→ Start of lecture 2 (or with p.81)

5.

Appendices

Appendix A: Black-Scholes vs Binomial

if you know this

Consider a binomial model for an option with a fixed time to maturity T and a fixed strike price K.

- Build a binomial model with n periods for each $n = 1, 2, ...$
- Use the standard formulas for scaling the jumps:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad \Delta t = T/n, \qquad \sigma > 0, \; d < 1 < u$$

- For a large n, the stock **price** at time T will then be a **product** of a large number of i.i.d. random variables.
- More precisely

0 ... $T = n\Delta t$ — n divisions of $[0,T]$

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

where n is the number of periods in the binomial model and $Z_i = u, d$. In S_T number of u's and d's matters only, not the order $\Rightarrow$ looks like successes/failures in Binomial models

Recall (this is the Cox-Ross-Rubinstein model)

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

- The stock **price** at time T will be a **product** of a large number of i.i.d. random variables.

- The log-**return** will be a large **sum** of i.i.d. variables.
$\log S_T = \log S_0 + \sum_{i=1}^{n} \log Z_i$

- The Central Limit Theorem will kick in. (details omitted)

- In the limit, **returns** will be **normally** distributed. $\Rightarrow$

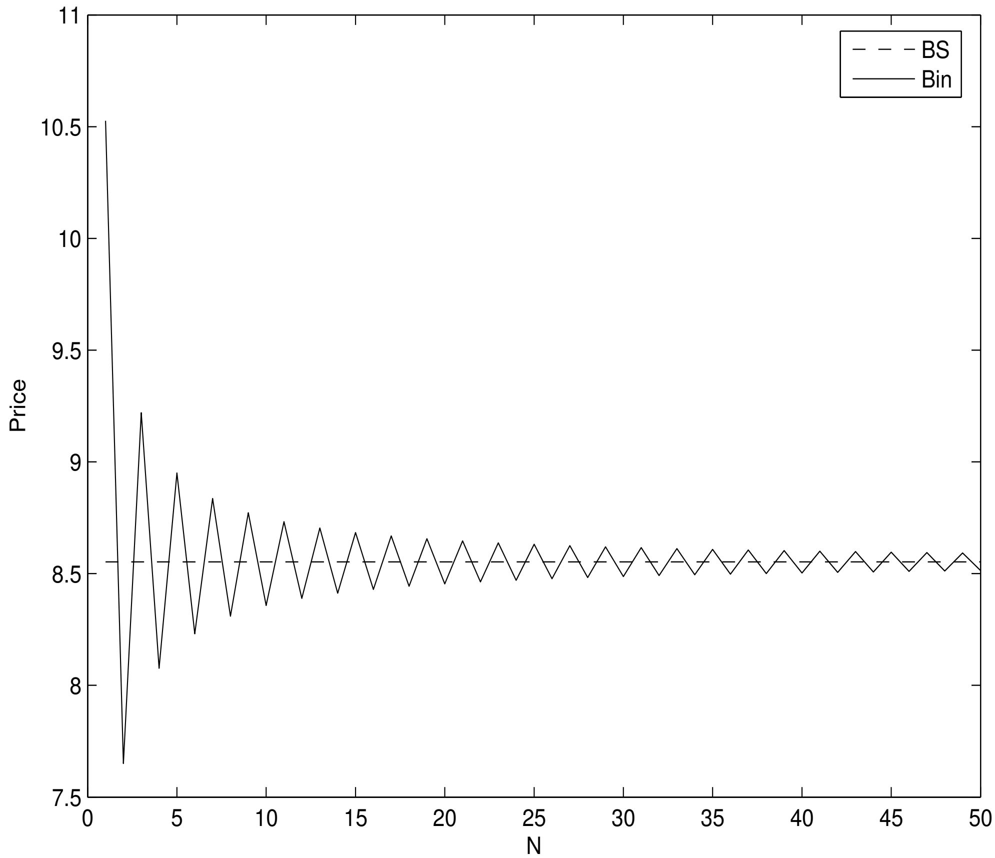
- Stock **prices** will be **lognormally** distributed.

- We are in the Black-Scholes model.

- The binomial price will converge to the Black-Scholes price.

Theorem! (no proof/details)

Binomial convergence to Black-Scholes



Binomial ∼ Black-Scholes

The intuition from the Binomial model carries over to Black-Scholes.

