→ Start of lecture 7a ←

# Continuous Time Finance

# Dividends,

# Forwards, Futures, and Futures Options

Ch 16 & 26

Tomas Björk

# Contents

# 1. Dividends

# Dividends

Black-Scholes model:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$
$$dB_t = rB_t dt.$$

**New feature:**
The underlying stock pays **dividends**.

don't confuse with notation for discount factor on p. 204

$D_t$ = The cumulative dividends over the interval $[0, t]$ (will depend on $S_u$, $u \le t$)

**Interpretation:**
Over the interval $[t, t + dt]$ you obtain the amount $dD_t$

Two cases

- Discrete dividends (realistic but messy): we skip!
- Continuous dividends (unrealistic but easy to handle).

in fact differentiable, as we shall see

# Portfolios and Dividends

Consider a market with $N$ assets.

$$\begin{aligned} S_t^i &= \text{price at } t\text{, of asset No } i \\ D_t^i &= \text{cumulative dividends for } S^i \text{ over} \\ &\quad \text{the interval } [0,t], \quad D_0^i = 0 \\ h_t^i &= \text{number of units of asset } i \\ V_t &= \text{market value of the portfolio } h \text{ at } t \end{aligned}$$

**Assumption:** We assume that $D$ has continuous differentiable trajctories.

**Definition:** The **value process** $V$ is defined by

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

(as before)

# Self financing portfolios
in presence of dividends

Recall:

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

New **Definition:** The strategy $h$ is **self financing** if

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$

where the **gain** process $G^i$ is defined by

if $D_t^i \equiv 0$, we are back in old case

$$dG_t^i = dS_t^i + dD_t^i$$

extra "income", compared to $S_t^i$

Interpret!

**Note:** The definitions above rely on the assumption that $D$ is continuous. In the case of a discontinuous $D$, the definitions are more complicated.

# Relative weights

$u_t^i =$ the relative share of the portfolio value, which is invested in asset No $i$.

$$u_t^i = \frac{h_t^i S_t^i}{V_t}$$ (as before)

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$ (previous page)

Substitute!

$$dV_t = V_t \sum_{i=1}^{N} u_t^i \frac{dG_t^i}{S_t^i}$$

[without dividends we had
$$dV_t = V_t \sum_{i=1}^{N} u_t^i \frac{dS_t^i}{S_t^i}$$ ]

# Continuous Dividend Yield

**Definition:** The stock $S$ pays a differentiable **continuous dividend yield** of $q$, if $D$ has the form (with $q \geq 0$)

$$dD_t = qS_tdt$$ , here the dividend growth $qS_t$ is proportional to $S_t$, with rate $q$.

**Problem:**
How does the dividend affect the price of a European Call? (compared to a non-dividend stock).

**Answer:**
The price is lower. (why?) you can guess....

# Black-Scholes with Cont. Dividend Yield

$$\begin{aligned} \text{as before} \rightarrow \quad dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ \text{new} \rightarrow \quad dD_t &= q S_t dt \end{aligned} \quad \Bigg\} \Rightarrow$$

Gain process:

$$dG_t = (\alpha + q) S_t dt + \sigma S_t dW_t$$

Consider a fixed claim

$$X = \Phi(S_T)$$

and assume that

$$\Pi_t [X] = F(t, S_t),$$

justified by some Markov property as before.

# Standard Procedure , familiar by now

- Assume that the derivative price is of the form

$$\Pi_t[X] = F(t, S_t).$$

- Form a portfolio based on underlying $S$ and derivative $F$, with portfolio dynamics with SF property:

$$dV_t = V_t \left\{ u_t^S \cdot \boxed{\frac{dG_t}{S_t}} + u_t^F \cdot \frac{dF}{F} \right\}$$

new

(compare p.212)

- Choose $u^S$ and $u^F$ such that the $dW$-term is wiped out. This gives us eventually, after computations,

$$dV_t = V_t \cdot k_t dt$$

- Absence of arbitrage implies

$$k_t = r$$

- This relation will say something about $F$, as before.

