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# Stochastic finance in continuous time
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1. In this exercise we consider the pricing of the $T$-claim $X = S_T \mathbf{1}_{\{S_T \geq K\}}$, so the owner of the claim receives nothing at time $T$ when the stock price at that moment has dropped below the threshold $K$. In the Black-Scholes framework, the asset price $S_t$ follows under the *risk neutral measure* $\mathbb{Q}$ the dynamics
$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$
where $W$ is a Wiener process under $\mathbb{Q}$. As usual, the bank account is given by $B_t = e^{rt}$. It follows that $S_t = S_0 \exp\big((r - \frac{1}{2}\sigma^2)t + \sigma W_t\big)$. The filtration $\{\mathcal{F}_t, t \leq T\}$ is the one generated by $W$. Define the process $R$ by $R_t = \frac{B_t}{S_t}$. Note that $R_0 = \frac{1}{S_0}$.

   (a) Show that
$$\mathrm{d}R_t = R_t(\sigma^2\mathrm{d}t - \sigma\,\mathrm{d}W_t).$$

   Next we make a change of measure by a likelihood ratio process $L$ that satisfies $\mathrm{d}L_t = \alpha L_t\,\mathrm{d}W_t$, where the constant $\alpha$ is to be determined. General theory tells us that the process $\tilde{W}$ defined by $\tilde{W}_t = W_t - \alpha t$ is a Wiener process under the measure $\tilde{\mathbb{Q}}$ defined on $\mathcal{F}_T$ by $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = L_T$.

   (b) Show that $\alpha = \sigma$ in order that $R$ becomes a martingale under $\tilde{\mathbb{Q}}$.

   We keep this value of $\alpha$ in all what follows.

   (c) Show that $R_t L_t = \frac{1}{S_0}$, for $t \leq T$.

   (d) Show also that the process $S$ satisfies
$$\mathrm{d}S_t = S_t(r + \sigma^2)\,\mathrm{d}t + S_t\sigma\,\mathrm{d}\tilde{W}_t,$$
   and that
$$\mathrm{d}\log S_t = (r + \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t.$$

   We now take the process $S$ as numéraire, traded assets are such that their prices divided by $S_t$ become martingales under $\tilde{\mathbb{Q}}$.

   (e) Show that the time $t$ price $\Pi_t$ of the claim is given by $\Pi_t = S_t\tilde{Q}(S_T \geq K|\mathcal{F}_t)$.

   (f) Show that $\Pi_t$ for $t < T$ can be computed as $\Pi_t = S_t\big(1 - \Phi(\frac{\log\frac{K}{S_t} - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}})\big)$, where $\Phi$ is the standard normal distribution function. Compute also $\Pi_T$.

2. The setting in this exercise is that of a usual Black-Scholes market (which is known to be complete), with an asset price $S$ evolving under the physical probability measure $\mathbb{P}$ as
$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$
where $W$ is a Wiener process under $\mathbb{P}$. The bank account is given by $B_t = e^{rt}$. We assume that the time horizon is $T$, and that we deal with the filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ generated by $W$.

   The problem is to maximize $\mathbb{E}^{\mathbb{P}}\Phi(Z)$, where we make the *special choice* $\Phi(z) = 2\sqrt{z}$, where $z \geq 0$. So the problem becomes the maximization of
$$\mathbb{E}^{\mathbb{P}}[2\sqrt{Z}].$$

Here the $Z$ are $\mathcal{F}_T$-measurable random variables, which may have the interpretation as wealths arising from a self-financing portfolio process and an initial capital $x_0 > 0$. So we look at the maximization of $\mathbb{E}^{\mathbb{P}}[2\sqrt{Z}]$ under the budget constraint implied by $x_0$.

Recall that $Z$ is an attainable claim, it can be replicated by a self-financing strategy, and hence the corresponding discounted value process is a martingale under the equivalent martingale measure $\mathbb{Q}$. The likelihood ratio process is denoted $L$ and $L_T$ is the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ of the two probability measures on $\mathcal{F}_T$.

(a) Show that the budget constraint is given by $e^{-rT}\mathbb{E}^{\mathbb{Q}}Z = x_0$.

