→ Start of lecture 8a ←

2. Forward and Futures Contracts

Forward Contracts

A **forward contract** on the T-claim X, **contracted at** t, is defined by the following payment scheme.

- The holder of the forward contract receives, at time T, the stochastic amount X from the underwriter.
  ↳ $\mathcal{F}_T$ - measurable
- The holder of the contract pays, at time T, the **forward price** $f(t;T,X)$ to the underwriter.
- The forward price $f(t;T,X)$ is determined at time t. (will be $\mathcal{F}_t$ - measurable)
- The forward price $f(t;T,X)$ is determined in such a way that the price of the forward contract equals zero, at the time t when the contract is made.
  (compensating cash flows, swap of products that net to zero)

General Risk Neutral Formula

Suppose we have a bank account B with dynamics

$$dB_t = r_t B_t dt, \quad B_0 = 1$$

with a (possibly stochastic) short rate r_t. Then

adapted process

$$B_t = e^{\int_0^t r_s ds}$$

and we have the following risk neutral valuation for a T-claim X

$$\Pi_t[X] = E^Q\left[\overbrace{e^{-\int_t^T r_s ds}}^{B_t/B_T,\ B_t \in \mathcal{F}_t} \cdot X \middle| \mathcal{F}_t\right] \quad (*)$$

$$= B_t E^Q\left[e^{-\int^T r_s ds} \cdot X \middle| \mathcal{F}_t\right]$$

Setting $X = 1$ we have the price, at time t, of a zero coupon bond maturing at T as

$$p(t,T) = E^Q\left[e^{-\int_t^T r_s ds}\middle| \mathcal{F}_t\right] = B_t E^Q\left[\frac{1}{B(T)}\middle| \mathcal{F}_t\right]$$

$\frac{p(t,T)}{B(t)}$ is Q-martingale

[see also book, Section 29.1]

generalizes $e^{-r(T-t)}$!

Forward Price Formula

Theorem: The forward price of the claim X is given by

$$f(t,T) = \frac{1}{p(t,T)} E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]$$

where $p(t,T)$ denotes the price at time t of a zero coupon bond maturing at time T.

In particular, if the short rate r is deterministic we have

$$f(t,T) = E^Q \left[X \middle| \mathcal{F}_t \right]$$

note the normalization factor,

not B_t !

Proof

definition of forward contract

The net cash flow at maturity is $X - f(t,T)$. If the value of this at time t equals zero we obtain

$$\Pi_t[X] = \Pi_t[f(t,T)]$$

We have (from p. 234)

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t\right]$$

and, since $f(t,T)$ is known at t, we obviously (why?) have (see definition of $p(t,T)$ on p. 234)

$$(\Pi_t[X] =)\ \Pi_t[f(t,T)] = p(t,T)f(t,T).$$

This proves the main result. If r is deterministic and constant then $p(t,T) = e^{-r(T-t)}$ which gives us the second formula.

$$\Pi_t[f(t,T)] = \underbrace{f(t,T)}_{\mathcal{F}_t\text{-measurable}} \underbrace{E^Q\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right]}_{p(t,T)}.$$

Futures Contracts

A **futures contract** on the T-claim X, is a financial asset with the following properties.

$\mathcal{F}_t$-measurable

(i) At every point of time t with $0 \leq t \leq T$, there exists in the market a quoted object $F(t; T, X)$, known as the **futures price** for X at t, for delivery at T.

(ii) At the time T of delivery, the holder of the contract pays $F(T; T, X)$ and receives the claim X, both $\mathcal{F}_T$-measurable
So $F(T;T,X) = X$

(iii) During an arbitrary time interval $(s, t]$ the holder of the contract receives the amount $F(t; T, X) - F(s; T, X)$. The cashflow $F(t;T,X)$ looks like a dividend

(iv) The spot price, at any time t prior to delivery, for buying or selling the futures contract, is by definition equal to zero.

this is not the futures price

Futures Price Formula (Section 29.2)

in particular (iii)

From the definition it is clear that a futures contract is a **price-dividend pair** (S, D) with

$$S \equiv 0, \quad dD_t = dF(t, T)$$

(not of the form $q S_t dt$)

$D_t = F(t,T)$

From general theory, (p. 228) the normalized gains process

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a Q-martingale.

Since $S \equiv 0$ and $dD_t = dF(t, T)$ this implies that

$$\frac{1}{B_t} dF(t, T)$$

usually of the type "$h_t \, dW_t^Q$"

"general theory"

is a martingale increment, which implies (why?) that $\beta_t h_t dW_t^Q =$ $dF(t, T)$ is a martingale increment. Thus F is a Q-martingale (in t) and we have

martingale int

$$F(t, T) = E^Q \left[F(T, T) | \mathcal{F}_t \right] = E^Q \left[X | \mathcal{F}_t \right]$$

see p.237 (ii)

This proves:

Theorem: The futures price process is given by

$$F(t,T) = E^{Q}\left[X \middle| \mathcal{F}_t\right].$$

(no discounting!)

