→ Start of lecture 6a ←

Go back to p.180 ("NA ⟺ EMM")

Proof that EMM implies no arbitrage

Assume that there exists an EMM denoted by Q. Assume that (existence of an arbitrage opportunity,) $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$. *)

Note: $E^Q(V_T) > 0$!

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

This is a proof by contradiction.

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale (Itô theory)

$$\Downarrow$$

$$V_0 \underset{B_0 = 1}{=} V_0^Z = E^Q\left[V_T^Z\right] > 0$$

, contradicts $V_0 = 0$ on p. 172

$$\Downarrow$$

No arbitrage

*) All these statements also true for V_T^Z instead of V_T

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

(generalizes $B_t = e^{rt}$, $dB_t = r B_t dt$, which we assume in most examples.)

Example: The Black-Scholes Model

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Look for martingale measure. We set $Z_t = S_t/B_t = S_t e^{-rt}$

Standard calculus gives, differentiate a product:

no Itô is needed as B_t contains no Brownian term

$$dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t, \quad (6a)$$

Girsanov transformation on $[0, T]$:

$$
\begin{cases}
dL_t = L_t \varphi_t dW_t, \\
L_0 = 1.
\end{cases}
$$

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T$$

Girsanov: (see p.166)

$$dW_t = \varphi_t dt + dW_t^Q, \quad (6b)$$

where W^Q is a Q-Wiener process. (whereas W is P-Wiener)

Insert (6b) into (6a).

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

!

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma},$$ then $dZ_t = Z_t \sigma dW_t^Q$: martingale!

Q-dynamics of S: insert (6b) into equation for dS_t:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage, as follows from p.182 since we have now shown that Q is an EMM: Z is a martingale under Q.

Pricing

We consider a market $B_t, S_t^1, \dots, S_t^N$, (many risky assets).

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent (T)-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

New Approach: use the change of measure framework.

Def: $\Pi_t(X)$ is an arbitrage free price process if the extended market is arbitrage free

Solution

The $\boxed{\text{extended}}$ market

extra asset

$$B_t, S_t^1, \ldots, S_t^N, \Pi_t[X]$$

FTAP 1.

must be arbitrage free, so there must exist a martingale measure Q for $(S_t, \Pi_t[X])$. In particular

$$\frac{\Pi_t[X]}{B_t}$$

must be a Q-martingale, i.e. it has the martingale property,

$$\frac{\Pi_t[X]}{B_t} = E^Q\left[\left.\frac{\Pi_T[X]}{B_T}\right|\mathcal{F}_t\right] \qquad (6c)$$

$$\Longleftrightarrow \quad \Pi_t(X) = B_t E^Q\left[\frac{\Pi_T(X)}{B_T}\middle|\mathcal{F}_t\right] = E^Q\left[\frac{B_t}{B_T} X \middle| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T[X] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula , rewrite (6c),

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right],$$

if $dB_t = r_t B_t dt$,

NB: if $r_t \equiv r$, then

$$\Pi_t(X) = e^{-r(T-t)} E^Q[X | \mathcal{F}_t],$$

which we have encountered on p. 76.

Note: here have <u>not</u> used the Feynman-Kac formula to arrive at the same $\Pi_t(X)$, but used a martingale argument.

Example: The Black-Scholes Model

Q-dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q. \qquad (*)$$

NB: S is a Markov process under Q!

Simple claim:

$$X = \Phi(S_T),$$

Markov property gives

$$\Pi_t[X] = e^{-r(T-t)} E^Q[\Phi(S_T)|\mathcal{F}_t] \qquad (p.188)$$

$$= e^{-r(T-t)} E^Q[\Phi(S_T)|S_t]$$

Kolmogorov $\Rightarrow$

$\Pi_t(X)$ is function of (t and) S_t:

$$\Pi_t[X] = F(t, S_t)$$

where $F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T, s) &= \Phi(s). \end{cases}$$

use Feynman-Kac and the model (*).

Problem

Recall the valuation formula
(p.188)

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \,\middle|\, \mathcal{F}_t \right]$$, depends on Q !

What if there are several different martingale measures Q?

This is connected with the **completeness** of the market.

Hedging (recall p. 100)

Def: A portfolio h is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X$, $P - a.s.$: $P(V_T = X) = 1$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural (arbitrage free) way of pricing X is

$$\Pi_t[X] = V_t^h$$

(see p. 101 for a justification)

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

$$\Downarrow$$

" martingale representation theorem "

Fix T-claim X.

