→ Start of lecture 3 ←

Parity Relations

Let Φ and Ψ be contract functions for the T-claims $Z = \Phi(S_T)$ and $Y = \Psi(S_T)$. Then for any real numbers α and β we have the following price relation.

$$\Pi_t [\alpha\Phi + \beta\Psi] = \alpha\Pi_t [\Phi] + \beta\Pi_t [\Psi] .$$

see Feynman-Kac, p.76

or use a no-arbitrage argument

Proof. Linearity of mathematical $\boxed{\text{expectation.}}$

Consider the following "basic" contract functions.

$$\begin{aligned}
\Phi_S(x) &= x, \\
\Phi_B(x) &\equiv 1, \\
\Phi_{C,K}(x) &= \max [x - K, 0] .
\end{aligned}$$

(think of x as a value of S_T)

Prices:

$$\left.\begin{aligned}
\Pi_t [\Phi_S] &= S_t, \\
\Pi_t [\Phi_B] &= e^{-r(T-t)},
\end{aligned}\right\} \text{think a bit!}$$

$$\Pi_t [\Phi_{C,K}] = c(t, S_t; K, T).$$

just notation to express that the price of the call depends on t, S_t, K, T (and more)

If we have for more options, with strike K_i

$$\Phi = \alpha \Phi_S + \beta \Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i},$$

then

$$\Pi_t[\Phi] = \alpha \Pi_t[\Phi_S] + \beta \Pi_t[\Phi_B] + \sum_{i=1}^{n} \gamma_i \Pi_t[\Phi_{C,K_i}]$$

We may replicate the claim Φ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a **buy-and hold** portfolio:

the α, β, γ_i

- α shares of the underlying stock,
- β zero coupon T-bonds with face value $\$1$,:
- γ_i European call options with strike price K_i, all maturing at T.

pay off at time T is $\$1$;
value at time $t<T$: $e^{-r(T-t)}$ (in $\$$)

Put-Call Parity

Consider a European put contract

$$\Phi_{P,K}(s) = \max\left[K - s, 0\right]$$

It is easy to see (draw a figure) that
(or simple algebra)

$$\begin{aligned}\Phi_{P,K}(x) &= \Phi_{C,K}(x) - s + K \\ &= \Phi_{\cancel{P}\,C,K}(x) - \Phi_S(x) + \underline{\Phi_B(x)}\,K\end{aligned}$$

We immediately get

Put-call parity:

$$e^{-r(T-t)}$$

$$p(t, s; K, T) = c(t, s; K, T) - s + Ke^{-r(T-t)}$$

Thus you can construct a synthetic put option, using a buy-and-hold portfolio. (with a call option)

(see Prop. 9.3 in the book).

Delta Hedging

name has to do with the "Greeks", see further down

Consider a fixed claim

$$X = \Phi(S_T)$$

with pricing function

$$F(t, s).$$ (or $F(t, S_t)$ with $S_t = s$)

Setup:
We are at time t, and have a short (interpret!) position in the contract. ("debt" in the contract)

Goal:
Offset the risk in the derivative by buying (or selling) the (highly correlated) underlying, find how much to buy/sell

Definition:
A position in the underlying (S_t) is a **delta hedge** against the derivative if the portfolio (underlying + derivative) is immune against small changes in the underlying price. calls for differentiation, derivatives in the sense of calculus

Formal Analysis

minus because of the short position

$$
\begin{aligned}
-1 &= \text{number of units of the derivative product} \\
x &= \text{number of units of the underlying} \\
s &= \text{today's stock price} \\
t &= \text{today's date}
\end{aligned}
$$

, $S_t = s$

Value of the portfolio:

$$V = -1 \cdot F(t, s) + x \cdot s$$

A delta hedge is characterized by the property that

$$\frac{\partial V}{\partial s} = 0.$$

(insensitive to changes in s)

We obtain

$$-\frac{\partial F}{\partial s} + x = 0$$

Solve for x!

see also on p.106 lect 5

Result:
We should have

$$\hat{x} = \frac{\partial F}{\partial s}$$

shares of the underlying in the delta hedged portfolio.

Definition:
For any contract, its "delta" is defined by

$$\Delta = \frac{\partial F}{\partial s}.$$

Recall: (F = pricing function)

Result:
We should have

$$\hat{x} = \Delta$$

shares of the underlying in the delta hedged portfolio.

Warning:
The delta hedge must be rebalanced over time. (why?)

