SF CT 2024/12/19 Very concise solutions

1(a) $dV = h(dS + dD) + k\,dF \overset{Q}{=} h((r-\delta)S\,dt + \sigma S\,dW^Q + \delta S\,dt) + k\,dF = (\text{Itô}) =$

$= h(\dots)dt + k\left(F_t\,dt + F_s\,dS + \frac{1}{2}F_{ss}(dS)^2\right) = h(\dots)dt + k\left[F_t\,dt + F_s((r-\delta)S\,dt + \sigma S\,dW^Q) + \frac{1}{2}F_{ss}\sigma^2 S^2\,dt\right]$

$= \left[hrS + kF_t + kF_s(r-\delta)S + \frac{1}{2}\sigma^2 S^2 k\right]dt + (h + kF_s)\sigma S\,dW^Q$

(b) $h + kF_s = 0$, $h = -kF_s$

(c) Combine with $dV = rV\,dt$; gives $-kF_s rS + kF_t + kF_s(r-\delta)S + \frac{1}{2}\sigma^2 S^2 k = r(\overbrace{-kF_s S + kF}^{V})$

Divide by k (assume non zero) to get the PDE. $F(T,s) = \Phi(s)$

2(a) First $d\left(\frac{1}{S}\right) = -\frac{1}{S^2}dS + \frac{1}{2}\cdot\frac{2}{S^3}(dS)^2 = -\frac{1}{S^2}(\mu S\,dt + \sigma S\,dW) + \frac{1}{S^3}\sigma^2 S^2\,dt$ } combine to get

As B has no Brownian term $d\left(\frac{B}{S}\right) = \frac{1}{S}dB + B\,d\left(\frac{1}{S}\right)$

$d\left(\frac{B}{S}\right) = \frac{B}{S}\underbrace{(-\mu + \sigma^2 - \sigma\varphi + r)}_{=0!}dt + -\frac{B\sigma}{S}dW^1$, using $dW = \varphi\,dt + dW^1$.

(b) Let L^0 denote the likelihood ratio process for $P \to Q$. Theory says $\frac{L^1(t)}{L^0(t)} = \frac{1}{S(0)}\frac{S(t)}{B(t)}$.

Then $E^Q[\Phi|\mathcal{F}_t] = E^{Q^1}\left[\Phi\frac{L^0(T)}{L^1(T)}\Big|\mathcal{F}_t\right] \Big/ \frac{L^0(t)}{L^1(t)} = \frac{B(T)}{B(t)}S(t)\,E^{Q^1}\left[\frac{\Phi}{S(T)}\Big|\mathcal{F}_t\right]$.

So $\pi(t) = S(t)\,E^{Q^1}\left[\frac{\Phi}{S(T)}\Big|\mathcal{F}_t\right]$.

(c) From (b), or realizing that S is used as numèraire: $\frac{\Pi(t)}{S(t)}$ is a Q^1-martingale $\Rightarrow$ Result.

(d) Use $dW^Q = \varphi\,dt + dW^1$, with φ as in (a) and $dS = \dots$ to get the result.

(e) $\Pi_1(t) = e^{-(r+\sigma^2)(T-t)}E^{Q^1}[\Phi(T)|\mathcal{F}_t]$. Plug this into the expression in (c).

(f) Equation for $dS(t)$ suggests $B_1(t) = e^{(r+\sigma^2)t}$. You may use $d\left(\frac{S(t)}{B_1(t)}\right) = \frac{dS(t)}{B_1(t)} - \frac{S(t)}{B_1(t)^2}dB_1(t)$.

4(a). Start with $dX = h^S dS + h^B dB = h^S(\mu S\,dt + \sigma S\,dW) + h^B rB\,dt$ and use $h^S S = wX$, $h^B B = (1-w)X$.

(b) General theory with $dX = b\,dt + \sigma\,dW$ gives $V_t + \sup\{bV_x + \frac{1}{2}\sigma^2 V_{xx}\} = 0$. Use now $b \to x(w\mu + (1-r)w)$ and $\sigma^2 \to \sigma^2 x^2 w^2$.

(c) Educated guess is $V(t,x) = f(t)\log x + g(t) \to V_x = \frac{f}{x}$, $V_{xx} = -\frac{f}{x^2}$. Compute derivative w.r.t. w in $\{\dots\}$: $xV_x(\mu - r) + \sigma^2 x^2 V_{xx} w = 0$, use V_x, V_{xx}, to get $w = \frac{\mu - r}{\sigma^2}$

(d) Solve HJB with w and V as above, i.e. solve $f'(t)\log x + g'(t) + \left\{\frac{\mu-r}{\sigma^2}\mu + \left(1 - \frac{\mu-r}{\sigma^2}\right)r - \frac{\sigma^2}{2}f(t)\left(\frac{\mu-r}{\sigma^2}\right)^2\right\} = 0$

Separate terms with x and without: $f'(t)\log x = 0$, $g'(t) + \{\dots\} = 0$.

We see $f'(t) = 0$. But $V(T,x) = f(T)\log x + g(T) = \log x \Rightarrow f(T) = 1$, $g(T) = 0$.

So $f(t) = 1$, $\forall t$. Then $g'(t) + \{\dots\} = g'(t) + c = 0$, with $c = \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} + r \Rightarrow g(t) = c(T-t)$.

The optimal $E\log X(T)$ is $V(0,x) = \log x + cT = \log x + \left[\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} + r\right]T$.

(e) Use Itô: $dY(t) = d[\log X(t)] = (w\mu + r(1-w))dt + \sigma w\,dW + -\frac{1}{2}\sigma^2 w^2\,dt$, or $Y(T) = Y(0) + \int_0^T \dots$

The Itô integral has expectation zero. So $EY(T) = \log x + E\int_0^T\left[w(t)\mu + r(1-w(t)) - \frac{1}{2}\sigma^2 w(t)^2\right]dt$

(f) Optimize under E and the integral, differentiate w.r.t. $w(t)$ the term in $[\dots]$. This gives $\mu - r - \sigma^2 w(t) = 0 \Rightarrow w(t) = \frac{\mu - r}{\sigma^2}$, as in (c)!

3(a) Suppose h is SF, $dV = h\,dS$. Then $dV^Z = d(RV) = R\,dV + V\,dR + dR\,dV = Rh\,dS + hS\,dR + dR\,h\,dS$
$= h(R\,dS + S\,dR + dR\,dS) = h\,d(RS) = h\,dZ \Rightarrow h$ is Z-self financing.

Conversely, if h is Z-SF, $dV^Z = h\,dZ$. Then $dV = d(S^1 V^Z) = S^1 dV^Z + V^Z dS^1 + dS^1 dV^Z =$
$= S^1 h\,dZ + hZ\,dS^1 + dS^1 h\,dZ) = h(S^1 dZ + Z\,dS^1 + dS^1 dZ) = h\,d(S^1 Z) = h\,dV$.

(b) If X is replicable, there is SF h s.t. $X = V(T)$, so $XR(T) = V(T)R(T) = V^Z(T) \Rightarrow XR(T)$ is replicable in the Z-market, as h is Z-SF in view of (a). Converse is similar.