or think that we live in, a risk neutral world.
- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.

Market price of risk



Diversification argument about λ

- If the risk factor is **idiosyncratic** and **diversifiable**, then one can argue that the factor should not be priced by the market. Compare with APT, *Arbitrage Pricing Theory, e.g. GAPM*
product dependent, not on market properties
- Mathematically this means that $\lambda = 0$, i.e. $P = Q$, i.e. **the risk neutral distribution coincides with the objective distribution.**
see p. 314
- We thus have the “actuarial pricing formula”

$$f(t, x) = e^{-r(T-t)} E_{t,x}^P [\Phi(X_T)] \quad (\text{so, if } P=Q)$$

where we use the objective ^{probabiliy} measure P .

Modeling Issues

Temperature:

A standard model is given by (Ornstein-Uhlenbeck process)

$$dX_t = \{m(t) - bX_t\} dt + \sigma dW_t,$$

where m is the mean temperature capturing seasonal variations. This often works reasonably well. Here X_t has a normal distribution if X_0 is normal (independent of W)

Electricity:

A (naive) model for the spot electricity price is

$$dS_t = S_t \{m(t) - a \ln S_t\} dt + \sigma S_t dW_t$$

Use Ito to show: $\log S_t$ is an OU process

This implies lognormal prices (why?). Electricity prices are however very far from lognormal, because of “spikes” in the prices. Complicated. in reality!

CAT bonds:

Here we have to use the theory of point processes and the theory of extremal statistics to model natural disasters. Complicated. natural concept here, like Poisson process; beyond scope of this course

Martingale Analysis

(like on p.314)

Model: Under P we have

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t,$$

$$dB_t = rB_t dt,$$

We look for martingale measures. Since B is the only traded asset we need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q martingale.

Result: In this model, every $Q \sim P$ is a martingale measure.

Girsanov

$$dL_t = L_t \varphi_t dW_t$$

P -dynamics

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad (a)$$

$$dL_t = L_t \varphi_t dW_t$$

$$dQ = L_t dP \text{ on } \mathcal{F}_t$$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q \quad (b)$$

Martingale pricing: for appropriate T -claim Z

$$F(t, x) = e^{-r(T-t)} E^Q[Z | \mathcal{F}_t]$$

Q -dynamics of X : (combine (a) and (b)):

$$dX_t = \{\mu(t, X_t) + \sigma(t, X_t) \varphi_t\} dt + \sigma(t, X_t) dW_t^Q,$$

Result: We have $\lambda_t = -\varphi_t$, i.e., the Girsanov kernel φ equals minus the market price of risk.

Several Risk Factors

We recall the dynamics of the f -derivative

$$df = f\mu_f dt + f\sigma_f dW_t$$

and the Market Price of Risk

$$\frac{\mu_f - r}{\sigma_f} = \lambda, \quad \text{i.e.} \quad \mu_f - r = \lambda\sigma_f.$$

also

$$df = f(\mu_f + \sigma_f \varphi) dt + f\sigma_f dW_t$$

In a multifactor model of the type

$$dX_t = \mu(t, X_t) dt + \sum_{i=1}^n \sigma_i(t, X_t) dW_t^i,$$

for multiple Wiener processes (not treated)

it follows from Girsanov that for every risk factor W^i there will exist a market price of risk $\lambda_i = -\varphi_i$ such that

New equation:

$$\mu_f - r = \sum_{i=1}^n \lambda_i \sigma_i$$

Compare with CAPM. (if you know what that is)

→ End of lecture 10! ←