→ Start of lecture 5 ←

The likelihood process on a filtered space

We now consider the case when we have a probability measure P on some space Ω and that instead of just one σ-algebra $\mathcal{F}$ we have a **filtration**, i.e. an increasing family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$.

The interpretation is as usual that $\mathcal{F}_t$ is the information available to us at time t, and that we have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Now assume that we also have another measure Q, and that for some fixed T, we have $Q << P$ on $\mathcal{F}_T$. We define the random variable L_T by

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

as on p.150

Since $Q << P$ on $\mathcal{F}_T$ we also have $Q << P$ on $\mathcal{F}_t$ for all $t \leq T$ and we define

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \leq t \leq T$$

part of the R-N theorem, p.154

For every t we have $\boxed{L_t \in \mathcal{F}_t}$, so L is an adapted process, known as the **likelihood process**.

A process X is adapted (to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$) if for every $t\geq 0$ X_t is $\mathcal{F}_t$-measurable.

The L process is a P martingale

We recall that

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \le t \le T$$

p. 153

Since $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$ we can use Proposition 5 and deduce that

$$L_s = E^P\left[L_t \middle| \mathcal{F}_s\right] \quad s \le t \le T$$

and we have thus proved the following result.

Proposition: Given the assumptions above, the likelihood process L is a P-martingale.

A process X is a $(P, \mathcal{F}_t)$-martingale if it
(i) it is adapted (to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$),
(ii) $E|X_t| < \infty$, $\forall t \geq 0$,
(iii) $E[X_t | \mathcal{F}_s] = X_s$, $\forall t \geq s \geq 0$ ← martingale property

Here $E = E^P$, expectation using P.

Where are we heading?

(and why do we have to know this?)

We are now going to perform measure transformations on Wiener spaces, where P will correspond to the objective measure and Q will be the risk neutral measure.

For this we need define the proper likelihood process L and, since L is a P-martingale, we have the following natural questions.

- What does a martingale look like in a Wiener driven framework? (like Black-Scholes setting)
- Suppose that we have a P-Wiener process W and then change measure from P to Q. What are the properties of W under the new measure Q?

These questions are handled by the Martingale Representation Theorem, and the Girsanov Theorem respectively.

Recall BS framework, with $dS = \mu S dt + \sigma S dW$ (under P)

and $dS = rS dt + \sigma S dW^Q$ (under Q)

We will see that $P \sim Q$ on $\mathcal{F}_T$ (and then also on $\mathcal{F}_t$, $t \le T$)

4.

The Martingale Representation Theorem

(Section 11.1)

Intuition

from general Itô theory

Suppose that we have a Wiener process W under the measure P. We recall that if h is adapted (and integrable enough) and if the process X is defined by

$E\int_0^T h_s^2 ds < \infty$

$$X_t = x_0 + \int_0^t h_s dW_s$$

then X is a a martingale. We now have the following natural question:

Question: Assume that X is an arbitrary martingale. Does it then follow that X has the form

!

$$X_t = x_0 + \int_0^t h_s dW_s$$

for some adapted process h?

In other words: Are **all** martingales stochastic integrals w.r.t. W?

Answer : No, but .....

It is immediately clear that all martingales can **not** be written as stochastic integrals w.r.t. W. Consider for example the process X defined by

$$X_t = \begin{cases} 0 & \text{for} \quad 0 \le t < 1 \\ Z & \text{for} \quad t \ge 1 \end{cases}$$

where Z is an random variable, independent of W, with $E[Z] = 0$.

You(!) compute $E[X_t | \mathcal{F}_s]$ for different $t > s$

X is then a martingale (why?) but it is clear (how?) that it cannot be written as

?

which filtration?
X adapted $\Rightarrow$
Z should be $\in \mathcal{F}_t$, for $t \geq 1$.

$$X_t = x_0 + \int_0^t h_s dW_s$$

for any process h. See next page

Simplest case:

$$\mathcal{F}_t = \begin{cases} \{\emptyset, \Omega\}, & t \leq 1 \\ \sigma(Z), & t > 1. \end{cases}$$

Intuition

The intuitive reason why we cannot write

$$X_t = x_0 + \int_0^t h_s dW_s$$

in the example above is of course that the random variable Z "has nothing to do with" the Wiener process W. In order to exclude examples like this, we thus need an assumption which guarantees that our probability space only contains the Wiener process W and nothing else.

This idea is formalized by assuming that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ **is the one generated by the Wiener process** W,

$\mathcal{F}_t = \sigma(W_s, s \leq t)$.

not the case with the example on p. 159

Note that the X of p. 159 is NOT adapted to this filtration.