- The B-S model is "just" a binomial model where we rebalance the portfolio infinitely often.
- The B-S model is "thus" complete. (notion comes later)
- Completeness explains the unique prices for options in the B-S model.
- The B-S price for a derivative is the limit of the binomial price when the number of periods is very large.

These statements are actually theorems.
Take them for granted.

- Remark: Binomial models have been used in practice (even in Excel)

Appendix B: Portfolio theory

(this is a copy of page 53)

We consider a market with N assets.

$$S_t^i = \text{price at } t\text{, of asset No } i.$$

A **portfolio** strategy is an adapted vector process

$$h_t = (h_t^1, \cdots, h_t^N)$$

where

$$\begin{aligned} h_t^i &= \text{number of units of asset } i, \\ V_t &= \text{market value of the portfolio} \end{aligned}$$

$$V_t = \sum_{i=1}^N h_t^i S_t^i$$

The portfolio is typically of the form

$$h_t = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

Self financing portfolios

We want to study **self financing** portfolio strategies, i.e. portfolios where

- There is now external infusion and/or withdrawal of money to/from the portfolio.
- Purchase of a "new" asset must be financed through sale of an "old" asset.

How is this formalized?

the one on p.54

the one on p.54

Problem: Derive an expression for dV_t for a self financing portfolio.

We analyze in discrete time, and then go to the continuous time limit.

Discrete time portfolios

We trade at discrete points in time $t = 0, 1, 2, \ldots$.

Price vector process:

$$S_n = (S_n^1, \cdots, S_n^N), \quad n = 0, 1, 2, \ldots$$

Portfolio process:

$$h_n = (h_n^1, \cdots, h_n^N), \quad n = 0, 1, 2, \ldots$$

Interpretation: At time n we buy the portfolio h_n at the price S_n, and keep it until time $n + 1$.

Value process:

$$V_n = \sum_{i=1}^{N} h_n^i S_n^i = \underbrace{h_n S_n}_{\text{inner}}$$

inner product notation

The self financing condition

- At time $n-1$ we buy the portfolio h_{n-1} at the price S_{n-1}.
- At time n this portfolio is worth $h_{n-1}S_n$. [prices jump from S_{n-1} to S_n]
- At time n we buy the new portfolio h_n at the price S_n.
- The cost of this new portfolio is h_nS_n.
- The self financing condition is the **budget constraint**

$$h_{n-1}S_n = h_nS_n$$

The self financing condition

Recall:

$$V_n = h_n S_n$$

Definition: For any sequence $x_1, x_2, \ldots$ we define the sequence Δx_n by

$$\Delta x_n = x_n - x_{n-1}$$

Problem: Derive an expression for ΔV_n for a self financing portfolio.

Lemma: For any pair of sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ we have the relation

$$\Delta(xy)_n = x_{n-1}\Delta y_n + y_n \Delta x_n$$

(Abel's summation formula:

$x_n y_n - x_0 y_0 = \sum x_{i-1} \Delta y_i + \sum y_i \Delta x_i$)

like integration by parts!

Proof: Do it yourself.

Recall

$$V_n = h_n S_n$$

From the Lemma we have

$$\Delta V_n = \Delta(hS)_n = h_{n-1}\Delta S_n + S_n \Delta h_n$$

Recall the self financing condition

$$h_{n-1} S_n = h_n S_n$$

which we can write as

$$S_n \Delta h_n = 0$$

Inserting this into the expression for ΔV_n gives us.

Proposition: The dynamics of a self financing portfolio are given by

$$\Delta V_n = h_{n-1} \Delta S_n$$

Note the forward increments!

Portfolios in continuous time

Price process:

$$S_t^i = \text{price at } t \text{, of asset No } i.$$

Portfolio:

$$h_t = (h_t^1, \cdots, h_t^N)$$

Value process

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

From the self financing condition in discrete time

$$\Delta V_n = h_{n-1} \Delta S_n$$

we are led to the following definition. (by analogy!)

Definition: The portfolio h is self financing if and only if

the one on page 54

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

need that the S^i are "semimartingales" (or diffusions)

[copy of page 55]

Relative weights

Definition:

ω_t^i = relative portfolio weight on asset No i.

