→ start of lecture 9a ←

Martingale Analysis

$$Q^d = \text{domestic martingale measure}$$
$$Q^f = \text{foreign martingale measure}$$

$$L_t = \frac{dQ^f}{dQ^d}, \quad L_t^d = \frac{dQ^d}{dP}, \quad L_t^f = \frac{dQ^f}{dP}$$

P-dynamics of X

$$dX_t = X_t\alpha_t dt + X_t\sigma_t dW_t$$

where α and σ are arbitrary adapted processes and W is P-Wiener.

Problem: How are Q^d and Q^f related?

through L_t of course,
but how exactly?
via L_t^d, L_t^f?

Main Idea

Fix an arbitrary foreign T-claim Z.

- Compute foreign price and change to domestic currency. The price at $t = 0$ will be

$$\Pi_0[Z] = X_0 E^{Q^f}\left[e^{-\int_0^T r_s^f ds} Z\right]$$

foreign price

This can be written as

domestic price

$$\Pi_0[Z] = X_0 E^{Q^d}\left[L_T e^{-\int_0^T r_s^f ds} Z\right]$$

$\Rightarrow Z \to Z X_T$

OR

- Change into domestic currency at T and then compute domestic arbitrage free price. This gives us

$$\Pi_0[Z] = E^{Q^d}\left[e^{-\int_0^T r_s^d ds} X_T \cdot Z\right]$$

- These expressions must be equal for all choices of $Z \in \mathcal{F}_T$.

Proposition: $E[XZ] = E[YZ], \forall Z \Rightarrow X = Y$ a.s.

This returns on p. 278.

We thus obtain

$$E^{Q^d}\left[e^{-\int_0^T r_s^d ds} X_T \cdot Z\right] = X_0 E^{Q^d}\left[L_T e^{-\int_0^T r_s^f ds} Z\right]$$

for all T-claims Z. This implies the following result. (replace T with t)

Theorem: The exchange rate X is given by

$$X_t = X_0 e^{\int_0^t (r_s^d - r_s^f) ds} L_t = X_0 \frac{B_t^d}{B_t^f} \frac{L_t^f}{L_t^d}$$

alternatively by

$$X_t = X_0 \frac{D_t^f}{D_t^d}$$

where D_t^d is the domestic stochastic discount factor etc.

↳ see p. 204 : $D_t^f = \frac{L_t^f}{B_t^f}$, $D_t^d = \frac{L_t^d}{B_t^d}$ etc.

Proof: The last part follows from

$$L = \frac{dQ^f}{dQ^d} = \frac{dQ^f}{dP} \Big/ \frac{dQ^d}{dP} = \frac{L^f}{L^d}$$

and $B_t^f = \exp\left(\int_0^t r_s^f ds\right)$ etc.

Compare to general theory on p.155.

Here: Q^d plays the role of P, Q^f that of Q

Q^d-Dynamics of X

positive

In particular, since L is a Q^d-martingale the Q^d dynamics of L are of the form

(a) $$dL_t = L_t \varphi_t dW_t^d$$

where W^d is Q^d-Wiener. From (Thm on p.264)

(b) $$X_t = X_0 e^{\int_0^t (r_s^d - r_s^f) ds} L_t$$

the Q^d-dynamics of X follows ~~as~~ from (a), (b) and Itô as

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \overset{\boxed{\sigma_t}}{\varphi_t} dW_t^d$$

(bit of work)

Compare to p.257 (use $W^Q = W^d$) and conclude that

~~so~~ the Girsanov kernel φ equals the exchange rate volatility σ and we have the general Q^d dynamics.

