SFCT 20250129 Very concise solutions

1(a) $dU = x(dS + dG) + y\,dB = rU\,dt + \sigma x \tilde{S}\,dW^Q$

(b) As $U(t) = F(t, S(t))$, we have $dU \overset{PDE}{=} (rF + \delta S F_S)dt + \sigma S F_S\,dW^Q$. The Brownian terms should coincide, thus $x(t) = F_S(t, S(t))$; $y(t) = [U(t) - x(t)S(t)]/B(t)$.

(c) Compute $d\tilde{U} = \frac{dU}{B} - \frac{U}{B^2}dB = \dots = \sigma x \tilde{S}\,dW^Q \rightarrow$ martingale. Likewise,
$d\tilde{S} = \frac{dS}{B} - \frac{S}{B^2}dB = (r-\delta)\tilde{S}\,dt + \sigma\tilde{S}\,dW^Q - r\tilde{S}\,dt = -\delta\tilde{S}\,dt + \sigma\tilde{S}\,dW^Q \Rightarrow$
$d\tilde{S} + \delta\tilde{S}\,dt = \sigma\tilde{S}\,dW^Q$, martingale increment.

(d) The martingale property yields $E^Q[\tilde{S}(T) + \delta\int_0^t \tilde{S}(u)du + \delta\int_t^T \tilde{S}(u)\,du \mid \mathcal{F}_t] = \tilde{S}(t) + \delta\int_0^t \tilde{S}(u)du$ which gives the first result as $\int_0^t \tilde{S}(u)\,du$ is $\mathcal{F}_t$-measurable. Depart from that and write $E^Q[\tilde{S}(T)|\mathcal{F}_t] + \delta\int_t^T E^Q[\tilde{S}(u)|\mathcal{F}_t]\,du = \tilde{S}(t)$, or

(e) $h(T) + \delta\int_t^T h(u)\,du = \tilde{S}(t)$. Differentiate $h'(T) + \delta h(T) = 0 \Rightarrow h(T) = c.e^{-\delta T}$
But, $h(t) = \tilde{S}(t)$, so $\tilde{S}(t) = c\,e^{-\delta t} \Rightarrow c = \tilde{S}(t)e^{\delta t}$ and $h(T) = \tilde{S}(t)\,e^{-\delta(T-t)}$.
Then $E^Q[S(T)|\mathcal{F}_t] = e^{rT}E^Q[\tilde{S}(T)|\mathcal{F}_t] = e^{rT}e^{-\delta(T-t)}S(t)e^{-rt} = e^{(r-\delta)(T-t)}S(t)$

(f) Take $x(t) = e^{-\delta(T-t)}$, $y(t) = 0$

2(a) First, $d\left(\frac{1}{S^1}\right) = -\frac{1}{(S^1)^2}dS^1 + \frac{1}{(S^1)^3}(dS^1)^2 = -\frac{\cancel{\mu_1}}{S^1}dt - \frac{\sigma_1}{S^1}dW^1 + \frac{\sigma_1^2}{S^1}dt$.
Then (no Itô covariation terms, as W^1 and W^2 are independent):
$dR = \frac{dS_2}{S_1} + S_2\,d\left(\frac{1}{S_1}\right) = \cancel{\mu_2}R\,dt + \boxed{\sigma_2 R\,dW^2} - \cancel{\mu_1}R\,dt\,\boxed{-\sigma_1 R\,dW^1} + \boxed{\sigma_1^2 R\,dt}$.

(b) We have $S^1(t) = S^1(0)\exp((r - \frac{1}{2}\sigma^2)t + \sigma W^1(t)) = S^1(0)e^{rt}\exp(\sigma W^1(t) - \frac{1}{2}\sigma^2 t)$.
But, from $d\tilde{L} = \sigma_1\tilde{L}\,dW^1$, $\tilde{L}(t) = \exp(\sigma W^1(t) - \frac{1}{2}\sigma^2(t))$, so $S^1(t) = S^1(0)e^{rt}\tilde{L}(t)$

(c) From $d\tilde{W}^1(t) = dW^1(t) - \sigma_1\,dt$, we get $dS^1(t) = (r + \sigma_1^2)S^1(t)dt + S^1(t)\sigma_1\,d\tilde{W}^1(t)$
and $dR(t) = R(t)(\sigma_2\,dW^2(t) - \sigma_1\,dW^1(t))$, a martingale!

(d) $\pi_t = \frac{B(t)}{B(T)}E^Q[M|\mathcal{F}_t] = \frac{B(t)}{B(T)}E^{\tilde{Q}}\left[\left(\frac{dQ}{d\tilde{Q}}\right)_T M \,\middle|\, \mathcal{F}_t\right] \Big/ \left(\frac{dQ}{d\tilde{Q}}\right)_t$. But $\left(\frac{dQ}{d\tilde{Q}}\right)_t = \frac{1}{\tilde{L}(t)}$,
and with (c) we get $\pi_t = \frac{B(t)}{B(T)}E^{\tilde{Q}}\left[\frac{M}{S^1(T)}\middle|\mathcal{F}_t\right]S^1(0)e^{rT}\Big/\left(\frac{S^1(0)e^{rt}}{S^1(t)}\right) =$
$S^1(t)\,E^{\tilde{Q}}\left[\frac{M}{S^1(T)}\middle|\mathcal{F}_t\right]$.

(e) Write $M = S^1(T)\min\{1, R(T)\} = S^1(T)(1 + \min\{R(T)-1, 0\}) = S^1(T)(R(T) - \max\{R(T)-1, 0\})$
$\Rightarrow \pi_t = S^1(t)E^{\tilde{Q}}[R(T) - \max\{R(T)-1, 0\}|\mathcal{F}_t]$

(f) Note that $\max\{R(T)-1, 0\}$ is a call option on $R(T)$ with strike $K=1$.
Under $\tilde{Q}$, R is a martingale so the drift is zero, see (c). Write $dR = R\sigma\,d\tilde{W}$,
where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\tilde{W} = \frac{\sigma_2 W^2 - \sigma_1 W^1}{\sigma}$ is a Brownian motion. Then
$\pi_t = S^1(t)\left[R(t) - C_t(R(t), \sigma, 0, 1)\right]$.

3(a) $dX = h^S dS + h^B dB$, plug in dS, dB and $h^S(t)S(t) = w(t)X(t)$.

(b) X is like a time varying GBM, or one can write $X(t) = \exp(Y(t))$ etc.

(c) Immediately follows from (a) and the general HJB theory.

(d) With the ansatz, $xV_x = \gamma x^\gamma f$, $x^2V_{xx} = x^\gamma\gamma(\gamma-1)f$. Then sup over w becomes maximizing $(w(\mu - r) + r)\gamma + \frac{1}{2}\sigma^2 w^2\gamma(\gamma-1) \rightarrow w(t) = \frac{\mu - r}{\sigma^2(1-\gamma)}$. (Note coeff. of w^2 is <0)

(e) Plug in the w from c in the HJB equation:
$f'x^\gamma + g' + \gamma x^\gamma f\left(\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)} + r\right) = 0$, valid for all x: $g'(t) = 0$, $f'(t) + \gamma f(t) \cdot c = 0$,
with $c = \frac{(\mu-r)^2}{2\sigma^2(1-\gamma)} + r$. $g(T) = 0$, $f(T) = 1 \Rightarrow g(t) \equiv 0$, $f(t) = e^{c\gamma(T-t)}$. $\max E X(T)^\gamma = V(0, x)$.