Value dynamics (repeat):

$$dV = V \cdot \left\{ u^S \frac{dG}{S} + u^F \frac{dF}{F} \right\},$$

applied to $F(t, S_t)$

$$dG = S(\alpha + q)dt + \sigma S dW.$$ (previous page)

From Itô we obtain

$$dF = \alpha_F F dt + \sigma_F F dW,$$

where

we use the shorthand notation

$$\alpha_F = \frac{1}{F}\left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\},$$

$$\sigma_F = \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}.$$

Collecting terms gives us

$$\begin{aligned} dV &= V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt \\ &+ V \cdot \left\{ u^S \sigma + u^F \sigma_F \right\} dW, \end{aligned}$$

Define $u^S$ and $u^F$ by the system

$$\begin{aligned} u^S\sigma + u^F\sigma_F &= 0, \\ u^S + u^F &= 1. \end{aligned}$$

to wipe out the Wiener term

Solution (if $\sigma_F \neq \sigma$)

$$u^S = \frac{\sigma_F}{\sigma_F - \sigma},$$
$$u^F = \frac{-\sigma}{\sigma_F - \sigma},$$

Value dynamics (dW term wiped out in previous equation):

$$dV = V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt.$$

Absence of arbitrage implies [usual argument]

$$u^S(\alpha + q) + u^F \alpha_F = r,$$

We get, using $\alpha_F$ and $\sigma_F$ of p.216 in $u^S$ and $u^F$,

$$\frac{\partial F}{\partial t} + (r - q)S\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.$$

(verify!)

# Pricing PDE

**Proposition:** The pricing function $F$ is given as the solution to the PDE

$$\begin{cases} \dfrac{\partial F}{\partial t} + (r-q)s\dfrac{\partial F}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

q only here

We can now apply Feynman-Kac to the PDE in order to obtain a risk neutral valuation formula.

under a risk-neutral measure Q:

new: $dS_t = (r-q)S_t\,dt + \sigma S_t\,dW_t^Q$

$dB_t = rB_t\,dt$

$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\,\Phi(S_T)$

differences with previous results are due to q.

If $q=0$ we are back in the old situation.



→ End of lecture 7a ←

→ Start of lecture 7b ←

# Risk Neutral Valuation

The pricing function has the representation still discounting with bank account

$$F(t, s) = e^{-r(T-t)} E_{t,s}^{Q} \left[ \Phi(S_T) \right],$$

where the $Q$-dynamics of $S$ are given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^Q.$$

**Question:** Which object is a martingale under the measure $Q$? Is it $\frac{S_t}{B_t}$ as before?

Answer should be no, as there is "q" in the equation for $S_t$.

# Martingale Property

**Proposition:** Under the martingale measure $Q$ the **normalized gain process**

$$G_t^Z = e^{-rt}S_t + \underbrace{\int_0^t e^{-ru}dD_u}_{\text{extra term compared to previous case}}$$

is a $Q$-martingale.

**Proof:** Exercise: Show $dG_t^Z = e^{-rt}\sigma S_t dW_t^Q$; no $dt$ term

**Note:** The result above holds in great generality, see p.228.

**Interpretation:**
In a risk neutral world, today's stock price should be the expected value of all future discounted earnings which arise from holding the stock, these include dividends,

$$S_0 = E^Q\left[\int_0^t e^{-ru}dD_u + e^{-rt}S_t\right], \geqslant \underbrace{e^{-rt}E^Q S_t}_{\text{the "old" price}}!$$

from Proposition upon noticing $G_0^Z = S_0$

# Pricing formula

Find

Pricing formula for claims of the type

$$\mathcal{Z} = \Phi(S_T).$$

We are standing at time $t$, with dividend yield $q$. Today's stock price is $s$.

- Suppose that you have the pricing function

$$F^0(t, s) = \Pi_t(\mathcal{Z}), \text{ when } S_t = s.$$

for a non dividend stock.

- Denote the pricing function for the dividend paying stock by

$$F^q(t, s)$$

**Proposition:** With notation as above we have

$$F^q(t, s) = F^0\left(t, se^{-q(T-t)}\right)$$

This is Exercise 16.5.

# Moral

Use your old formulas, but replace today's stock price $s$ with $se^{-q(T-t)}$.

$\Rightarrow$ explicit expressions in the Black-Scholes case

NB: $se^{-q(T-t)} \leq s$.

If $F^0(t,s)$ is increasing in $s$, then $F^q(t,s) \leq F^0(t,s)$; this happens for a European call.

For a European put, $F^0(t,s)$ is decreasing in $s$.