(see p. 257)

Theorem: The Q^d dynamics of X are of the form

$$dX_t = (r_t^d - r_t^f) X_t dt + X_t \sigma_t dW_t^d ,$$

known from p.257

Market Prices of Risk

Recall

$$D_t^d = e^{-\int_0^t r_s^d ds} L_t^d \quad \left(= \frac{L_t^d}{B_t^d} \right)$$

We also have a representation like

$$dL_t^d = L_t^d \varphi_t^d dW_t \leftarrow \text{P-Wiener process}$$

where $-\varphi_t^d = \boxed{\lambda^d}$ is the domestic market price of risk and similar for φ^f etc. From

new terminology (returns later, see p. 305 and p. 319)

$$X_t = X_0 \frac{D_t^f}{D_t^d}$$

we now easily obtain ? (exercise in Itô-calculus)

$$dX_t = X_t \alpha_t dt + X_t \left(\lambda_t^d - \lambda_t^f \right) dW_t,$$

where we do not care about the exact shape of α. We thus have

we don't specify it

Theorem: The exchange rate volatility is given by

$$\sigma_t = \lambda_t^d - \lambda_t^f,$$

a relation between volatility and market prices of risk.

Siegel's Paradox

Assume that the domestic and the foreign markets are both risk neutral and assume constant short rates. We now have the following surprising (?) argument.

A: Let us consider a T claim of 1 dollar. The arbitrage free dollar value at $t = 0$ is of course

$$e^{-r^f T}$$

so the Euro value at at $t = 0$ is given by

$$X_0 e^{-r^f T}.$$

The 1-dollar claim is, however, identical to a T-claim of X_T euros. Given domestic risk neutrality, the Euro value at $t = 0$ is then

$Q^d = P$ by assumption

$$e^{-r^d T} E^P \left[X_T \right].$$

We thus have

$$X_0 e^{-r^f T} = e^{-r^d T} E^P \left[X_T \right]$$

Siegel's Paradox ct'd

B: We now consider a T-claim of one Euro and compute the dollar value of this claim. The Euro value at $t = 0$ is of course

$$e^{-r^d T}$$

so the dollar value is (at $t=0$)

$$\frac{1}{X_0} e^{-r^d T}.$$

The 1-Euro claim is identical to a T-claim of X_T^{-1} Euros so, by foreign risk neutrality, we obtain the dollar price as $Q^f = P$

$$e^{-r^f T} E^P \left[\frac{1}{X_T} \right]$$

which gives us

$$\frac{1}{X_0} e^{-r^d T} = e^{-r^f T} E^P \left[\frac{1}{X_T} \right]$$

Siegel's Paradox ct'd

Recall our earlier results [pp 267 and 268]

$$
\begin{aligned}
X_0 e^{-r^f T} &= e^{-r^d T} E^P \left[X_T \right] \\
\frac{1}{X_0} e^{-r^d T} &= e^{-r^f T} E^P \left[\frac{1}{X_T} \right]
\end{aligned}
$$

[multiply]

Combining these gives us

$$
E^P \left[\frac{1}{X_T} \right] = \frac{1}{E^P \left[X_T \right]}
$$

which, by Jensen's inequality, *) is impossible unless X_T is deterministic. This is sometimes referred to as (one formulation of) "Siegel's paradox."

It thus seems that Americans cannot be risk neutral at the same time as Europeans.

What is going on?

*) If φ is convex, then $E\varphi(X) \geq \varphi(EX)$. Example: $\varphi(x) = \frac{1}{x}$

If φ is strictly convex: strict inequality, if X not degenerate,

equality iff X is degenerate: $\exists x_0 : P(X = x_0) = 1$.

Formal analysis of Siegel's Paradox

Question: Can we assume that both the domestic and the foreign markets are risk neutral?

Answer: Generally no, because of Jensen's inequality

Proof: The assumption would be equivalent to assuming the $P = Q^d = Q^f$ i.e.

$$\lambda_t^d = \lambda_t^f = 0$$

(must have $L_t^d \equiv 1 \equiv L_t^f$)

However, we know that (see p. 266)

$$\sigma_t = \lambda_t^d - \lambda_t^f$$

so we would need to have $\sigma_t = 0$ i.e. a non-stochastic exchange rate, which is not realistic.