($\hat{x}_t = \Delta_t = \frac{\partial F}{\partial s}(t, S_t)$, time dependent)

Black Scholes

For a European Call in the Black-Scholes model we have

$$\Delta = N[d_1] = P(N(0,1) \le d_1)$$

NB This is **not** a trivial result! But see p. 71, Black-Scholes case.

From put call parity it follows (how?) that Δ for a European Put is given by

$$\Delta = N[d_1] - 1$$

$$= -P(N(0,1) > d_1) < 0$$

Check signs and interpret!

Rebalanced Delta Hedge

- Sell one call option at time $t = 0$ at the B-S price F.
- Compute Δ and by Δ shares. (Use the income from the sale of the option, and borrow money if necessary.)
- Wait one day (week, minute, second..). The stock price has now changed.
- Compute the new value of Δ, and borrow money in order to adjust your stock holdings.
- Repeat this procedure until $t = T$. Then the value of your portfolio (B+S) will match the value of the option almost exactly.

and better if you rebalance more frequently

- Lack of perfection comes from discrete, instead of continuous, trading.

- You have created a "synthetic" option. (Replicating portfolio).

Formal result:
The relative weights in the replicating portfolio are

$$u_S = \frac{S \cdot \Delta}{F},$$

$$u_B = \frac{F - S \cdot \Delta}{F}$$

(See p.106, with $\Delta = F_s(t, S_t)$)

Portfolio Delta

Assume that you have a portfolio consisting of derivatives

$$\Phi_i(S_{T_i}), \quad i = 1, \cdots, n$$

all **written on the same underlying** stock S.

$$\begin{aligned} F_i(t,s) &= \text{pricing function for i:th derivative } (S_t = s) \\ \Delta_i &= \frac{\partial F_i}{\partial s} \\ h_i &= \text{units of i:th derivative} \end{aligned}$$

Portfolio value:

$$\Pi = \sum_{i=1}^{n} h_i F_i$$

Portfolio delta:

$$\Delta_\Pi = \sum_{i=1}^{n} h_i \Delta_i$$

Gamma

A problem with discrete delta-hedging is.

- As time goes by, S will change.
- This will cause $\Delta = \frac{\partial F}{\partial s}$ to change, see page 115
- Thus you are sitting with the "wrong" value of delta. (at a later time instant)

Moral:

- If delta is sensitive to changes in S, then you have to rebalance often.
- If delta is insensitive to changes in S you do not need to rebalance so often, or perhaps not at all

Definition:
Let Π be the value of a derivative (or portfolio). **Gamma** (Γ) is defined as

$$\Gamma = \frac{\partial \Delta}{\partial s}$$

i.e.

$$\Gamma = \frac{\partial^2 \Pi}{\partial s^2},$$ Π is the pricing function, often $F(t, s)$

Gamma is a measure of the sensitivity of Δ to changes in S.

Result: For a European Call in a Black-Scholes model, Γ can be calculated as

$$\Gamma = \frac{N'[d_1]}{S\sigma\sqrt{T-t}}$$ (Exercise!)

Important fact:
For a position in the underlying stock itself we have

$$\Gamma = 0$$ (trivial!)

Gamma Neutrality

A portfolio Π is said to be **gamma neutral** if its gamma equals zero, i.e.

$$\Gamma_{\Pi} = 0$$

- Since $\Gamma = 0$ for a stock you can not gamma-hedge using only stocks. Typically you use some derivative to obtain gamma neutrality.

→ End of lecture 3a ←

General procedure

Given a portfolio Π with underlying S. Consider two derivatives with pricing functions F and G.

$$\begin{aligned} x_F &= \text{number of units of } F \\ x_G &= \text{number of units of } G \end{aligned}$$

Problem:
Choose x_F and x_G such that the entire portfolio is delta- and gamma-neutral.

