→ Start of lecture 12a ←

Back to finance:

2. Investment Theory

(Section 19.6)

- Problem formulation.
- An extension of HJB.
- The simplest consumption-investment problem.
- The Merton fund separation results.

Recap of Basic Facts

We consider a market with n assets.

$$
\begin{aligned}
S_t^i &= \text{price of asset No } i, \\
h_t^i &= \text{units of asset No } i \text{ in portfolio} \\
w_t^i &= \text{portfolio weight on asset No } i \quad (\text{previously } u_t^i) \\
X_t &= \text{portfolio value} \quad [\text{previously denoted } V_t] \\
c_t &= \text{consumption rate} \quad \geq 0
\end{aligned}
$$

New concept

We have the relations

$$
X_t = \sum_{i=0}^{n} h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=0}^{n} w_t^i = 1.
$$

(The 0-asset is the bank account in what follows)

Basic equation: when consumption is present:
Dynamics of self financing portfolio in terms of relative weights

$$
dX_t = X_t \sum_{i=0}^{n} w_t^i \frac{dS_t^i}{S_t^i} - c_t dt
$$

(opposite to dividends, see pp. 212, 214, now we have a "minus term")

Simplest model (n=1)

Assume a scalar risky asset and a constant short rate.

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t \\
dB_t &= r B_t dt
\end{aligned}
$$

We want to maximize expected utility of consumption over time

$\boxed{\text{note the control variables } w^0, w^1, c}$

utility function $F(t;)$

$$
\max_{w^0, w^1, c} \quad E\left[\int_0^T F(t, c_t) dt\right]
$$

[also a term $\Phi(T, c_T)$ can be included, or $\Phi(T, X_T)$]

Dynamics

$$
dX_t = X_t \left[w_t^0 r + w_t^1 \alpha\right] dt - c_t dt + w_t^1 \sigma X_t dW_t,
$$

(verify!)

Constraints

$$
\begin{aligned}
c_t &\geq 0, \ \forall t \geq 0, \\
w_t^0 + w_t^1 &= 1, \ \forall t \geq 0.
\end{aligned}
$$

Sensible problem (formulation)?

.... become suspicious ....

Nonsense!

What are the problems?

- We can obtain unlimited utility by simply consuming arbitrary large amounts.

- The wealth will go negative, but there is nothing in the problem formulations which prohibits this.

- We would like to impose a constraint of type $X_t \geq 0$ but this is a **state constraint** and DynP does not allow this. (see p.324)

  [leading to HJB]

Good News:
DynP can be generalized to handle (some) problems of this kind.

The use of stopping times helps!

Generalized problem

Let D be a nice open subset of $[0, T] \times R^n$ and consider the following problem.

note τ, $\boxed{\text{stopping time}}$

$$\max_{u \in U} \quad E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi\left(\tau, X_\tau^{\mathbf{u}}\right)\right].$$

Dynamics:

$$\begin{aligned} dX_t &= \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t, \\ X_0 &= x_0, \in D. \end{aligned}$$

The **stopping time** τ is defined by

$$\tau = \inf\left\{t \geq 0 \,|(t, X_t) \in \partial D\right\} \wedge T. \quad \leq T \;!$$

a random time!

boundary of D

So, the problem looks as before, but with the difference that the horizon τ is random!

Generalized HJB

Theorem: Given enough regularity the follwing hold.

1. The optimal value function satisfies

as before

boundary condition on ∂D, not just at $t=T$

$$
\begin{cases}
\dfrac{\partial V}{\partial t}(t,x) + \sup\limits_{u \in U} \left\{ F(t,x,u) + \mathcal{A}^u V(t,x) \right\} = 0, & \forall (t,x) \in D \\
V(t,x) = \Phi(t,x), & \boxed{\forall (t,x) \in \partial D.}
\end{cases}
$$

2. We have an obvious verification theorem.

replace e.g. $H(T,x)$ on p.340 by $H(t,x) = \Phi(t,x)$, $\forall (t,x) \in \partial D$

Reformulated problem

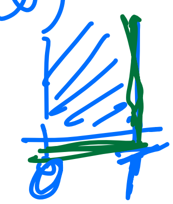
$$\max_{c \geq 0,\ w \in R} \quad E\left[\int_0^{\tau} F(t, c_t)dt + \Phi(X_T)\right]$$

or c_T? now a "premium" on remaining wealth

open set in $[0,\infty) \times \mathbb{R}$

The "ruin time" τ is defined by

corresponds to $D = [0,T) \times (0,\infty)$



$$\tau = \inf\left\{t \geq 0 \,| X_t = 0\right\} \wedge T.$$

$t \leq \tau \Rightarrow X_t \geq 0$!