Corollary. If the short rate is deterministic, then the futures and forward prices coincide.

see p. 235

3. Futures Options

Futures Options

We denote the futures price process, at time t with delivery time at T by

$$F(t, T).$$

When T is fixed we sometimes suppress it and write F_t, i.e. $F_t = F(t, T)$

not a partial derivative (in spite of same notation)

Definition:

A European futures call option, with strike price K and exercise date T, on a futures contract with delivery date T_1 will, if exercised at T, pay to the holder:

X

$\boxed{T < T_1}$

- The amount $F(T, T_1) - K$ in **cash**.

$X = (F(T, T_1) - K)^+$

- A long postition in the underlying futures contract.

NB! The long position above can immediately be closed at no cost, so focus on $[F(T, T_1) - K)^+$

the spot price of a future is zero

Institutional fact:
The exercise date T of the futures option is typcally very close to the date of delivery of the underlying T_1 futures contract.

Why do Futures Options exist?

- On many markets (such as commodity markets) the futures market is much more liquid than the underlying market.
- Futures options are typically settled in **cash**. This relieves you from handling the underlying (tons of copper, hundreds of pigs, etc.). tons of potatoes
- The market place for futures and futures options is often the same. This facilitates hedging etc.

Quote from Björk's book, page 455:

> • If the reader thinks that a futures contract conceptually is a somewhat complicated object, then the author is inclined to agree.

For more explanation you may want to consult:

https://www.investopedia.com/terms/f/futurescontract.asp

or a similar page on Wikipedia:

https://en.wikipedia.org/wiki/Futures_contract

Pricing Futures Options – Black-76

We consider a futures contract with delivery date T_1 (fixed) and use the notation $F_t = F(t, T_1)$. We assume the following dynamics for F.

$$dF_t = \mu F_t dt + \sigma F_t dW_t$$

Now suppose we want to price a derivative with exercise date T with the T_1-futures price F as underlying, i.e. a claim of the form

$T_1 > T$

$$\Phi(F_T)$$

This turns out to be quite easy.

From risk neutral valuation we know that the price process $\Pi_t[\Phi]$ is of the form (by analogy, replace S_t with F_t):

$$\Pi_t[\Phi] = f(t, F_t)$$

where f is given by

$$f(t, F) = e^{-r(T-t)} E^Q_{t,F}[\Phi(F_T)]$$

conditional expectation under Q, with $F_t = F$

so it only remains to find the Q-dynamics for F.

We now recall from p.238

Proposition: The futures price process F_t is a Q-martingale.

! Thus the Q-dynamics of F are given by

$$dF_t = \sigma F_t dW_t^Q \text{ :}$$

No dt term, and

Note that the diffusion coefficient is the same for $Q \sim P$, see previous page.

We thus have

$$f(t,F) = e^{-r(T-t)} E^Q_{t,F}\left[\Phi(F_T)\right]$$

with Q-dynamics

$$dF_t = \sigma F_t dW^Q_t$$

from p. 230

Now recall the formula for a stock with continuous dividend yield q.

$$f(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right]$$

with Q-dynamics

$$dS_t = (r-q)S_t + \sigma S_t dW^Q_t$$

Note: If we set $q = r$ the formulas are **identical**!
and replace S_t with F_t

Pricing Formulas

Let $f^0(t, s)$ be the pricing function for the contract $\Phi(S_T)$ for the case when S is a stock without dividends. Let $f(t, F)$ be the pricing formula for the claim $\Phi(F_T)$.

Proposition: With notation as above we have
(book Prop 7.13)

$$f(t, F) = f^0(t, Fe^{-r(T-t)})$$

Moral: Reset today's futures price F to $Fe^{-r(T-t)}$ and use your formulas for stock options.

Compare to p. 222 for dividends,
similar story, replace q with r.

Black-76 Formula

The price of a futures option with exercise date T and exercise price K is given by

$$c = e^{-r(T-t)} \left\{ FN\left[d_1\right] - KN\left[d_2\right] \right\}.$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t) \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

(Use the formula on p. 224 with $q = r$, and $s = F$.

→ End of lecture 8a ←

→ start of lecture 8b ←

Continuous Time Finance

Currency Derivatives

Ch 17

Tomas Björk

Pure Currency Contracts

Consider two markets, domestic (England) and foreign (USA).

$$
\begin{aligned}
r^d &= \text{domestic short rate} \\
r^f &= \text{foreign short rate} \\
X &= \text{exchange rate}
\end{aligned}
$$

generally different!

NB! The exchange rate X is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}}$$

If 1 EUR = 1,09 USD, then

$$X = \frac{1}{1{,}09}$$

Simple Model (Garman-Kohlhagen)

for the exchange rate

The P-dynamics are given as:

different bank accounts

$$\begin{aligned} dX_t &= X_t \alpha dt + X_t \sigma dW_t, \\ dB_t^d &= r^d B_t^d dt, \\ dB_t^f &= r^f B_t^f dt, \end{aligned}$$

in which currency?