The Martingale Representation Theorem

Theorem. Let W be a P-Wiener process and assume that the filtation is the **internal** one i.e.

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\left\{W_s;\ 0 \le s \le t\right\}$$

Then, for every $(P, \mathcal{F}_t)$-martingale X, there exists a real number x and an adapted process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

i.e.

$$dX_t = h_t dW_t.$$

Proof: Hard. This is very deep result.

Crucial is that X is adapted to this special filtration

Note

For a given martingale X, the Representation Theorem above guarantees the existence of a process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

The Theorem does **not**, however, tell us how to find or construct the process h.

5.

The Girsanov Theorem

Sections 11.2, 11.3

Setup

Let W be a P-Wiener process and fix a time horizon T. Suppose that we want to change measure from P to Q on $\mathcal{F}_T$. For this we need a P-martingale L with $L_0 = 1$ to use as a likelihood process, and a natural way of constructing this is to choose a process g and then define L by

see p. 155

$$\begin{cases} dL_t &= g_t dW_t \\ L_0 &= 1 \end{cases}$$ (to get a martingale)

This definition does not guarantee that $L \geq 0$, so we make a small adjustment. We choose a process φ and define L by

take $g_t = L_t \varphi_t$ for some φ_t

$$(*) \quad \begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

under integrability conditions!

The process L will again be a martingale and we easily obtain

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

Apply the Itô formula to L_t to see that $(*)$ holds:

$L_t = f(X_t)$, with $f(x) = e^x$, $X_t = \int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds$,

and $(dX_t)^2 = \varphi_t^2 dt$

Thus we are guaranteed that $L \geq 0$. We now change measure form P to Q by setting

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \leq t \leq T$$

"The" main problem is to find out what the properties of W are, under the new measure Q. This problem is resolved by the **Girsanov Theorem**.

Recall: $dQ = L_T dP$ and

$$L_t = E^P[L_T | \mathcal{F}_t]$$

The Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem: Choose an adapted process φ, and define the process L by

OK if L is a P-martingale

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases} \quad (*)$$

Assume that $E^P[L_T] = 1$, *) and define a new mesure Q on $\mathcal{F}_T$ by

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

Then $Q << P$ and the process W^Q, defined by

$$W_t^Q = W_t - \int_0^t \varphi_s ds$$

is Q-Wiener. We can also write this as

$$dW_t = \varphi_t dt + dW_t^Q$$

*) In fact, with $(*)$, L is a martingale $(t \le T)$ iff $E^P[L_T] = 1$

Changing the drift in an SDE

(Section 11.5)

The single most common use of the Girsanov Theorem is as follows. (related to BS like models)

Suppose that we have a process X with P dynamics

$$(*) \qquad dX_t = \mu_t dt + \sigma_t dW_t$$

where μ and σ are adapted and W is P-Wiener.

We now do a Girsanov Transformation as above, and the question is what the Q-dynamics look like.

From the Girsanov Theorem we have

$$dW_t = \varphi_t dt + dW_t^Q \qquad \text{(page 166)}$$

and substituting this into the P-dynamics $(*)$ we obtain the Q dynamics as

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t dW_t^Q$$

Moral: The drift changes but the diffusion is unaffected, meaning that we keep on having the same σ_t in front of the new Brownian motion W^Q

The Converse Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem. Assume that:

- $Q << P$ on $\mathcal{F}_T$, with likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t \; 0, \leq t \leq T$$

- The filtation is the **internal** one .i.e.

$$\mathcal{F}_t = \sigma\left\{W_s;\; 0 \leq s \leq t\right\}$$

Then there exists a process φ such that

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

note (p.155) that
L is a P-martingale.

→ end of lecture 5a ←

→ start of lecture 5b ←

Continuous Time Finance

The Martingale Approach

II: Pricing and Hedging

(Ch 10-12)

Tomas Björk

Financial Markets (a recap)

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N\right]$$

Example: (Black-Scholes, $S^0 := B$, $S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N\right]$$

$h_t^i =$ number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = \overbrace{h_t}^{\text{row vector}} \underbrace{S_t}_{\text{column vector}}$$

or interpret as an inner product

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight: (from general theory):
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**. (needs assumptions)

NB! This simple observation is in fact the basis of the following theory.

Itô theory!

discrete time: $\Delta Y_t = h_t \Delta X_t$ predictable:

$E[\Delta Y_t | \mathcal{F}_{t-1}] = E[h_t \Delta X_t | \mathcal{F}_{t-1}]$
$= h_t E[\Delta X_t | \mathcal{F}_{t-1}]$
$h_t \in \mathcal{F}_{t-1}$

Arbitrage

The portfolio h is an **arbitrage** portfolio if (with $V = V^h$)

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T \geq 0, \ P - a.s.$ → $P(V_T \geq 0) = 1$
- $P(V_T > 0) > 0$

(weaker than the earlier definition on p. 59)

This implies $E_P[V_T] > 0$.

