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More (alternative) theory

Continuous Time Finance

The Martingale Approach to Optimal Investment Theory

Ch 20

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essential ingredient is completeness of the market

Contents

- Decoupling the wealth profile from the portfolio choice.
- Lagrange relaxation. (seen before, but will be explained again)
- Solving the general wealth problem.
- Example: Log utility.
- Example: The numeraire portfolio.

Problem Formulation

Standard model with internal filtration (see pp. 360, 368)

$$
\begin{aligned}
dS_t &= D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \\
dB_t &= rB_t dt.
\end{aligned}
$$

Assumptions:

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is **complete**. ("N = R")
- We have a given initial wealth x_0

Problem:

$$
\max_{h \in \mathcal{H}} \quad E^P \left[\Phi(X_T) \right]
$$

(only terminal wealth)

where

$$
\mathcal{H} = \{\text{self financing portfolios}\}
$$

given the initial wealth $X_0 = x_0$.

Some observations

- In a complete market, there is a unique martingale measure Q.

- Every claim Z satisfying the budget constraint

$$e^{-rT}E^Q[Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

self-financing

$h^1_T S_T = Z$ and

$x_0 = h^1_0 S_0 = e^{-rT} E^Q h^1_T S_T$, $h^1_T S_T =: V^h_T$

and the value process V^h of a self-financing portfolio process is a Q-martingale.

- We can thus write our problem as

Note: E^P

$$\max_Z \quad E^P[\Phi(Z)]$$

subject to the constraint

Note: E^Q

$$e^{-rT}E^Q[Z] = x_0.$$

- We can forget the wealth dynamics! (for the time being, see step 2 below)

Basic Ideas

Our problem was

$$\max_{Z} \quad E^P \left[\Phi(Z)\right]$$

subject to
$$e^{-rT} E^Q \left[Z\right] = x_0.$$

Idea I:

We can **decouple** the optimal portfolio problem into:

1. Finding the optimal wealth profile $\hat{Z}$.
2. Given $\hat{Z}$, find the replicating portfolio. (Here the dynamics come in!)

Idea II:

- Rewrite the constraint under the measure P (instead of Q).
- Use Lagrangian techniques to relax the constraint.

Lagrange formulation

Recall

Problem:

$$\max_{Z} \quad E^P [\Phi(Z)]$$

$Z \in \mathcal{F}_T$

subject to

$$e^{-rT} E^P [L_T Z] = x_0.$$

Now: constraint in terms of measure P!

Here L is the likelihood process, i.e.

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t, \quad 0 \le t \le T$$

, and

recall $E^Q Z = E^P[L_T Z]$

The Lagrangian of the problem is

$$\mathcal{L} = E^P [\Phi(Z)] + \lambda \left\{ x_0 - e^{-rT} E^P [L_T Z] \right\}$$

i.e.

$$\mathcal{L} = E^P \left[\Phi(Z) - \lambda e^{-rT} L_T Z \right] + \lambda x_0$$

note: both expectations under P!

The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable λ, to find the optimal Z by maximizing

$$\mathcal{L} = E^P\left[\Phi(Z) - \lambda e^{-rT} L_T Z\right] + \lambda x_0$$

over unconstrained Z, i.e. to maximize the Lebesgue integral

$$\int_\Omega \left\{\Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega)\right\} dP(\omega)$$

This is a trivial problem! (if you look at it the right way)

We can simply maximize $Z(\omega)$ for each ω separately.

$$\max_z \quad \left\{\Phi(z) - \lambda e^{-rT} L_T z\right\},$$ with $L_T = L_T(\omega)$, $z = Z(\omega)$

under the integral

The optimal wealth profile

Our problem: (repeat from previous slide)

$$\max_z \quad \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\}$$

First order condition

$$\Phi'(z) = \lambda e^{-rT} L_T$$

The optimal Z is thus given by

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right),$$

$\hat{Z}$ depends on λ!

where

$$G(y) = \left[\Phi'\right]^{-1}(y).$$

[if Φ' has inverse with y in its domain]

The dual variable λ is determined by the constraint

$$e^{-rT} E^P\left[L_T \hat{Z}\right] = x_0.$$

implicitly

$\Downarrow$

you have to solve λ from this equation, $\hat{Z}$ depends on λ, and hope/prove that a unique solution exists

→ end of lecture 13a ←

—start of lecture 13b

Example – log utility

Assume that

$$\Phi(x) = \ln(x) \quad , \ \Phi'(x) = \frac{1}{x} \Rightarrow$$

Then inverse of Φ' is

$$g(y) = \frac{1}{y} \quad , \text{ for all } y > 0$$

Thus

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right) = \frac{1}{\lambda} e^{rT} L_T^{-1}$$

Finally λ is determined by

$$e^{-rT} E^P \left[L_T \hat{Z} \right] = x_0.$$

i.e.

$$e^{-rT} E^P \left[L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0.$$

so $\lambda = x_0^{-1}$ and

$$\hat{Z} = x_0 e^{rT} L_T^{-1},$$

to be interpreted as optimal wealth at time T, given the budget constraint.