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

repeat:

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

We get the equations

$$\frac{\partial V}{\partial s} = 0,$$ (delta neutral)

$$\frac{\partial^2 V}{\partial s^2} = 0.$$ (gamma neutral)

i.e.

$$\Delta_\Pi + x_F \Delta_F + x_G \Delta_G = 0,$$

$$\Gamma_\Pi + x_F \Gamma_F + x_G \Gamma_G = 0$$

Solve for x_F and x_G! (linear system, has a unique solution?)

in general yes, if G is sufficiently different from F

Particular Case

- In many practical cases the original portfolio Π is already delta neutral.
- Then it is natural to use a derivative to obtain gamma-neutrality.
- This will destroy the delta-neutrality. for the new portfolio
- Therefore we use the underlying stock (with zero gamma!) to delta hedge in the end: next page

additional derivative

stock needed for Δ-neutrality

Formally:

$$V = \Pi + x_F \cdot F + x_S \cdot S$$

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S \Delta_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F + x_S \Gamma_S &= 0 \end{aligned}$$

We have

(assumption was Π is Δ neutral)

$$\begin{aligned} \Delta_\Pi &= 0, \\ \Delta_S &= 1 \\ \Gamma_S &= 0. \end{aligned}$$

i.e.

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F &= 0 \end{aligned}$$

Solution is
$$\begin{cases} x_F = -\dfrac{\Gamma_\Pi}{\Gamma_F} \\ x_S = \dfrac{\Delta_F \Gamma_\Pi}{\Gamma_F} - \Delta_\Pi \end{cases}$$

Further Greeks

$$\Theta = \frac{\partial \Pi}{\partial t},$$

$$V = \frac{\partial \Pi}{\partial \sigma},$$

$$\rho = \frac{\partial \Pi}{\partial r}$$

V is pronounced "Vega".

NB!

- A delta hedge is a hedge against the movements in the underlying stock, given a **fixed model**.
- A Vega-hedge is not a hedge against movements of the underlying asset. It is a hedge against a **change of the model itself**: σ is a model parameter.

Continuous Time Finance

The Martingale Approach

I: Mathematics

(Ch 10-12)

Tomas Björk

a purely theoretical lecture

Introduction

which you probably know already

In order to understand and to apply the martingale approach to derivative pricing and hedging we will need to some basic concepts and results from measure theory. These will be introduced below in an informal manner - for full details see the textbook.

Many propositions below will be proved but we will also present a couple of central results without proofs, and these must then be considered as dogmatic truths. You are of course not expected to know the proofs of such results (this is outside the scope of this course) but you are supposed to be able to **use** the results in an operational manner.

They are working knowledge, part of your toolkit.

Contents

1.

Events and sigma-algebras

Events and sigma-algebras

(book: Section A.2)

contains all "relevant" outcomes

Consider a probability measure P on a sample space Ω. An **event** is simply a subset $A \subseteq \Omega$ and $P(A)$ is the probability that the event A occurs.

For technical reasons, a probability measure can only be defined for a certain "nice" class $\mathcal{F}$ of events, so for $A \in \mathcal{F}$ we are allowed to write $P(A)$ as the probability for the event A.

For technical reasons the class $\mathcal{F}$ must be a **sigma-algebra**, which means that $\mathcal{F}$ is closed under the usual set theoretic operations like complements, countable intersections and countable unions. !

Interpretation: We can view a σ-algebra $\mathcal{F}$ as formalizing the idea of information. More precisely: A σ-algebra $\mathcal{F}$ is a collection of events, and if we assume that we have access to the information contained in $\mathcal{F}$, this means that for every $A \in \mathcal{F}$ we know exactly if A has occured or not.

Probability space is the triple $(\Omega, \mathcal{F}, P)$

Borel sets

Definition: The **Borel algebra** $\mathcal{B}$ is the smallest sigma-algebra on R which contains all intervals. A set B in $\mathcal{B}$ is called a **Borel set**.

Remark: There is no constructive definition of $\mathcal{B}$, but almost all subsets of R that you will ever see will in fact be Borel sets, so the reader can without danger think about a Borel set as "an arbitrary subset of R".

alternatively, contains all open sets, or
contains all closed sets
(these are topological concepts)

Random variables

Section B.1

An $\mathcal{F}$-**measurable random variable** X is a a mapping

$$X : \Omega \rightarrow R$$

such that $\{X \in B\} = \{\omega \in \Omega : \; X(\omega) \in B\}$ belongs to $\mathcal{F}$ for all Borel sets B. This guarantees that we are allowed to write $P(X \in B)$. Instad of writing "X is $\mathcal{F}$-measurable" we will often write $X \in \mathcal{F}$.

This means that if $X \in \mathcal{F}$ then the value of X is completely determined by the information contained in $\mathcal{F}$.

If we have another σ-algebra $\mathcal{G}$ with $\mathcal{G} \subseteq \mathcal{F}$ then we interpret this as "$\mathcal{G}$ contains less information than $\mathcal{F}$".

→ End of lecture 3b ←