We have

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t}$$

Insert this into the self financing condition

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

We obtain

Portfolio dynamics:

$$dV_t = V_t \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$$

Interpret!

recall model: $dS_t = \mu S_t dt + \sigma S_t dW_t$

Appendix C: The original Black-Scholes PDE argument

Consider the following portfolio.

- "borrow"
  Short one unit of the derivative, with pricing function $f(t,s)$: you have -1 as a quantity
- Hold x units of the underlying S. (or x_t at time t)
  [later we find a "good" x]

The portfolio value is given by

$$V = -f(t, S_T) + x_t S_t$$

($x_t = x$, short hand notation)

The object is to choose x_t such that the portfolio is risk free for an infinitesimal interval of length dt.

self financing!

We have $dV = -df + x_t dS$ and from Itô we obtain (for df, use dS_t)

$$
\begin{aligned}
dV &= -\left\{\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial s} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}\right\}dt \\
&- \sigma S\frac{\partial f}{\partial s}dW + \underbrace{x\mu S dt + x\sigma S dW}_{x_t dS_t}
\end{aligned}
$$

Rearrange:

$$
\begin{aligned}
dV &= \left\{ x\mu S - \frac{\partial f}{\partial t} - \mu S \frac{\partial f}{\partial s} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\
&+ \sigma S \left\{ x - \frac{\partial f}{\partial s} \right\} dW
\end{aligned}
$$

We obtain a risk free portfolio if we choose x as

$$
x = \frac{\partial f}{\partial s}
$$

(the "good" x)

and then we have, after simplification, (inserting this x in $dV = \dots$)

$$
dV = \left\{ -\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt
$$

Using $V = -f + xS$ and x as above, the return dV/V is thus given by

$$
\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}}{-f + S \frac{\partial f}{\partial s}} dt
$$

Remark: not clear what the "logical problems" of page 62 are.

We had (previous page)

$$\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}}dt$$

This portfolio is risk free, so absence of arbitrage implies that

$$\frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}} \overset{!}{=} r$$

(see p.60)

Simplifying this expression gives us the Black-Scholes PDE.

$$\begin{aligned}
\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2} - rf &= 0, \\
f(T,s) &= \Phi(s).
\end{aligned}$$

→ end of lecture 2a ←

Continuous Time Finance

Completeness and Hedging

(Ch 8-9)

Tomas Björk

Problems around Standard Black-Scholes

"over the counter"

- We **assumed** that the derivative was traded. How do we price $\boxed{\text{OTC}}$ products?

- Why is the option price independent of the expected rate of return α of the underlying stock?

  previously, we used μ instead of α as notation

- Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with **completeness**.

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

Definition:
We say that a T-claim X can be **replicated**, alternatively that it is **reachable** or **hedgeable**, if there exists a self financing portfolio h such that

$$V_T^h = X, \quad P - a.s.$$

$V_t^h = V_t =$
$= h_t S_t$, $t \leq T$

In this case we say that h is a **hedge** against X. Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**

Basic Idea: If X can be replicated by a portfolio h then the arbitrage free price for X is given by

$$\Pi_t [X] = V_t^h.$$

(law of one price for reachable claim)

[If $\Pi_t(X) < V_t^h$, you sell the portfolio, buy the claim and put the surplus aside. At time T you sell the claim and buy the portfolio back: net cost is zero.]

Similar argument for $t = 0$

Consider the following

Trading Strategy

Consider a replicable claim X which we want to sell at $t = 0$..

- Compute the price $\Pi_0[X]$ and sell X at a slightly (well) higher price. [Suppose you are able to do that]
- Buy the hedging portfolio and invest the surplus in the bank.
- Wait until expiration date T.
- The liabilities stemming from X is exactly matched by V_T^h, and we have our surplus in the bank.

Completeness of Black-Scholes

Theorem: The Black-Scholes model is complete.

"arbitrary" function

Proof. Fix a claim $X = \Phi(S_T)$. We want to find processes V, u^B and u^S such that

relative portfolios!