$\partial D = \{(t,x) \in [0,T] \times [0,\infty) : t = T \text{ or } x = 0\}$

Notation:

$$\begin{aligned} w^1 &= w, \\ w^0 &= 1 - w \end{aligned}$$

Thus no constraint on w, only w and c remain as controls.

Dynamics of simple model on p.350 become

$$dX_t = w_t\left[\alpha - r\right] X_t dt + \left(rX_t - c_t\right) dt + w\sigma X_t dW_t,$$

now c, w are the control variables

for optimal $\hat{c}, \hat{w}$: $V_t + F(t, \hat{c}) + \hat{w}\, x(\alpha - r)\frac{\partial V}{\partial x}(\cdot\cdot) + \dots = 0$ $(*)$

HJB Equation

$$
\frac{\partial V}{\partial t} + \sup_{c \geq 0, w \in R} \left\{ F(t,c) + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2} \right\} = 0,
$$

$\Phi = 0 \Rightarrow$

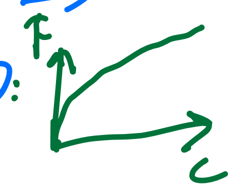
$$
\begin{cases} V(T,x) = 0, \\ V(t,0) = 0. \end{cases}
$$

?

We now specialize (why?) to

$$
F(t,c) = e^{-\delta t} c^{\gamma},
$$

with $0 < \gamma < 1 \Rightarrow$ $F_c(t,0) = +\infty \Rightarrow$ optimal $c > 0$:



and for simplicity we assume that

$$
\Phi = 0,
$$

so we have to maximize

$$
e^{-\delta t} c^{\gamma} + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},
$$

w.r.t. $c \geq 0$ and $w \in R$

Analysis of the HJB Equation

In the embedded static problem we maximize, over c and w, (repeat from p.356)

$$G(c,w) := e^{-\delta t}c^{\gamma} + wx(\alpha - r)V_x + (rx - c)V_x + \frac{1}{2}x^2w^2\sigma^2V_{xx},$$

First order conditions:

$$\begin{array}{llcll} (1) & \gamma c^{\gamma-1} & = & e^{\delta t}V_x, & \left(\text{from } \frac{\partial G}{\partial c} = 0\right) \\ (2) & w & = & \dfrac{-V_x}{x \cdot V_{xx}} \cdot \dfrac{\alpha - r}{\sigma^2}, & \left(\text{from } \frac{\partial G}{\partial w} = 0\right) \end{array}$$

Ansatz:

$$V(t,x) = e^{-\delta t}h(t)x^{\gamma}, \quad \left(\text{like } F(t,c)\right)$$

Because of the boundary conditions, we must demand that

$$h(T) = 0. \tag{5}$$

$\Phi = 0$

alternatively,
you could try $V(t,x) = k(t)x^{\gamma}$,
uld work as well

Given a V of this form we have (using $\cdot$ to denote the time derivative)

$$\begin{aligned} V_t &= e^{-\delta t}\dot{h}x^{\gamma} - \delta e^{-\delta t}hx^{\gamma}, \\ V_x &= \gamma e^{-\delta t}hx^{\gamma-1}, \\ V_{xx} &= \gamma(\gamma-1)e^{-\delta t}hx^{\gamma-2}. \end{aligned}$$

[note: $h = h(t)$]

giving us

use p.357, (2): $\widehat{w}(t,x) = \dfrac{\alpha - r}{\sigma^2(1-\gamma)}$, (constant!)

use p.357, (1): $\widehat{c}(t,x) = xh(t)^{-1/(1-\gamma)}$. (linear in x)

Plug all this into HJB! and try to solve

(*) on ~~top~~ of p. 356, or the one on the bottom with $\hat{w}$ and $\hat{c}$ etc.

tedious work, but you can check!

After rearrangements we obtain

$$x^{\gamma}\left\{\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)}\right\}=0,$$

where the constants A and B are given by

more tedious computations,

$$\begin{aligned} A &= \frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}+r\gamma-\frac{1}{2}\frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}-\delta = \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\gamma-\delta \\ B &= 1-\gamma. \end{aligned}$$

If this equation is to hold for all x and all t, then we see that h must solve the ODE

$$\begin{aligned} \dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)} &= 0, \\ h(T) &= 0. \end{aligned}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly. See Exercises 19.2, 19.3

We are done.