If h is a hedge for X then $V_T = X$ and

- $V_T^Z = \frac{X}{B_T}$; recall the normalized prices $Z_t^i = \frac{S_t^i}{B_t}$, which are Q-martingales;
- h is self financing, i.e.

$$dV_t^Z = \sum_1^K h_t^i dZ_t^i$$

, see p. 176.

Thus V^Z is a Q-**martingale**, $V_t^Z = E^Q\left[V_T^Z \middle| \mathcal{F}_t\right]$:

$$V_t^Z = E^Q\left[\left.\frac{X}{B_T}\right|\mathcal{F}_t\right],$$

if X can be hedged by h.

→ End of lecture 6a ←

→ Start of lecture 66 ←

We reverse the previous argument, which led to p.193.

Lemma:
Fix T-claim X. Define martingale M by

$$M_t = E^Q\left[\frac{X}{B_T}\middle| \mathcal{F}_t\right]$$ *)

Suppose that there exist predictable processes $h^1, \cdots, h^N$ such that

$$M_t = x + \sum_{i=1}^{N} \int_0^t h_s^i dZ_s^i,$$

Then X can be replicated.

*) Note that then $M_T B_T = X$ (take $t=T$)

Proof

We guess that (for replication)

$$M_t = V_t^Z = h_t^B \cdot 1 + \sum_{i=1}^{N} h_t^i Z_t^i \quad {*})$$

normalized bank account

Define: h^B by

$$h_t^B = M_t - \sum_{i=1}^{N} h_t^i Z_t^i. \quad \text{(the } h^i \text{ are given)}$$

We have $M_t = V_t^Z$, and we get , by assumption,

$$dV_t^Z = dM_t = \sum_{i=1}^{N} h_t^i dZ_t^i, \quad \text{by assumption on p. 194,}$$

so the portfolio is self financing. Furthermore:

$$V_T^Z = M_T \underset{\text{def of } M_T}{=} E^Q\left[\left.\frac{X}{B_T}\right| \mathcal{F}_T\right] = \frac{X}{B_T}.$$

Hedge, as now $V_T = X$!

*) Note that then $X = V_T^Z B_T = h_T B_T + \sum_{i=1}^{N} h_T^i S_T^i$

Second Fundamental Theorem

[FTAP 2]

The second most important result in arbitrage theory is the following.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

?

Proof: It is obvious (why?) that if the market is complete, then Q must be unique. The other implication is very hard to prove. It basically relies on duality arguments from functional analysis.

For all $A \in \mathcal{F}_T$, $1_A B_T$ can be hedged and hence has a unique price: for any Q:

$\Pi_t(1_A B_T) = \dfrac{E^Q[1_A | \mathcal{F}_t]}{B_t}$. For $t=0$: unique

$\Pi_0(1_A B_T) = E^Q[1_A | \mathcal{F}_0] = E^Q[1_A] = Q(A)$.

Here we used $\mathcal{F}_0$ is the trivial σ-algebra, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $E[X|\mathcal{F}_0] = EX$ for any X.

Black-Scholes Model

Q-dynamics (recall $Z_t = \frac{S_t}{B_t}$)

$$dS_t = rS_tdt + \sigma S_t dW_t^Q,$$
$$dZ_t = Z_t \sigma dW_t^Q$$

, see p. 185.

Consider the martingale (!)

$$M_t = E^Q\left[e^{-rT}X \middle| \mathcal{F}_t \right],$$

Here X is an arbitrary claim

Representation theorem for Wiener processes

$$\Downarrow$$

there exists g such that

(if we know that the $\mathcal{F}_t$ are generated by W_t^Q)

$$M_t = M(0) + \int_0^t g_s dW_s^Q.$$

Thus

$$M_t = M_0 + \int_0^t h_s^1 dZ_s,$$

same

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result: from lemma on p. 194, 195:

X can be replicated using the portfolio defined by

$$\begin{aligned} h_t^1 &= g_t/\sigma Z_t, \\ h_t^B &= M_t - h_t^1 Z_t. \end{aligned}$$

Moral: The Black Scholes model is complete.

Here we didn't need (as on p. 102) that X is of the form $X = \Phi(S_T)$, but see next page(s).

Special Case: Simple Claims

Assume X is of the form $X = \Phi(S_T)$, and

martingale! $\rightarrow$
$$M_t = E^Q\left[e^{-rT}\Phi(S_T)\middle| \mathcal{F}_t\right],$$
normalized price!