The optimal wealth process

- We have computed the optimal **terminal** wealth profile

$$\widehat{Z} = \widehat{X}_T = x_0 e^{rT} L_T^{-1} \qquad (1)$$

- What does the optimal wealth **process** $\widehat{X}_t$ look like?

We have (why?) (discounted traded assets are Q-martingales)

$$\widehat{X}_t = e^{-r(T-t)} E^Q \left[\left. \widehat{X}_T \right| \mathcal{F}_t \right] \qquad (2)$$

so we obtain from (1) and (2):

Q is now the base measure

$$\widehat{X}_t = x_0 e^{rt} E^Q \left[\left. L_T^{-1} \right| \mathcal{F}_t \right]$$

abstract theory $\Rightarrow$ $L_T^{-1} = dP/dQ$ on $\mathcal{F}_T$

But L^{-1} is a Q-martingale (why?) so we obtain

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

The Optimal Portfolio

- We have computed the optimal wealth process: $\hat{X}_t$,
  once we know L_t.
- How do we compute the optimal portfolio?

Assume for simplicity that we have a standard Black-Scholes model (complete model!)

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt
\end{aligned}
$$

Recall that

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

L_t^{-1} satisfies (Girsanov theory) an equation like $dL_t^{-1} = L_t^{-1} \varphi_t dW_t^Q$ for some φ_t

Q-Wiener

$\boxed{W \text{ is P-Wiener}}$

$= L_t^{-1}(\dots)dt + L_t^{-1}\varphi_t dW_t$, as we know $dW^Q = (\dots)dt + dW_t$.

Basic Program

$z_t = z_0 e^{rt} L_t^{-1}$, you "know" L_t

1. Use Ito and the formula for $\widehat{X}_t$ to compute $d\widehat{X}_t$ like

$$d\widehat{X}_t = \widehat{X}_t(\quad)dt + \widehat{X}_t \beta_t dW_t$$

(find β_t later)

where we do not care about $(\quad)$.

2. Recall that (for some $\hat{u}_t$, portfolio weight)

involves dW_t

$$d\widehat{X}_t = \widehat{X}_t \left\{ (1 - \hat{u}_t)\frac{dB_t}{B_t} + \hat{u}_t \frac{dS_t}{S_t} \right\}$$

see p.382

which we write as

$$d\widehat{X}_t = \widehat{X}_t \{\quad\}\, dt + \widehat{X}_t \hat{u}_t \sigma dW_t$$

3. We can identify $\hat{u}$ as

$$\hat{u}_t = \frac{\beta_t}{\sigma}$$

(if we know β_t, see further down)

ψ_t on p. 382 is thus $\psi_t = -\varphi_t$

We recall

(1) $$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

We also recall that

(L_t is $\frac{dQ}{dP}$ on $\mathcal{F}_t$) $$dL_t = L_t \varphi dW_t,$$

where

$$\varphi = \frac{r - \mu}{\sigma}$$ (many lectures ago)

From this we have (Itô for L_t^{-1})

(2) $$dL_t^{-1} = \varphi^2 L_t^{-1} dt - L_t^{-1} \varphi dW_t = -\varphi L_t^{-1} dW_t^P$$

and we obtain from (1) and (2), do the calculus!,

$$d\widehat{X}_t = \widehat{X}_t \{ \quad \} dt - \widehat{X}_t \varphi dW_t \rightarrow \beta_t = -\varphi$$

Result: The optimal portfolio is given by $\hat{u}_t = \frac{\beta_t}{\sigma}$

$$\Rightarrow \quad \hat{u}_t = \frac{\mu - r}{\sigma^2}$$ (which we have seen as Market price of risk)

Note that $\hat{u}$ is a "myopic" portfolio in the sense that it does not depend on the time horizon T.

(Another example)

A Digression: The Numeraire Portfolio

Standard approach:

- Choose a fixed numeraire (portfolio) N.
- Find the corresponding martingale measure, i.e. find Q^N s.t.

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are Q^N -martingales.

Alternative approach: (swap the two steps above)

- Choose a fixed measure $Q \sim P$.
- Find numeraire N such that $Q = Q^N$:

$\frac{X_t}{N_t}$ is Q^N-martingale if X_t is value of traded asset

Special case:

- Set $Q = P$, our choice
- Find numeraire N such that $Q^N = P$ i.e. such that

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are $\overset{P}{\overset{\shortparallel}{Q^N}}$-martingales under the **objective** measure P.

- This N is called the **numeraire portfolio**.

Specialize further:

Log utility and the numeraire portfolio

Definition:
The **growth optimal portfolio** (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date T.

Theorem:
Assume that X is GOP. Then X is the numeraire portfolio.

wealth process is $X_t = x_0 e^{rt} L_t^{-1}$ (p. 381)

Proof:
We have to show that the process

see p. 385

$$Y_t = \frac{S_t}{X_t}$$

is a P martingale. (and likewise is $\frac{B_t}{X_t} = x_0^{-1} L_t$)

We have (see p. 381)

$$\frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t$$

likelihood ratio $\frac{dQ}{dP}$ on $\mathcal{F}_t$

which is a P martingale, since $x_0^{-1} e^{-rt} S_t$ is a Q martingale. use "Bayes" (Additional exercise 3 = exercise C.9 in the book)

→ End of lecture 13b ←