Merton's Mutal Fund Theorems

Section 19.7

1. The case with no risk free asset

We consider n risky assets with dynamics

$$dS_i = S_i\alpha_i dt + S_i\sigma_i dW, \quad i = 1, \ldots, n \quad , \ \sigma_i \in \mathbb{R}^{1\times k}$$

where W is Wiener in R^k. On vector form:

means every component in Wiener and they are all independent

$$dS = D(S)\alpha dt + D(S)\sigma dW.$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \text{ and } \sigma = \begin{bmatrix} - \ \sigma_1 \ - \\ \vdots \\ - \ \sigma_n \ - \end{bmatrix} \in \mathbb{R}^{n\times k}$$

$D(S)$ is the diagonal matrix

$$D(S) = diag[S_1, \ldots, S_n]. \quad \in \mathbb{R}^n$$

$$= \begin{bmatrix} S_1 & & \emptyset \\ & \ddots & \\ \emptyset & & S_n \end{bmatrix}$$

Formal problem

[Note $\Phi(\cdot\cdot)=0$]

$$\max_{c,w} \quad E\left[\int_0^T F(t, c_t)dt\right]$$

given the dynamics (use the ST condition on p.349)

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW,$$

additional to the dS_t^i equations

and constraints

$$\sum_{i=1}^n w_t^i = e'w_t = 1, \quad c_t \geq 0.$$

$w_t = (w_t^1, \ldots, w_t^n)'$

Assumptions:

- The vector α and the matrix σ are constant and deterministic.
- The volatility matrix σ has full row rank so $\sigma\sigma'$ is positive definite and invertible. $\Longrightarrow$ arbitrage free and complete market if $n=k$

Note: S does not (explicitly) turn up in the X-dynamics so V is of the form

$$\underbrace{V(t, x, s)}_{\text{would result from}} = V(t, x)$$

$d\begin{pmatrix} X_t \\ S_t \end{pmatrix} = \ldots dt + \ldots dW_t$

combined state vector

→ End of lecture 12a ←

→ Start of lecture 12B ←

see p. 328 for earlier $\mathcal{A}^u$

The HJB equation is

Note that the optimal c,w will depend on time

$$
\begin{cases}
V_t(t,x) + \sup_{e'w=1,\ c\geq 0} \{F(t,c) + \mathcal{A}^{c,w}V(t,x)\} & = & 0, \\
V(T,x) & = & 0, \\
V(t,0) & = & 0.
\end{cases}
$$

boundary conditions as on p. 356

where

$$
\mathcal{A}^{c,w}V = xw'\alpha V_x - cV_x + \frac{1}{2}x^2 w'\Sigma w\, V_{xx},
$$

property of W (k-dim):

$(dW)(dW)' = I_k\, dt$

$\Rightarrow dX = \dots dt + Xw'\sigma dW \Rightarrow$

$(dX)^2 = X^2 (w'\sigma)\, dW\, dW' \sigma' w$

$= X^2\, w'\sigma\sigma' w\, dt$

The matrix Σ is given by

$$
\Sigma = \sigma\sigma'.
$$

Symmetric!

The HJB equation ~~is~~ then becomes

$$\begin{cases} V_t + \sup_{w'e=1,\ c\geq 0} \left\{ F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} \right\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0. \end{cases}$$

where $\Sigma = \sigma\sigma'$.

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

standard technique for

$$L = F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\left(1 - w'e\right).$$

$$\| \\ L(c, w, \lambda)$$

optimization under
linear constraints

Repeat:

$$
\begin{aligned}
L &= F(t,c) + (xw'\alpha - c)V_x \\
&+ \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\left(1 - w'e\right).
\end{aligned}
$$

The first order condition for c is (we have not specified F)

$$F_c = V_x.$$

recall from calculus: $D(w'\Sigma w) = 2w'\Sigma$, w' is a row vector

The first order condition for w is

$$x\alpha' V_x + x^2 V_{xx} w'\Sigma = \lambda e', \quad \text{(row vectors)}$$

$$\Leftrightarrow x\alpha V_x + x^2 V_{xx}\Sigma w = \lambda e \quad \text{(column vectors)}$$

so we can solve for w in order to obtain

$$\hat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}} e - \frac{xV_x}{x^2 V_{xx}}\alpha\right]. \quad \text{(column vector)}$$

Using the relation $e'w = 1$ this gives λ as

$$\lambda = \frac{x^2 V_{xx} + xV_x e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

$$1 = e'\hat{w} = \lambda\frac{e'\Sigma^{-1}e}{x^2 V_{xx}} - \frac{\cancel{x}V_x e'\Sigma^{-1}\alpha}{x^{\cancel{2}} V_{xx}}$$

in $\hat{w} = \Sigma^{-1}[\dots]$ of previous page

Inserting λ gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \underbrace{\frac{V_x}{xV_{xx}}}\Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right].$$

