

Stochastic finance in continuous time
19 December 2024

1. Consider a Black-Scholes market where asset holders receive dividends according to a constant rate δ . That means that under the physical (real world) measure \mathbb{P} the asset dynamics are given by

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t),$$

where W is a Brownian motion under \mathbb{P} . Moreover, the cumulative dividends $D(t)$ satisfy $dD(t) = \delta S(t) dt$. The *gains process* G is given by $G(t) = S(t) + D(t)$. In addition there is the bank account with constant spot rate r . The goal is to price a T -claim of the form $\Phi(S_T)$, where T denotes the horizon. The time t price of the claim, a derivative, is $F(t, S(t))$, for a smooth enough function F with arguments t and s . The risk neutral measure is denoted \mathbb{Q} .

We consider a self-financing portfolio with (at time t) absolute investments $h(t)$ in stock and $k(t)$ in the derivative. The resulting portfolio value is $V(t)$ and self-financing means $dV(t) = h(t) dG(t) + k(t) dF(t, S(t))$.

- (a) Write down the dynamics of V (in terms of $dV(t)$) under the risk neutral measure, use the Itô formula.
- (b) Derive a relation between $h(t)$ and $k(t)$ such that the value process is riskless (involves no Brownian term).
- (c) Derive the pricing partial differential equation (PDE)

$$F_t(t, s) + (r - \delta)sF_s(t, s) + \frac{1}{2}\sigma^2 s^2 F_{ss}(t, s) - rF(t, s) = 0,$$

and give the terminal condition for this PDE, so what is $F(T, s)$?

2. Consider a standard Black-Scholes market, under the physical (real world) measure \mathbb{P} the asset dynamics are given by

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t),$$

where W is a Brownian motion under \mathbb{P} . The objective is to price a derivative of the special form $\Phi(T) = S(T)\Psi(T)$, which is such that all relevant expectations exist. In addition there is the usual bank account with constant spot rate r . We consider a Girsanov transformation with likelihood process L^1 , for $0 \leq t \leq T$, resulting in a probability measure \mathbb{Q}^1 on \mathcal{F}_T that is given by $\frac{d\mathbb{Q}^1}{d\mathbb{P}} = L^1(T)$. As L^1 is a martingale under \mathbb{P} , we know that $L^1(t) = 1 + \int_0^t \phi^1(s) L^1(s) dW(s)$, for an appropriate process ϕ^1 to be chosen later. From general theory we know that the process W^1 satisfying $dW^1(t) = dW(t) - \phi^1(t) dt$ is a Brownian motion under \mathbb{Q}^1 .

- (a) Show that $\phi^1(s) = \frac{r+\sigma^2-\mu}{\sigma}$ (for all s) to ensure that the process $\frac{B(t)}{S(t)}$ becomes a martingale under \mathbb{Q}^1 .
- (b) Under the usual risk-neutral measure \mathbb{Q} , one has the well known pricing formula $\Pi(t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\Phi(T) | \mathcal{F}_t]$. Use a change of numéraire result and the abstract Bayes formula to convert this formula of $\Pi(t)$ into one in terms of a conditional expectation under the measure \mathbb{Q}^1 .
- (c) Invoke a martingale argument (for the measure \mathbb{Q}^1) to show that $\Pi(t) = S(t) \mathbb{E}^{\mathbb{Q}^1}[\Psi(T) | \mathcal{F}_t]$.
- (d) Show that the dynamics under \mathbb{Q}^1 of S are given by

$$dS(t) = (r + \sigma^2)S(t) dt + \sigma S(t) dW^1(t).$$

- (e) Show that $\Pi(t) = S(t)e^{(r+\sigma^2)(T-t)}\Pi_1(t)$, where $\Pi_1(t)$ is the Black-Scholes price of $\Psi(T)$ when \mathbb{Q}^1 would be the risk-neutral measure and the spotrate would be $r + \sigma^2$.
- (f) Find a function B_1 such that the process $\frac{S(t)}{B_1(t)}$, $t \leq T$, is a martingale under \mathbb{Q}^1 .

4. Consider the standard Black-Scholes model for stock and bond, for $0 \leq t \leq T$,

$$\begin{aligned}dS(t) &= \mu S(t) dt + \sigma S(t) dW(t), \\dB(t) &= rB(t) dt.\end{aligned}$$

The problem to solve is the maximization of logarithmic utility, $\mathbb{E} \log X(T)$, starting from initial wealth $x > 0$, by investing in self-financing portfolios, at time t denoted $(h^S(t), h^B(t))$, with associated wealth process $X(t) = h^S(t)S(t) + h^B(t)B(t)$. The relative investment in the stock is denoted $w(t)$, so $w(t) = h^S(t)S(t)/X(t)$.

- (a) Show that $dX(t) = X(t)(w(t)\mu + r(1 - w(t))) dt + \sigma X(t)w(t) dW(t)$.
 (b) Derive the HJB equation for the value function $V(t, x)$ for this problem from the general theory of HJB equations. I.e., show that (suppressing dependence on t and x)

$$V_t + \sup_w \{x(w\mu + r(1 - w))V_x + \frac{1}{2}\sigma^2 x^2 w^2 V_{xx}\} = 0.$$

- (c) Show that the optimal weight $w(t)$ is constant in time (make an educated guess for V first).
 (d) Find the optimal value function and compute from this the final result, the maximal $\mathbb{E} \log X(T)$.
 (e) Go back to the initial problem with dynamics of X as in part (a) and let $Y(t) = \log X(t)$. Give the dynamics of Y . Express $\mathbb{E} Y(T)$ as an expectation of an integral over time in terms of the $w(t)$.
 (f) The goal is now to maximize $\mathbb{E} Y(T)$ as a function of the $w(t)$. Find the optimal $w(t)$ in this setting.