$dX_t = \alpha_t X_t dt + 0 dt$, from p. 266

Moral / solving the paradox

The previous slide gave us the mathematical result, but the intuitive question remains why Americans cannot be risk neutral at the same time as Europeans.

The solution is roughly as follows.

- Risk neutrality (or risk aversion) is always **defined in terms of a given numeraire**. (B^d_t and B^f_t).
- It is **not** an attitude towards **risk as such**.
  but also refers to a specific market
- You can therefore **not** be risk neutral w.r.t two different numeraires at the same time unless the ratio between them is deterministic.
- In particular we cannot have risk neutrality w.r.t. Dollars and Euros at the same time.

Convincing?

If you are risk neutral in one market, you cannot be so in the other one, due to random fluctuations of the exchange rate.

→ End of lecture gap

→ Start of lecture 9b ←

Continuous Time Finance

Change of Numeraire

Ch 26

Tomas Björk

Recap of General Theory

Consider a market with asset prices

$$S_t^0, S_t^1, \ldots, S_t^N$$

FTAP 1:

Theorem: The market is arbitrage free

iff

there exists an EMM, i.e. a measure Q such that

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \ldots, \frac{S_t^N}{S_t^0}$$

  are Q-martingales.

Recap continued

Recall the normalized market

$$\left(Z_t^0, Z_t^1, \ldots Z_t^N\right) = \left(\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0} \ldots, \frac{S_t^N}{S_t^0}\right)$$

- We obviously have

$$Z_t^0 \equiv 1$$

- Thus Z^0 is a risk free asset in the normalized economy.

- Z^0 is a bank account in the normalized economy.

- In the normalized economy **the short rate is zero**:

If $\left.\begin{array}{l} dZ_t^0 = r_t Z_t^0 dt \\ Z_0^0 \equiv 1 \end{array}\right\}$ then $Z_t^0 = \exp\left(\int_0^t r_s ds\right) = 1, \forall t \geq 0$

$\Rightarrow r_s = 0, \forall s \geq 0.$

Dependence on numeraire

- The EMM Q will obviously depend on the choice of numeraire, so we should really write Q^0 to emphasize that we are using S^0 as numeraire.

- So far we have only considered the case when the numeraire asset is the bank account, i.e. when $S_t^0 = B_t$. In this case, the martingale measure Q^B is referred to as "the risk neutral martingale measure".

- Henceforth the notation Q (without upper case index) will only be used for the risk neutral martingale measure, i.e. $Q = Q^B = Q^0$

- We will now consider the case of a general numeraire.

General change of numeraire.

- Consider a financial market, including a bank account B.

- Assume that the market is using a fixed risk neutral measure Q as pricing measure. (S^i_t / B_t are Q-martingales) if $Q = Q^B$

Alternative:

- Choose a fixed asset S as numeraire, and denote the corresponding martingale measure by Q^S. :

$\frac{S^i_t}{S_t}$ become martingales under Q^S

Problems:

- Determine Q^S, i.e. determine

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

(or determine $\frac{dQ^S}{dP}$ on $\mathcal{F}_t$)

- Develop pricing formulas for contingent claims using Q^S instead of Q.

Constructing $\mathbf{Q}^S$

Fix a T-claim X. From general theory we know that

$$\Pi_0[X] = E^Q\left[\frac{X}{B_T}\right]$$

$\left(\frac{\Pi_t(X)}{B_t} \text{ is } Q^B\text{-martingale}\right)$

Since Q^S is a martingale measure for the numeraire S, the normalized process

$$\frac{\Pi_t[X]}{S_t}$$

note the different measures and numeraires

is a Q^S-martingale. We thus have

with $L_T = \frac{dQ^S}{dQ}$ on $\mathcal{F}_T$

$$\frac{\Pi_0[X]}{S_0} = E^S\left[\frac{\Pi_T[X]}{S_T}\right] = E^S\left[\frac{X}{S_T}\right] = E^Q\left[L_T\frac{X}{S_T}\right]$$