(b) The Lagrangian for this problem is given by

$$\begin{aligned}\mathcal{L} &= \mathbb{E}^{\mathbb{P}}\{2\sqrt{Z} + \lambda(x_0 - L_T e^{-rT} Z)\} \\ &= \int_\Omega \{2\sqrt{Z(\omega)} + \lambda(x_0 - L_T(\omega)e^{-rT}Z(\omega))\}\mathbb{P}(d\omega).\end{aligned}$$

Show that, for fixed $\lambda$, the optimal $Z$ is given by $\hat{Z} = \frac{e^{2rT}}{\lambda^2 L_T^2}$.

(c) Show that $\lambda$ satisfies $\frac{1}{\lambda^2} = \frac{x_0}{e^{rT}\mathbb{E}^{\mathbb{P}}[L_T^{-1}]}$.

(d) Conclude that $\hat{Z} = \frac{x_0 e^{rT}}{L_T^2 \mathbb{E}^{\mathbb{P}}[L_T^{-1}]}$, and that the optimal $\mathbb{E}^{\mathbb{P}}[2\sqrt{Z}]$ equals $2\sqrt{x_0 e^{rT}\mathbb{E}^{\mathbb{P}}[L_T^{-1}]}$.

3. We continue the previous exercise, as we are interested in portfolio optimization by *the martingale method*. In the previous exercise we have seen in item 2d that the optimal $Z$ is $\hat{Z}$, which we now interpret as the optimal final wealth $\hat{X}_T = \frac{x_0 e^{rT}}{L_T^2\mathbb{E}^{\mathbb{P}}[L_T^{-1}]}$. This is the first step of the martingale method.

The purpose is now to perform the second step, which is to find the self-financing strategy that results in this wealth, as well is the intermediate wealths $\hat{X}_t$. Recall that the discounted optimal wealth process, given by $\tilde{X}_t = e^{-rt}\hat{X}_t$, is a martingale under the measure $\mathbb{Q}$. The likelihood ratio process $L$ is given by $L_t = \exp(\phi W_t - \frac{1}{2}\phi^2 t)$, with $\phi = \frac{r-\mu}{\sigma}$.

(a) Show, use the abstract Bayes formula, that $\hat{X}_t = \frac{x_0 e^{rt}\mathbb{E}^{\mathbb{P}}[L_T^{-1}|\mathcal{F}_t]}{L_t \mathbb{E}^{\mathbb{P}}[L_T^{-1}]}$.

(b) Show (you may try a direct computation, or use the Itô formula) that $M_t = \exp(-\phi W_t - \frac{1}{2}\phi^2 t)$ gives a martingale under $\mathbb{P}$. Conclude that $\mathbb{E}^{\mathbb{P}}[L_T^{-1}|\mathcal{F}_t] = e^{\phi^2(T-t)}L_t^{-1}$ and $\mathbb{E}^{\mathbb{P}}[L_T^{-1}] = e^{\phi^2 T}$.

(c) Show that $\hat{X}_t = x_0 R_t e^{rt-\phi^2 t}$, where $R_t = \frac{1}{L_t^2}$.

(d) Show that $\mathrm{d}\hat{X}_t = \hat{X}_t(-2\phi\,\mathrm{d}W_t + (\ldots)\,\mathrm{d}t)$, where you don't have to specify the term $(\ldots)$. [You may want to do this by first showing that $\mathrm{d}R_t = R_t(-2\phi\,\mathrm{d}W_t + (\ldots)\,\mathrm{d}t)$, where also here you don't have to specify the (different) term $(\ldots)$.]

(e) Let $w_t$ be the fraction of wealth that at time $t$ is invested in the stock by using a self-financing strategy to obtain the optimal wealth process $\hat{X}$. Show that

$$\mathrm{d}\hat{X}_t = \hat{X}_t\big((w_t\mu + r(1-w_t))\,\mathrm{d}t + w_t\sigma\,\mathrm{d}W_t\big).$$

(f) Find $w_t$ in terms of the parameters $\mu, \sigma, r$.