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Stochastic finance in continuous time
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1. Consider a Black-Scholes market where asset holders receive dividends according to a constant rate δ. That means that under the physical (real world) measure $\mathbb{P}$ the asset dynamics are given by

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t),$$

where W is a Brownian motion under $\mathbb{P}$. Moreover, the cumulative dividends $D(t)$ satisfy $\mathrm{d}D(t) = \delta S(t)\,\mathrm{d}t$. The *gains process* G is given by $G(t) = S(t) + D(t)$. In addition there is the bank account with constant spot rate r. The goal is to price a T-claim of the form $\Phi(S_T)$, where T denotes the horizon. The time t price of the claim, a derivative, is $F(t, S(t))$, for a smooth enough function F with arguments t and s. The risk neutral measure is denoted $\mathbb{Q}$.

We consider a self-financing portfolio with (at time t) absolute investments $h(t)$ in stock and $k(t)$ in the derivative. The resulting portfolio value is $V(t)$ and self-financing means $\mathrm{d}V(t) = h(t)\,\mathrm{d}G(t) + k(t)\,\mathrm{d}F(t, S(t))$.

   (a) Write down the dynamics of V (in terms of $\mathrm{d}V(t)$) under the risk neutral measure, use the Itô formula.

   (b) Derive a relation between $h(t)$ and $k(t)$ such that the value process is riskless (involves no Brownian term).

   (c) Derive the pricing partial differential equation (PDE)

$$F_t(t,s) + (r-\delta)sF_s(t,s) + \frac{1}{2}\sigma^2 s^2 F_{ss}(t,s) - rF(t,s) = 0,$$

   and give the terminal condition for this PDE, so what is $F(T,s)$?

2. Consider a standard Black-Scholes market, under the physical (real world) measure $\mathbb{P}$ the asset dynamics are given by

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t),$$

where W is a Brownian motion under $\mathbb{P}$. The objective is to price a derivative of the special form $\Phi(T) = S(T)\Psi(T)$, which is such that all relevant expectations exist. In addition there is the usual bank account with constant spot rate r. We consider a Girsanov transformation with likelihood process L^1, for $0 \le t \le T$, resulting in a probability measure $\mathbb{Q}^1$ on $\mathcal{F}_T$ that is given by $\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}} = L^1(T)$. As L^1 is a martingale under $\mathbb{P}$, we know that $L^1(t) = 1 + \int_0^t \phi^1(s)L^1(s)\,\mathrm{d}W(s)$, for an appropriate process ϕ^1 to be chosen later. From general theory we know that the process W^1 satisfying $\mathrm{d}W^1(t) = \mathrm{d}W(t) - \phi^1(t)\,\mathrm{d}t$ is a Brownian motion under $\mathbb{Q}^1$.

   (a) Show that $\phi^1(s) = \frac{r+\sigma^2-\mu}{\sigma}$ (for all s) to ensure that the process $\frac{B(t)}{S(t)}$ becomes a martingale under $\mathbb{Q}^1$.

   (b) Under the usual risk-neutral measure $\mathbb{Q}$, one has the well known pricing formula $\Pi(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(T)|\mathcal{F}_t]$. Use a change of numéraire result and the abstract Bayes formula to convert this formula of $\Pi(t)$ into one in terms of a conditional expectation under the measure $\mathbb{Q}^1$.

   (c) Invoke a martingale argument (for the measure $\mathbb{Q}^1$) to show that $\Pi(t) = S(t)\mathbb{E}^{\mathbb{Q}^1}[\Psi(T)|\mathcal{F}_t]$.

   (d) Show that the dynamics under $\mathbb{Q}^1$ of S are given by

$$\mathrm{d}S(t) = (r+\sigma^2)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^1(t).$$

   (e) Show that $\Pi(t) = S(t)e^{(r+\sigma^2)(T-t)}\Pi_1(t)$, where $\Pi_1(t)$ is the Black-Scholes price of $\Psi(T)$ when $\mathbb{Q}^1$ would be the risk-neutral measure and the spotrate would be $r+\sigma^2$.

   (f) Find a function B_1 such that the process $\frac{S(t)}{B_1(t)}$, $t \le T$, is a martingale under $\mathbb{Q}^1$.

3. Consider a market with N assets with prices processes S^i $(i = 1, \ldots, N)$. These price processes constitute the column vector process S, which then has elements S^i. This market is called S-market. We assume that the S^i are (continuous) *semimartingales*, i.e. they admit stochastic differentials, at time t denoted $\mathrm{d}S^i(t)$. Recall that for semimartingales A and B the Itô product formula holds, $\mathrm{d}(A(t)B(t)) = A(t)\,\mathrm{d}B(t) + B(t)\,\mathrm{d}A(t) + \mathrm{d}A(t)\,\mathrm{d}B(t)$, briefly $\mathrm{d}(AB) = A\,\mathrm{d}B + B\,\mathrm{d}A + \mathrm{d}A\,\mathrm{d}B$. We take one of the price process as a numéraire, say S^1, and write $R = \frac{1}{S^1}$, then also R is a semimartingale. Along with the S^i, we consider the scaled processes Z^i, where $Z^i = S^i R$, which then constitute the column vector process Z. This describes the Z-market.

   We consider a (row vector) portfolio process $h = (h^1, \ldots, h^N)$. The associated value process V is given by $V(t) = h(t)S(t)$, a product of a row and a column vector. Recall that h is called self-financing when $\mathrm{d}V = h\,\mathrm{d}S$. Consider also the scaled value process V^Z defined by $V^Z = RV$. Call h a Z-self-financing process if $\mathrm{d}V^Z = h\,\mathrm{d}Z$

   (a) Show that h is self-financing if and only if it is Z-self-financing.

   (b) Show that for some $T > 0$ a T-claim X is replicable (which means the same as hedgeable, reachable) in the S-market iff $XR(T)$ is replicable in the Z-market.

4. Consider the standard Black-Scholes model for stock and bond, for $0 \leq t \leq T$,

$$\begin{aligned} \mathrm{d}S(t) &= \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t), \\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t. \end{aligned}$$

   The problem to solve is the maximization of logarithmic utility, $\mathbb{E}\log X(T)$, starting from initial wealth $x > 0$, by investing in self-financing portfolios, at time t denoted $(h^S(t), h^B(t))$, with associated wealth process $X(t) = h^S(t)S(t) + h^B(t)B(t)$. The relative investment in the stock is denoted $w(t)$, so $w(t) = h^S(t)S(t)/X(t)$.

   (a) Show that $\mathrm{d}X(t) = X(t)\big(w(t)\mu + r(1 - w(t))\big)\,\mathrm{d}t + \sigma X(t)w(t)\,\mathrm{d}W(t)$.

   (b) Derive the HJB equation for the value function $V(t, x)$ for this problem from the general theory of HJB equations. I.e., show that (suppressing dependence on t and x)

$$V_t + \sup_w \{x(w\mu + r(1 - w))V_x + \frac{1}{2}\sigma^2 x^2 w^2 V_{xx}\} = 0.$$

   (c) Show that the optimal weight $w(t)$ is constant in time (make an educated guess for V first).

   (d) Find the optimal value function and compute from this the final result, the maximal $\mathbb{E}\log X(T)$.

   (e) Go back to the initial problem with dynamics of X as in part (a) and let $Y(t) = \log X(t)$. Give the dynamics of Y. Express $\mathbb{E}\,Y(T)$ as an expectation of an integral over time in terms of the $w(t)$.

   (f) The goal is now to maximize $\mathbb{E}\,Y(T)$ as a function of the $w(t)$. Find the optimal $w(t)$ in this setting.