Main Question: When is the market free of arbitrage?

Remark: If the market is free of arbitrage, and h is a SF financing portfolio with $P(V_T \geq 0) = 1$,
then $P(V_T > 0) = 0$, equivalently $P(V_T = 0) = 1$

First Attempt

not realistic

Proposition: If $S_t^0, \cdots, S_t^N$ are P-martingales, then the market is free of arbitrage.

Proof:

Assume that $\cancel{V}$ h is an arbitrage strategy. Since

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i,$$

V is a P-martingale, so (because then expectations remain the same, $E^P V_T = E^P V_0$)

$$V_0 = E^P[V_T] > 0.$$

see previous page

This contradicts $V_0 = 0$.

True, but useless: next page

(but, as we'll see, there is a point in the argument)

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

not realistic

(We would have to assume that $\boxed{\alpha = r = 0}$)

We now try to improve on this result.

for S and B martingales

Choose S_0 as numeraire

(look at "normalized" prices)

Definition:
The **normalized price vector** Z is given by

$$Z_t = \frac{S_t}{S_t^0} = \left[1, Z_t^1, ..., Z_t^N\right]$$

$Z_t^0 = 1, \forall t \leq T$: certainly a martingale

The **normalized value process** V^Z is given by

$$V_t^Z = \sum_0^N h_t^i Z_t^i.$$

Idea:
The arbitrage and self financing concepts should be independent of the accounting unit.

Invariance of numeraire

Proposition: One can show (see the book) that

- S-arbitrage $\iff$ Z-arbitrage.
- S-self-financing $\iff$ Z-self-financing: so we can just talk of self-financing

$dV_t = h_t dS_t \iff$

$dV^Z_t = h_t dZ_t$

Insight:

- If h self-financing then

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

(note that we don't need $i=0$)

- Thus, if the **normalized** price process Z is a P-martingale, then V^Z is a martingale, argument as before.

Second Attempt

the normalized processes

Proposition: If $Z_t^0, \cdots, Z_t^N$ are P-martingales, then the market is free of arbitrage.

True, but still fairly useless.

by argument as on p. 173

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Use the quotient rule for differentiation of $Z_t^1 = \frac{S_t}{B_t}$

to get

$$
\begin{aligned}
dZ_t^1 &= (\alpha - r) Z_t^1 dt + \sigma Z_t^1 dW_t, \\
dZ_t^0 &= 0dt.
\end{aligned}
$$

We would have to assume "risk-neutrality", i.e. that $\alpha = r$. to have Z^1 is a P-martingale

But, in principle, $\alpha \neq r$. Later we'll see what happens under Q.

Arbitrage

Recall that h is an arbitrage if
p.172

- h is self financing
- $V_0 = 0$.
- $V_T \geq 0, \;\; P - a.s.$
- $P\left(V_T > 0\right) > 0$

$$\begin{cases} V_T \geq 0, \; Q\text{-a.s.} \\ Q(V_T > 0) > 0 \end{cases}$$

Major insight

This concept is invariant under an **equivalent change of measure!**

$P \sim Q$ iff $\begin{cases} P(A)=0 \iff Q(A)=0 \\ P(A)=1 \iff Q(A)=1 \\ P(A)>0 \iff Q(A)>0 \end{cases}$, as noted on p.143

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e. (usually defined on $\mathcal{F}_T$)

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \ldots, N$$

  are **Q-martingales**.

We now state the main result of arbitrage theory.

First Fundamental Theorem
of Asset Pricing (FTAP 1)

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

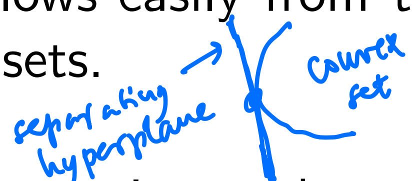
This theorem was already announced on p.81.

Comments

- It is very easy to prove that existence of EMM imples no arbitrage (see below).

- The other implication is technically very hard.

- For discrete time and finite sample space Ω the hard part follows "easily" from the separation theorem for convex sets.



- For discrete time and more general sample space we need the Hahn-Banach Theorem. (formulated as an infinite dimensional version of the separation theorem

- For continuous time the proof becomes technically very hard, mainly due to topological problems. See the textbook.

→ End of lecture 5 ←