From this we obtain (use S_0 is a constant)

$$\Pi_0[X] = E^Q\left[L_T\frac{X \cdot S_0}{S_T}\right],$$

For all $X \in \mathcal{F}_T$ we thus have

$$E^Q\left[\frac{X}{B_T}\right] = E^Q\left[L_T \frac{X \cdot S_0}{S_T}\right]$$

Recall the following basic result from probability theory.
(see again p. 264)

Proposition: Consider a probability space $(\Omega, \mathcal{F}, P)$ and assume that

$$E\left[Y \cdot X\right] = E\left[Z \cdot X\right], \quad \text{for all } \cancel{Z}\, X \in \mathcal{F}.$$ s.t. expectations exist

Then we have

$$Y = Z, \quad P - a.s.$$ (Prove this!)

From this result we conclude that

$$\frac{1}{B_T} = L_T \frac{S_0}{S_T}$$

Can do the same for t instead of T:

Main result

Proposition: The likelihood process for the change Q to Q^S,

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

is given by

$$L_t = \frac{S_t}{B_t} \cdot \frac{1}{S_0}$$

p.155:
General theory says L is a Q-martingale

NB Also $L_t = \dfrac{S_t/S_0}{B_t/B_0}$ (as $B_0 = 1$)

Easy exercises

1. Convince yourself that L is a Q-martingale.
   also follows from formula of L_t, and property of Q.

2. Assume that a process A_t has the property that A_t/B_t is a Q martingale. Show that this implies that A_t/S_t is a Q^S-martingale. Interpret the result.

   Prove by Bayes rule (as one possibility)

   There is a general result in the Exercise class, Exercise 3 in the "additional exercises".

Pricing

Theorem: For every T-claim X we have the pricing formula

$$\Pi_t[X] = S_t E^S \left[\left. \frac{X}{S_T} \right| \mathcal{F}_t \right]$$

Proof: Follows directly from the Q^S-martingale property of $\Pi_t[X]/S_t$. ∎ (parallel to the usual property under Q)

Note 1: We observe S_t directly on the market.

Note 2: The pricing formula above is particularly useful when X is of the form

$$X = S_T \cdot Y$$

In this case we obtain

$$\Pi_t[S_T Y] = \Pi_t[X] = S_t E^S[Y | \mathcal{F}_t]$$

only one random variable here, instead of the ratio $\frac{X}{S_T}$

Important example

Consider a claim of the form

$$X = \Phi\left[S_T^0, S_T^1\right]$$

We assume that Φ is **linearly homogeneous**, i.e.

$$\Phi(\lambda x, \lambda y) = \lambda \Phi(x, y), \quad \text{for all } \lambda > 0$$

Using Q^0 we obtain

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \frac{\Phi\left[S_T^0, S_T^1\right]}{S_T^0} \right| \mathcal{F}_t \right]$$

$$\Pi_t[X] = S_t^0 E^0 \left[\left. \Phi\left(1, \frac{S_T^1}{S_T^0}\right) \right| \mathcal{F}_t \right]$$

Φ is linearly homogeneous

Important example cnt'd

Proposition: For a claim of the form

$$X = \Phi\left[S_T^0, S_T^1\right],$$

where Φ is homogeneous, we have

$$\Pi_t[X] = S_t^0 E^0\left[\varphi(Z_T) \middle| \mathcal{F}_t\right]$$

where

$$\varphi(z) = \Phi[1, z], \quad Z_t = \frac{S_t^1}{S_t^0}$$

notation for normalized process, here with S^0 as numéraire

Exchange option

has nothing to do with exchange rates

Consider an exchange option, i.e. a claim X given by

explain the name!

$$X = \max\left[S_T^1 - S_T^0,\ 0\right]$$

Since $\Phi(x, y) = \max[x - y, 0]$ is homogeneous we obtain

$$\Pi_t[X] = S_t^0 E^0\left[\underbrace{\max\left[Z_T - 1,\ 0\right]}\middle|\, \mathcal{F}_t\right]$$

- This is a European Call on Z with strike price $K = 1$
- Zero interest rate. ($S_t^0 = B_t$?).
- Piece of cake!
- If S^0 and S^1 are both GBM, then $\boxed{\text{so is } Z}$ and the price will be given by the Black-Scholes formula.

