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Stochastic finance in continuous time
9 January 2025

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1. Consider a Black-Scholes market where asset holders receive dividends according to a constant rate δ. That means that under the physical (real world) measure $\mathbb{P}$ the asset dynamics are given by

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^{\mathbb{P}}(t),$$

where $W^{\mathbb{P}}$ is a Brownian motion under $\mathbb{P}$. Moreover, the cumulative dividends $D(t)$ satisfy $\mathrm{d}D(t) = \delta S(t)\,\mathrm{d}t$. The *gains process* G is defined by $G(t) = S(t)+D(t)$. In addition there is the bank account $B(t)$ with constant spot rate r. The goal is to price a T-claim of the form $\Phi(S_T)$, where T denotes the horizon. The time t price of the claim, a derivative, is $F(t, S(t))$, for a smooth enough function F with arguments t and s. The relevant filtration is denoted $\mathcal{F}_t, t \in [0, T]$ and the risk neutral measure is denoted $\mathbb{Q}$. It is known that F satisfies a pricing partial differential equation (PDE),

$$F_t(t, s) + (r - \delta)sF_s(t, s) + \frac{1}{2}\sigma^2 s^2 F_{ss}(t, s) - rF(t, s) = 0.$$

The objective is to hedge the claim by a self-financing portfolio consisting of stocks and the money account. So we consider $U(t) = x(t)S(t) + y(t)B(t)$. Hedging means that $U(T) = \Phi(T)$.

   (a) Write down the dynamics of U under the risk neutral measure.

   (b) Use the PDE and the self-financing condition to conclude that $x(t) = F_s(t, S(t))$. What is $y(t)$?

   (c) Let $\tilde{U}(t) = \frac{U(t)}{B(t)}$ and $\tilde{S}(t) = \frac{S(t)}{B(t)}$. Show that $\tilde{U}(t)$ is a martingale under $\mathbb{Q}$. Show also that $\tilde{S}(t) + \delta \int_0^t \tilde{S}(u)\,\mathrm{d}u$ is a martingale under $\mathbb{Q}$.

In the situation where there are no dividend payments, we know that $\mathbb{E}^{\mathbb{Q}}[S(T)|\mathcal{F}_t] = e^{r(T-t)}S(t)$. In the current situation, *with dividends*, we expect something similar, but different, by exploiting the martingale property in terms of $\tilde{S}$ above.

   (d) Show that $\mathbb{E}^{\mathbb{Q}}[\tilde{S}(T) + \delta \int_t^T \tilde{S}(u)\,\mathrm{d}u|\mathcal{F}_t] = \tilde{S}(t)$, and by taking conditional expectations under the integral, $h(T) + \delta \int_t^T h(u)\,\mathrm{d}u = \tilde{S}(t)$, for $h(u) = \mathbb{E}^{\mathbb{Q}}[\tilde{S}(u)|\mathcal{F}_t]$, $T \geq u \geq t$.

   (e) Differentiate (allowed!) w.r.t. T to get a differential equation for h with an initial condition for $h(t)$. Solve this equation. What is $\mathbb{E}^{\mathbb{Q}}[S(T)|\mathcal{F}_t]$ now?

   (f) What is the corresponding hedge strategy for the claim $S(T)$, i.e., what are in this case the $x(t)$ and $y(t)$?

2. Consider a market consisting of two assets with prices $S^1(t)$ and $S^2(t)$ and a bank account $B(t)$ with constant short rate r. We assume the following model, under the risk neutral measure $\mathbb{Q}$, for $0 \leq t \leq T$.

$$\begin{aligned}
\mathrm{d}S^1(t) &= rS^1(t)\,\mathrm{d}t + \sigma_1 S^1(t)\,\mathrm{d}W^1(t),\\
\mathrm{d}S^2(t) &= rS^2(t)\,\mathrm{d}t + \sigma_2 S^2(t)\,\mathrm{d}W^2(t),\\
\mathrm{d}B(t) &= rB(t)\,\mathrm{d}t.
\end{aligned}$$

Here W^1 and W^2 are independent standard Brownian motions (under $\mathbb{Q}$). The relevant filtration is denoted $\mathcal{F}_t, t \in [0, T]$. We let R be the ratio process of the two assets, $R(t) = \frac{S^2(t)}{S^1(t)}$.

