

SFCT 19 Dec 2025

- 1a. Write $R_t = \frac{1}{S_0} \exp(+\frac{1}{2}\sigma^2 t - \sigma W_t)$ and use the Itô formula
- b. Write $dR_t = R_t(\sigma^2 dt - \sigma(d\tilde{W}_t + \sigma dt)) = R_t((\sigma^2 - \sigma^2)dt - \sigma d\tilde{W}_t)$. Use $\sigma^2 = \alpha\sigma$.
- d. Similar computation: $ds_t = r s_t dt + \sigma s_t(d\tilde{W}_t + \sigma dt) = (r + \sigma^2)s_t dt + \sigma s_t d\tilde{W}_t$.
Use then the Itô formula for $f(s_t) = \log s_t$, with $f(x) = \log x$.
- e. Risk neutral pricing gives $\pi_t = e^{rt} E^Q[\lambda e^{-rT} | \mathcal{F}_t] = e^{r(t-T)} E^Q[\lambda | \mathcal{F}_t]$.
But $E^Q[\lambda | \mathcal{F}_t] = L_t E^{\tilde{Q}}[\lambda / L_T | \mathcal{F}_t] = \frac{S_t}{S_0} E^{\tilde{Q}}[\frac{S_0 B_T}{S_T} | \mathcal{F}_t] = S_t e^{r(T-t)} E^{\tilde{Q}}[1_{S_T \geq K} | \mathcal{F}_t]$
 $= S_t e^{r(T-t)} \tilde{Q}(S_T \geq K)$. So $\pi_t = e^{r(t-T)} S_t e^{r(T-t)} \tilde{Q}(S_T \geq K) = S_t \tilde{Q}(S_T \geq K)$.
- c. Compute under $\tilde{Q}$ dP , for $P = RL$. Then $dP = 0 \Rightarrow P_t = P_0 = \frac{1}{S_0}$.
- f. $\tilde{Q}(S_T \geq K | \mathcal{F}_t) = \tilde{Q}(\log \frac{S_T}{S_t} \geq \log \frac{K}{S_t} | \mathcal{F}_t) = \tilde{Q}(Z \geq z)$ by $\frac{S_T}{S_t}$ independent of $\mathcal{F}_t$.
But Z is normal with mean $(r + \frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$, see (d).

- 2a. As z is attainable, we know $E^Q[z e^{-rT} | \mathcal{F}_t]$ is a Q -martingale and the value at $t=0$ must be the initial capital: $x_0 = E^Q[z e^{-rT}]$.
- b. The idea is to optimize under the integral: $z \sqrt{\frac{z(\omega)}{2}} + \lambda_0 (x_0 - L_T(\omega) e^{-rT} \frac{z(\omega)}{2})$
Differentiate w.r.t. z : $\frac{1}{\sqrt{z}} + \lambda_0 L_T e^{-rT} \frac{1}{2} = 0 \Rightarrow \hat{z} = \left(\frac{e^{rT}}{\lambda_0 L_T}\right)^2 = \frac{e^{2rT}}{\lambda_0^2 L_T^2}$
- c. Use the budget constraint: $x_0 = e^{-rT} E^Q\left[\frac{e^{2rT}}{\lambda_0^2 L_T^2}\right] = \frac{e^{rT}}{\lambda_0^2} E^P\left[\frac{1}{L_T}\right]$
- d. Combine the answers to (b) and (c): $\hat{z} = \frac{e^{2rT}}{L_T^2} \frac{x_0}{e^{rT} E^P\left[\frac{1}{L_T}\right]} = \frac{x_0 e^{rT}}{L_T^2 E^P\left[\frac{1}{L_T}\right]}$
The optimal value is then $2 E^P \sqrt{\hat{z}} = 2 E^P \left[\sqrt{\frac{x_0 e^{rT}}{E^P\left[\frac{1}{L_T}\right]}} \frac{1}{L_T} \right] = 2 \sqrt{x_0 e^{rT}} \frac{E^P\left[\frac{1}{L_T}\right]}{\sqrt{E^P\left[\frac{1}{L_T}\right]}} = 2 \sqrt{x_0 e^{rT} E^P\left[\frac{1}{L_T}\right]}$

- 3a. We know $\tilde{X}_t = E^Q[\tilde{X}_T | \mathcal{F}_t] = E^P[\tilde{X}_T L_T | \mathcal{F}_t] / L_t = E^P\left[\frac{x_0}{L_T^2 E^P\left[\frac{1}{L_T}\right]} L_T | \mathcal{F}_t\right] / L_t$.
- b. Itô: $dM_t = -\phi M_t dW_t \rightarrow$ martingale (no drift)
- c. For $t \leq T$, one has $L_t^{-1} = M_t e^{-\phi^2 t}$. Hence $E[L_T^{-1} | \mathcal{F}_t] = e^{+\phi^2 T} E[M_T | \mathcal{F}_t] = e^{+\phi^2 T} M_t = e^{+\phi^2(T-t)} \frac{1}{L_t} \Rightarrow E L_T^{-1} = e^{+\phi^2 T} M_0 = e^{+\phi^2 T}$.
- c. Use (a) and (b): $\hat{X}_t = \frac{e^{rt} e^{+\phi^2(T-t)} / L_t}{L_t e^{+\phi^2 T}} = x_0 \frac{e^{rt - \phi^2 t}}{L_t^2} = x_0 e^{rt - \phi^2 t} R_t$.
- d. $dR_t = d(e^{-\phi W_t + \phi^2 t}) = R_t(-2\phi dW_t + \phi^2 dt + \frac{1}{2}\phi^2 (dW_t)^2) = R_t(\dots dt - 2\phi dW_t)$
 $\Rightarrow d\hat{X}_t = x_0 e^{rt - \phi^2 t} (r - \phi^2) R_t dt + x_0 e^{rt - \phi^2 t} (-2\phi) dR_t = (\dots) dt - 2\hat{X}_t \phi dW_t$.
- e. Self-financing means $d\hat{X}_t = \hat{X}_t \left(w_t \frac{dS_t}{S_t} + (1-w_t) \frac{dB_t}{B_t} \right) = \hat{X}_t (\mu dt + \sigma dW_t)$
 $+ \hat{X}_t (1-w_t) r dt = \hat{X}_t (w_t \mu + (1-w_t)r) dt + \hat{X}_t w_t \sigma dW_t$.
- f. The two expressions in (d) and (e) must be the same, in particular both dW_t terms: $-2\hat{X}_t \phi dW_t = \hat{X}_t w_t \sigma dW_t \Rightarrow w_t = -\frac{2\phi}{\sigma} = 2 \frac{\mu - r}{\sigma}$.