Main Problem:
Find arbitrage free price for currency derivative, Z, of the form

$$Z = \Phi(X_T)$$

Typical example: European Call on X.

$$Z = \max\left[X_T - K, 0\right]$$

Naive idea

For the European Call, use the standard Black-Scholes formula, with S replaced by X and r replaced by r^d.

Is this OK?

"suspicious question"

NO!

WHY?

Main Idea

but asset price keeps on fluctuating

- When you buy stock you just keep the asset until you sell it. (no interest on assets)

- When you buy dollars, these are put into a bank account, giving the interest r^f.

Moral:
Buying a currency is like buying a dividend-paying stock with dividend yield $q = r^f$.

many similarities

But exchange rate keeps on fluctuating, this does NOT affect what you have on your bank account in the foreign currency.

Technique

- Transform all objects into **domestically traded** asset prices.
- Use standard techniques on the transformed model.

Transformed Market

1. Investing foreign currency in the foreign bank gives value dynamics **in foreign currency** according to

$$dB_t^f = r^f B_t^f dt.$$

2. $B_{\cancel{f}}^{f}$ units of the foreign currency is worth $X \cdot B_{\cancel{f}}^{f}$ in the domestic currency. $(X_t B_t^f)$

3. Trading in the foreign currency is equivalent to trading in a domestic market with the domestic price process

$$\tilde{B}_t^f = B_t^f \cdot X_t \quad \leftarrow$$

this is the transformation

4. Study the domestic market consisting of

$$\tilde{B}^f, \quad B^d$$

Market dynamics

Summary:

$$
\begin{aligned}
dX_t &= X_t \alpha dt + X_t \sigma dW \\
\tilde{B}_t^f &= B_t^f \cdot X_t \text{, the domestic prices}
\end{aligned}
$$

Using Itô we have domestic market dynamics

$$
\begin{aligned}
d\tilde{B}_t^f &= \tilde{B}_t^f \left(\alpha + r^f\right) dt + \tilde{B}_t^f \sigma dW_t \\
dB_t^d &= r^d B_t^d dt
\end{aligned}
$$

you want $\frac{\tilde{B}^f}{B^d}$ to be Q-martingale

Standard results gives us Q-dynamics for domestically traded asset prices: (write down $\frac{dQ}{dP}$ on $\mathcal{F}_T$)

compare to $dS_t = rS_t dt + \sigma S_t dW_t^Q$

$$
\begin{aligned}
d\tilde{B}_t^f &= \tilde{B}_t^f r^d dt + \tilde{B}_t^f \sigma dW_t^Q \\
dB_t^d &= r^d B_t^d dt
\end{aligned}
$$

domestic interest rate!

Itô gives us Q-dynamics for $X_t = \tilde{B}_t^f / B_t^f$:

$$
dX_t = X_t (r^d - r^f) dt + X_t \sigma dW_t^Q
$$

$$
d\left(\frac{\tilde{B}^f}{B^f}\right) = \frac{d\tilde{B}^f}{B^f} + \tilde{B}^f d\left(\frac{1}{B^f}\right)
$$
(no cross terms)

Verify this!

Risk neutral Valuation

of a currency derivative

Theorem: The arbitrage free price $\Pi_t[\Phi]$ is given by $\Pi_t[\Phi] = F(t, X_t)$ where

$$F(t,x) = e^{-r^d(T-t)} E^Q_{t,x}[\Phi(X_T)]$$

The Q-dynamics of X are given by, see page 257,

$$dX_t = X_t(r^d - r^f)dt + X_t \sigma dW^Q_t$$

→ Feynman-Kac representation:

gives relation to PDE

Pricing PDE

Theorem: The pricing function F solves the boundary value problem

from the equation for X under Q

$$
\begin{aligned}
\frac{\partial F}{\partial t} + \underbrace{x(r^d - r^f)}\frac{\partial F}{\partial x} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 F}{\partial x^2} - r^d F &= 0, \\
F(T, x) &= \Phi(x)
\end{aligned}
$$

(analogy with usual B-S framework, also similarity with results for dividends)

Currency vs Equity Derivatives

Proposition: Introduce the notation:

- $F^0(t,x)$ = the pricing function for the claim $\mathcal{Z} = \Phi(X_T)$, where we interpret X as the price of an ordinary stock without dividends.
- $F(t,x)$ = the pricing function of the same claim when X is interpreted as an exchange rate.

Then the following holds

$$F(t,x) = F_0\left(t, xe^{-r^f(T-t)}\right).$$

like dividend case on p.222
with $F^q(t,x)$ and $F^0(t,x)$
and q replaced with r^f

Currency Option Formula

The price of a European currency call is given by

$$F(t,x) = xe^{-r^f(T-t)}N[d_1] - e^{-r^d(T-t)}KN[d_2],$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right) + \left(r^d - r^f + \frac{1}{2}\sigma_X^2\right)(T-t)\right\}$$

$$d_2 = d_1(t,x) - \sigma\sqrt{T-t}$$

Upon renaming the constants, this is the same formula as on p.224 for dividends.

→ End of lecture 8b ←