Kolmogorov backward equation $\Rightarrow M_t = f(t, S_t)$ $\Leftarrow$ (S is Q-Markov)

(as on p. 189)

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} & = & 0, \\ f(T, s) & = & e^{-rT}\Phi(s). \end{cases}$$

Homework!

Itô $\Rightarrow$ $dM_t = df(t, S_t) = f_t\,dt + f_s\,dS + \frac{1}{2} f_{ss}(dS)^2$; use $\boxed{\text{PDE}}$ to get

$$dM_t = \sigma S_t \frac{\partial f}{\partial s} dW_t^Q,$$

compare to pp 103–106, $F(t,s) = e^{rt} f(t,s) = B_t f(t,s)$ see also next page

so we know the "abstract" g_t:

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s},$$

Replicating portfolio h:

$$h_t^B = f - S_t \frac{\partial f}{\partial s},$$

$$h_t^1 = \frac{g_t}{\sigma Z_t} = \frac{S_t \frac{\partial f}{\partial s}}{Z_t} = h_t^1 = B_t \frac{\partial f}{\partial s}.$$

Interpretation: $f(t, S_t) = V_t^Z$, normalized price process of X

unnormalized, nominal pricing function

Define $F(t,s)$ by

$$F(t,s) = e^{rt} f(t,s) = B_t f(t,s)$$

so $F(t, S_t) = V_t$. Then, from previous page and $\frac{\partial F}{\partial s} = B_t \frac{\partial f}{\partial s}$,

$$\begin{cases} h_t^B &= \frac{F(t,S_t) - S_t \frac{\partial F}{\partial s}(t,S_t)}{B_t}, \\ h_t^1 &= \frac{\partial F}{\partial s}(t, S_t) \end{cases}$$

where F solves the **Black-Scholes equation**

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

Use PDE on p. 199 and

$\frac{\partial F}{\partial t} = rB_t f + B_t \frac{\partial f}{\partial t}$, $\frac{\partial F}{\partial s} = B_t \frac{\partial f}{\partial s}$ and PDE for f

Homework!

Summary: **Main Results**

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q (FTAP 1)

- The market is complete $\Leftrightarrow$ Q is unique. (FTAP 2)

- Every X must be priced by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$, complete or not

for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X, if X is not hedgeable

- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t[X] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

because $\Pi_t(X)$ = the value of a portfolio!

Completeness vs No Arbitrage
Rule of Thumb

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$
\begin{aligned}
R &= \text{number of random sources} \\
N &= \text{number of risky assets}
\end{aligned}
$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim. (as on p.108)

Rule of thumb

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

For B-S we have $N = R = 1$. Thus the Black-Scholes model is arbitrage free and complete.

Stochastic Discount Factors / pricing formula under P

Given a model under P. For every EMM Q we define the corresponding **Stochastic Discount Factor**, or **SDF**, by

$$D_t = e^{-\int_0^t r_s ds} L_t, \quad = L_t / B_t$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

they use L_X

There is thus a one-to-one correspondence between EMMs and SDFs.

The risk neutral valuation formula for a T-claim X can now be expressed under P instead of under Q.

Proposition: With notation as above we have a pricing formula under the measure P (but note that Q is "hidden" in D_t):

$$\Pi_t [X] = \frac{1}{D_t} E^P [D_T X | \mathcal{F}_t]$$

Abstract

Proof: Bayes' formula:

Start from $\frac{\Pi_t(X)}{B_t} = E^Q\left[\frac{X}{B_T}\middle|\mathcal{F}_t\right]$

$$= \frac{E^P\left[X L_T / B_T \middle| \mathcal{F}_t\right]}{L_t} \quad \text{(see p. 149)}$$

$$= E^P\left[X D_T \middle| \mathcal{F}_t\right] / L_t$$

etc.

Martingale Property of $S \cdot D$

Proposition: If S is an arbitrary price process, then the process

$$S_t D_t$$

is a P-martingale.

Proof: Bayes' formula again:

Same trick: we know $E^Q\left[\frac{S_T}{B_T}\middle|\mathcal{F}_t\right] = \frac{S_t}{B_t}$

$$\underbrace{E^P\left[\frac{S_T}{B_T} L_T \middle| \mathcal{F}_t\right]}_{L_t} = \frac{S_t}{B_t}$$

D_T

→ End of lecture 6b ←