We can write this as

[the quotient depends on t!]

$$\hat{w}(t) = g + Y(t)h,$$

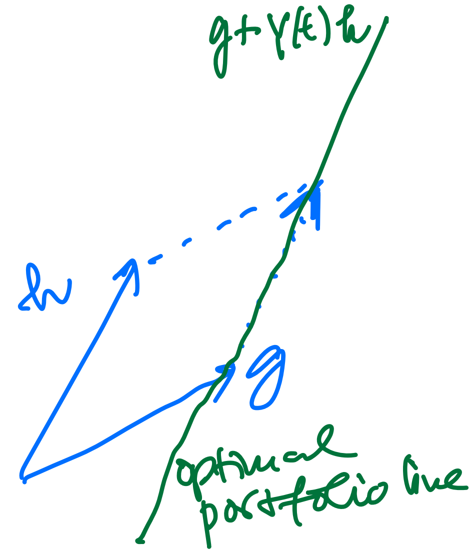
where the fixed vectors g and h are given by

not depending on t

$$\begin{aligned} g &= \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \\ h &= \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \end{aligned}$$

whereas Y is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}.$$



We had

$$\hat{w}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" Δ, where

$$\Delta = \{w \in R^n \, | e'w = 1\}.$$

If we fix two points on the optimal portfolio line, say $w^a = g + ah$ and $w^b = g + bh$, then any point w on the line can be written as an affine combination of the basis points w^a and w^b. An easy calculation shows that if $w^s = g + sh$ then we can write

do it!

$$w^s = \mu w^a + (1-\mu)w^b,$$

where

$$\mu = \frac{s-b}{a-b}.$$

Summary:

Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed s the portfolio w^s stays fixed over time.

2. For fixed a, b with $a \neq b$ the optimal portfolio $\hat{\mathbf{w}}(t)$ is, obtained by allocating all resources between the fixed funds w^a and w^b, i.e.

$$\hat{w}(t) = \mu^a(t) w^a + \mu^b(t) w^b,$$

$$\mu^a(t) = \frac{Y(t) - b}{b - a}, \quad \mu^b(t) = 1 - \mu^a(t)$$

(note $\mu^a(t) + \mu^b(t) = 1$)

Remark: These results do not follow from the previous case (see the book).

See pp 360-367 for the case "without"

Merton's mutual funds:

The case with a risk free asset

Again we consider the standard model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t),$$

We also assume the risk free asset B with dynamics

$$dB = rBdt.$$

We denote $B = S_0$ and consider portfolio weights $(w_0, w_1, \ldots, w_n)'$ where $\sum_0^n w_i = 1$. We then eliminate w_0 by the relation

extra term

$$w_0 = 1 - \sum_1^n w_i,$$

(method could also be used in previous case)

and use the letter w to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

Note: $w \in R^n$ without constraints. (no "Laplace" needed, as w is not constrained)

HJB

We obtain (again from the SF condition)

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW,$$

where $e = (1, 1, \ldots, 1)'$. (note: $w'e \neq 1$ in general here, because we removed w_0)

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \left\{ F(t,c) + \mathcal{A}^{c,w} V(t,x) \right\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}^c V &= xw'(\alpha - re)V_x(t,x) + (rx - c)V_x(t,x) \\ &+ \frac{1}{2}x^2 w' \Sigma w V_{xx}(t,x). \end{aligned}$$

again $\Sigma = \sigma\sigma'$

First order conditions

We maximize

$$F(t,c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx}$$

with $c \geq 0$ and $w \in R^n$.

The first order conditions are (parallel to p.364)

$$\begin{aligned} F_c &= V_x, \\ \hat{w} &= -\frac{V_x}{xV_{xx}} \underbrace{\Sigma^{-1}(\alpha - re)}_{=: w^f \in R^n\text{, only risky weights}}, \end{aligned}$$

with geometrically obvious economic interpretation,

like on p.366

see definition on p.371

Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds w^0 and w^f.

2. The fund w^0 consists only of the risk free asset.

3. The fund w^f consists only of the risky assets, and is given by

$$w^f = \Sigma^{-1}(\alpha - re).$$

and relative allocations of wealth are $u^f = -\frac{V_x}{xV_{xx}}$ (everything depending on $t, X(t)$)

$u_0 = 1 - u^f$

→ end of lecture 126 ←