(a) Show that the dynamics of R are given by

$$\mathrm{d}R(t) = \sigma_1^2 R(t)\,\mathrm{d}t + R(t)(\sigma_2 \mathrm{d}W^2(t) - \sigma_1 \mathrm{d}W^1(t)).$$

The next objective is to price the minimum option $M = \min\{S^1(T), S^2(T)\}$. We will see that this can be accomplished using a Girsanov transformation and taking the process S^1 as a numéraire. Consider the likelihood ratio process $\tilde{L}$ with $\mathrm{d}\tilde{L}(t) = \sigma_1 \tilde{L}(t)\,\mathrm{d}W^1(t)$, with $\tilde{L}(0) = 1$. Let $\tilde{\mathbb{Q}}$ be the probability measure on $\mathcal{F}_T$ with $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \tilde{L}(T)$. It follows from general theory that under $\tilde{\mathbb{Q}}$ the processes $\tilde{W}^1$ satisfying $\mathrm{d}\tilde{W}^1(t) = \mathrm{d}W^1(t) - \sigma_1\,\mathrm{d}t$ and (the original) W^2 are independent Brownian motions.

(b) Show that $\tilde{L}(t) = \frac{S^1(t)}{S^1(0)} e^{-rt}$. [You may want to use explicit expressions for $S^1(t)$ and $\tilde{L}(t)$.]

(c) Give the dynamics of S^1 and R under the measure $\tilde{\mathbb{Q}}$, use a Brownian motion under $\tilde{\mathbb{Q}}$.

(d) Show, using the abstract Bayes formula for conditional expectations, that the fair price Π_t of M at time t can be computed as $\Pi_t = S^1(t)\,\mathbb{E}^{\tilde{\mathbb{Q}}}[\frac{M}{S^1(T)}|\mathcal{F}_t]$ (involving a conditional expectation under $\tilde{\mathbb{Q}}$ as an alternative to using a conditional expectation under $\mathbb{Q}$).

(e) Show that Π_t can be written as $\Pi_t = S^1(t)\,\mathbb{E}^{\tilde{\mathbb{Q}}}[R(T) - \max\{R(T) - 1, 0\}|\mathcal{F}_t]$.

To further compute Π_t one can use the price of a standard call option on a stock $S(T)$, i.e. $\max\{S(T) - K, 0\}$. In a standard Black-Scholes model with short rate r and volatility σ, the time t price of such a call option is denoted $c_t(S(t), \sigma, r, K)$.

(f) By selecting the right parameters, express Π_t further in terms of the function $c_t(\cdot,\cdot,\cdot,\cdot)$.

3. Consider the standard Black-Scholes model for stock and bond, for $0 \le t \le T$,

$$\begin{aligned}\mathrm{d}S(t) &= \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t),\\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t.\end{aligned}$$

The problem to solve is the maximization of power utility, $\mathbb{E}X(T)^\gamma$, $\gamma \in (0,1)$, starting from initial wealth $x > 0$, by investing in self-financing portfolios, at time t denoted $(h^S(t), h^B(t))$, with associated wealth process $X(t) = h^S(t)S(t) + h^B(t)B(t)$. The relative investment in the stock is denoted $w(t)$, so $w(t) = h^S(t)S(t)/X(t)$. Note that, in general, $w(t)$ is random and depends on time.

(a) Show that $\mathrm{d}X(t) = X(t)\big(w(t)\mu + r(1 - w(t))\big)\,\mathrm{d}t + \sigma X(t)w(t)\,\mathrm{d}W(t)$.

(b) Give an argument that the $X(t)$ are positive.

(c) Derive the HJB equation for the value function $V(t,x)$ for this problem using the general theory of HJB equations. I.e., show that it results that (suppressing dependence on t and x)

$$V_t + \sup_w\{x(w\mu + r(1-w))V_x + \frac{1}{2}\sigma^2 x^2 w^2 V_{xx}\} = 0.$$

(d) Show that the optimal weight $w(t)$ is constant in time [start from an educated guess, an ansatz, $V(t,x) = f(t)x^\gamma + g(t)$ with positive f].

(e) Derive differential equations for f and g to show that the optimal value function is $V(t,x) = e^{c(T-t)}x^\gamma$ with $c = \gamma(\frac{(\mu-r)^2}{\sigma^2(1-\gamma)} + r)$ and compute the maximal $\mathbb{E}X(T)^\gamma$.