# European Call on Dividend-Paying-Stock

$$F^q(t,s) = se^{-q(T-t)}N\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)\right\} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Compare to p. 71, and observe the role of q.

# Martingale Analysis

**Basic task:** We have a general model for stock price $S$ and cumulative dividends $D$, under $P$. How do we find a martingale measure $Q$, and exactly which objects will be martingales under $Q$?

needed to define a martingale measure

**Main Idea:** We attack this situation by reducing it to the well known case of a market without dividends. Then we apply standard techniques.

# The Reduction Technique

- Consider the self financing portfolio where you keep 1 unit of the stock and invest all dividends in the bank. Denote the portfolio value by $V$.
- This portfolio can be viewed as a traded asset **without dividends**. (as they disappear into the bank account)
- Now apply the First Fundamental Theorem to the market $(B, V)$ instead of the original market $(B, S)$.
- Thus there exists a martingale measure $Q$ such that $\frac{\Pi_t}{B_t}$ is a $Q$ martingale for all traded assets (underlying and derivatives) without dividends.
- In particular the process

$$\frac{V_t}{B_t}$$

is a $Q$ martingale. Next we study V.

# The $V$ Process

Let $h_t$ denote the number of units in the bank account, where $h_0 = 0$. $V$ is then characterized by

from previous page

$$V_t = 1 \cdot S_t + h_t B_t \quad (1)$$

$$dV_t = \underbrace{dS_t + dD_t}_{\text{SF condition, p. 211}} + h_t dB_t \quad (2)$$

From (1) we obtain

$$dV_t = dS_t + h_t dB_t + B_t dh_t$$

(ordinary product rule, if $dh_t$ makes sense)

Comparing this with (2) gives us

$$B_t dh_t = dD_t \quad \text{and} \quad dh_t = \frac{1}{B_t} dD_t .$$

Integrating this gives us

$$h_t = \int_0^t \frac{1}{B_s} dD_s$$

We thus have

$$V_t = S_t + B_t \int_0^t \frac{1}{B_s} dD_s \tag{3}$$

and the first fundamental theorem gives us the following result.

**Proposition:** For a market with dividends, the martingale measure $Q$ is characterized by the fact that the **normalized gain process** $G_t^Z = \frac{V_t}{B_t}$ satisfies

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a $Q$ martingale. (as on p. 221, but from a different argument)

**Quiz:** Could you have guessed the formula (3) for $V$?

Another "quiz": Who is you?

# Continuous Dividend Yield

Model under $P$

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dD_t &= q S_t dt
\end{aligned}
$$

We recall (p.228, Proposition)

$$
G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s = Z_t + \int_0^t q Z_s \, ds
$$

You do!

Easy calculation gives us (under P)

$$
dG_t^Z = Z_t \left(\alpha - r + q\right) dt + Z_t \sigma dW_t,
$$

where $Z = S/B$.

Girsanov transformation $dQ = LdP$, where

$$
dL_t = L_t \varphi_t dW_t \quad \text{(for some } \varphi_t\text{)}
$$

We have

$$
dW_t = \varphi_t dt + dW_t^Q \quad \text{(see p.166)}
$$

Insert this into $dG^Z$

The $Q$ dynamics for $G^Z$ are

$$dG_t^Z = Z_t(\alpha - r + q + \sigma\varphi_t)\,dt + Z_t\sigma dW_t^Q$$

Martingale condition

$$\alpha - r + q + \sigma\varphi_t = 0$$

, $\alpha + \sigma\varphi_t = r - q$

$Q$-dynamics of $S$ : $dS = \alpha S_t\,dt + \sigma S_t(dW^Q + \varphi_t\,dt)$
gives

$$dS_t = S_t(\alpha + \sigma\varphi)\,dt + S_t\sigma dW_t^Q$$

Using the martingale condition this gives us the $Q$-dynamics of $S$ as

$$dS_t = S_t(r - q)\,dt + S_t\sigma dW_t^Q$$

, already seen on p. 219

(note again the role of the dividend rate $q$)

# Risk Neutral Valuation

**Theorem:** For a $T$-claim $X$, the price process $\Pi_t[X]$ is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q[X | \mathcal{F}_t],$$

where the $Q$-dynamics of $S$ are given by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t^Q.$$

note that q appears (only) in the Q-dynamics of S!.

→ end of lecture 7b ←