Recall p.55:
self financing condition:

$$
\begin{aligned}
dV_t &= V_t \left\{ u_t^B \frac{dB_t}{B_t} + u_t^S \frac{dS_t}{S_t} \right\} \\
V_T &= \Phi(S_T).
\end{aligned}
$$

i.e. (recall $dB_t = rB_t dt$, $dS_t = \alpha S_t dt + \sigma S_t dW_t$)

$$
\begin{aligned}
dV_t &= V_t \left\{ u_t^B r + u_t^S \alpha \right\} dt + V_t u_t^S \sigma dW_t, \\
V_T &= \Phi(S_T).
\end{aligned}
$$

$V_t = h_t^B B_t + h_t^S S_t \Rightarrow$

$u_t^B = \dfrac{h_t^B B_t}{V_t}$

Heuristics:
Let us **assume** that X is replicated by $\cancel{h} = (u^B, u^S)$ with value process V.

Ansatz: (reasonable, based on $X = \Phi(S_T)$ and X is Markov)

$$V_t = F(t, S_t)$$

for F to be found

Ito gives us

$$dV = \left\{ F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss} \right\} dt + \sigma S F_s dW,$$

Write this as

$$dV = V \left\{ \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{S F_s}{V} \sigma dW. \quad (*)$$

Compare with

$$dV = V \left\{ u^B r + u^S \alpha \right\} dt + V u^S \sigma dW$$

[dW, and dt, terms should coincide]

Define u^S by (time index t and S_t explicitly written)

$$u_t^S = \frac{S_t F_s(t, S_t)}{F(t, S_t)},$$

This gives us the eqn, from (*) on p.103,

$$dV = V\left\{\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^S \alpha\right\} dt + V u^S \sigma dW.$$

Again Compare with

$$dV = V\left\{u^B r + u^S \alpha\right\} dt + V u^S \sigma dW$$

Natural choice for u^B is given by (match the dt terms)

$$u^B = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF},$$

with u_B and u_S of p.104

The relation $u^B + u^S = 1$ gives us the Black-Scholes PDE

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0.$$

The condition

$$V_T = \Phi\left(S_T\right)$$

gives us the boundary condition

$$F(T, s) = \Phi(s)$$

even

Moral: The model is complete and we have explicit formulas for the replicating portfolio.

the u^B and u^S of p. 104

Main Result

Theorem: Define F as the solution to the boundary value problem

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF & = & 0, \\ F(T,s) & = & \Phi(s). \end{cases}$$

$= \Phi(S_T)$

Then X can be replicated by the relative portfolio

use u^B on p. 104 and the PDE

$$\begin{aligned} u_t^B &= \frac{F(t,S_t) - S_t F_s(t,S_t)}{F(t,S_t)}, \\ u_t^S &= \frac{S_t F_s(t,S_t)}{F(t,S_t)}. \end{aligned}$$

The corresponding absolute portfolio is given by

$h_t^B = u_t^B \frac{V_t}{B_t}$, $V_t = F(t, S_t)$

$$\begin{aligned} h_t^B &= \frac{F(t,S_t) - S_t F_s(t,S_t)}{B_t}, \\ h_t^S &= F_s(t,S_t), \end{aligned}$$

and the value process V^h is given by

$$V_t^h = F(t,S_t).$$

(see also book Lemma 8.4).

Notes

use the PDE

- Completeness explains unique price - the claim is superfluous! nothing "new" compared to S_t and B_t in the market
- Replicating the claim $P - a.s.$ $\Longleftrightarrow$ Replicating the claim $Q - a.s.$ for any $Q \sim P$. Thus the price only depends on the support of P.

determined by P

- Thus (Girsanov) it will not depend on the drift α of the state equation.
- The completeness theorem is a nice theoretical result, but the replicating portfolio is **continuously rebalanced**. Thus we are facing very high transaction costs.

- Proof only given for claims of the type $\Phi(S_T)$ and under the Ansatz $V_t = F(t, S_t)$. More general result still true: "Any" claim can be hedged

Completeness vs No Arbitrage

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$R = \text{number of random sources}$$
$$N = \text{number of risky assets}$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

for instance to annihilate (the) random sources and find arbitrage opportunities

Meta-Theorem

(Compare to solve $Ax=b$, with $A \in \mathbb{R}^{m \times n}$ etc.
when (unique) solution? $m \le n$ (if you ignore "rank" conditions)

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \le R$$

- The market is complete if and only if

$$N \ge R$$

Example:
The Black-Scholes model. R=N=1. Arbitrage free and complete.

→ End of lecture 2b ←