$Z_t = \frac{S_t^1}{S_t^0}$

$\boxed{\text{dangerous}}$ statement: "product or ratio of two lognormals is lognormal again"

[why dangerous?]

Related: Is the sum of two normals again?

Identifying the Girsanov Transformation

Assume the Q-dynamics of S are known as $\left(S = S^1 ! \right)$

$$dS_t = r_t S_t dt + S_t v_t dW_t^Q$$

p. 279 gives $$L_t = \frac{S_t}{S_0 B_t}$$ $\left(\frac{dQ^S}{dQ} \text{ on } \mathcal{F}_t \right)$

From this we immediately have $\left(\text{if } dB_t = r_t B_t dt \right)$

$$dL_t = L_t v_t dW_t^Q.$$

and we can summarize.

Theorem: The Girsanov kernel is given by the numeraire volatility v_t, i.e.

S

$$dL_t = L_t v_t dW_t^Q.$$

Application: or new theory? see also p.234.

Recap on zero coupon bonds

Recall: A zero coupon T-bond is a contract which gives you the claim

$$X \equiv 1$$

at time T.

The price process $\Pi_t[1]$ is denoted by $p(t,T)$, see also p.234

Allowing a stochastic short rate r_t we have

$$dB_t = r_t B_t dt.$$

r_t is adapted, $r_t \in \mathcal{F}_t$.

This gives us

$$B_t = e^{\int_0^t r_s ds}, \quad \in \mathcal{F}_t$$

and using standard risk neutral valuation we have

$$p(t,T) = E^Q\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = B_t E^Q\left[\frac{1}{B_T} \middle| \mathcal{F}_t \right]$$

Note:

$$p(T,T) = 1$$

Special choice of numéraire leads to

The forward measure Q^T !

- Consider a fixed T.
- Choose the bond price process $p(t, T)$ as numeraire.
- The corresponding martingale measure is denoted by Q^T and referred to as "the T-forward measure".

For any T claim X we obtain

$$\Pi_t[X] = p(t,T)E^{Q^T}\left[\left.\frac{\Pi_T[X]}{p(T,T)}\right|\mathcal{F}_t\right]$$

We have

$$\Pi_T[X] = X, \quad p(T,T) = 1$$

Theorem: For any T-claim X we have

only one random variable !

$$\Pi_t[X] = p(t,T)E^{Q^T}[X|\mathcal{F}_t]$$

"better" than $\Pi_t[X] = B_t E^Q\left[\frac{X}{B_T}\middle|\mathcal{F}_t\right]$

and $\Pi_0[X] = p(0,T)\, E^{Q^T}[X]$

but also note the different expectations

A general option pricing formula,
by use of two different numéraires, and Q^S and Q^T

the measure of p.276

European call on asset S with strike price K and maturity T.

$$X = \max\left[S_T - K,\ 0\right]$$

Write X as and use Note 2 on p.281

$$X = (S_T - K) \cdot I\left\{S_T \geq K\right\} = S_T I\left\{S_T \geq K\right\} - K\underbrace{I\left\{S_T \geq K\right\}}_{p(T,T)\,1\{S_T \geq K\}}$$

forward measure

Use Q^S on the first term and Q^T on the second.

(and p.281)

$$\Pi_0\left[X\right] = S_0 \cdot Q^S\left[S_T \geq K\right] - K \cdot p(0,T) \cdot Q^T\left[S_T \geq K\right]$$

Exercise: find similar expression for $\Pi_t[X]$, at time t instead of at time 0.

→ End of lecture 9b ←