Universidade de Lisboa
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# Selected exam questions
# The answers below are sometimes very brief!

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1. Consider a Black-Scholes market where asset holders receive dividends according to a constant rate $\delta$. Under the physical (real world) measure $\mathbb{P}$ the asset dynamics are given by

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where $W$ is a Brownian motion under $\mathbb{P}$. Moreover, the cumulative dividends $D_t$ satisfy $\mathrm{d}D_t = \delta S_t\,\mathrm{d}t$. The *gains process* $G$ is given by $G_t = S_t + D_t$. In addition there is the bank account with constant spot rate $r$. The goal is to price a $T$-claim of the form $\Phi(S_T)$, where $T$ denotes the horizon. The time $t$ price of the claim, a derivative, is $F(t, S_t)$, for a smooth enough function $F$ with arguments $t$ and $s$. The risk neutral measure is denoted $\mathbb{Q}$.

We consider a self-financing portfolio with (at time $t$) absolute investments $h_t$ in stock and $k_t$ in the derivative. The resulting portfolio value is $V_t$ and self-financing means $\mathrm{d}V_t = h_t\,\mathrm{d}G_t + k_t\,\mathrm{d}F(t, S_t)$.

   (a) Write down the dynamics of $V$ (in terms of $\mathrm{d}V_t$) under the risk neutral measure, use the Itô formula.
   **Answer.** Write, in abbreviated notation,

$$\begin{aligned}\mathrm{d}V &= h(\mathrm{d}S + \mathrm{d}D) + k\,\mathrm{d}F\\ &\overset{\mathbb{Q}}{=} h((r-\delta)S\,\mathrm{d}t + \sigma S\,\mathrm{d}W^{\mathbb{Q}} + \delta S\,\mathrm{d}t) + k\,\mathrm{d}F\\ &\overset{\text{Itô}}{=} h((r-\delta)S\,\mathrm{d}t + \sigma S\,\mathrm{d}W^{\mathbb{Q}} + \delta S\,\mathrm{d}t) + k\big(F_t\,\mathrm{d}t + F_s(r-\delta)S\,\mathrm{d}t + \sigma S\,\mathrm{d}W^{\mathbb{Q}}) + \frac{1}{2}F_{ss}\sigma^2S^2\,\mathrm{d}t\big)\\ &= \big(hrS + kF_t + kF_s(r-\delta)S + k\frac{1}{2}F_{ss}\sigma^2S^2\big)\,\mathrm{d}t + (h + kF_s)\sigma S\,\mathrm{d}W^{\mathbb{Q}}.\end{aligned}$$

   (b) Derive a relation between $h_t$ and $k_t$ such that the value process is riskless (involves no Brownian term).
   **Answer.** $h + kF_s = 0$, so $h = -kF_s$. This results into

$$\mathrm{d}V = \big(hrS + kF_t + kF_s(r-\delta)S + \frac{1}{2}kF_{ss}\sigma^2S^2\big)\,\mathrm{d}t.$$

   (c) Derive the pricing partial differential equation (PDE), with $F_t = \frac{\partial F}{\partial t}$, etc.,

$$F_t(t,s) + (r-\delta)sF_s(t,s) + \frac{1}{2}\sigma^2s^2F_{ss}(t,s) - rF(t,s) = 0,$$

   and give the terminal condition for this PDE, so what is $F(T, s)$?
   **Answer.** Use the previous answer, and combine with $\mathrm{d}V = rV\,\mathrm{d}t$ to get from the no arbitrage condition

$$-kF_srS + kF_t + kF_s(r-\delta)S + \frac{1}{2}kF_{ss}\sigma^2S^2 = rV.$$

   Use that $V = hS + kF = -kF_sS + kF$, insert this into the previous equation to get

$$-kF_srS + kF_t + kF_s(r-\delta)S + \frac{1}{2}kF_{ss}\sigma^2S^2 = r(-kF_sS + kF).$$

   Simplify to

$$kF_t + kF_s(r-\delta)S + \frac{1}{2}kF_{ss}\sigma^2S^2 = rkF.$$

   This has to hold for all values $s$ of $S$ and for all values of $k$. Divide then by $k \neq 0$ to get

$$F_t + F_s(r-\delta)s + \frac{1}{2}F_{ss}\sigma^2s^2 = rF.$$

   The terminal condition is of course $F(T, s) = \Phi(s)$.

2. Consider a market consisting of two assets with prices $S_t^1$ and $S_t^2$ and a bank account $B_t$ with constant short rate $r$. We assume the following model, under the risk neutral measure $\mathbb{Q}$, for $0 \le t \le T$.

$$\begin{aligned} \mathrm{d}S_t^1 &= rS_t^1\,\mathrm{d}t + \sigma_1 S_t^1\,\mathrm{d}W_t^1, \\ \mathrm{d}S_t^2 &= rS_t^2\,\mathrm{d}t + \sigma_2 S_t^2\,\mathrm{d}W_t^2, \\ \mathrm{d}B_t &= rB_t\,\mathrm{d}t. \end{aligned}$$

Here $W^1$ and $W^2$ are independent standard Brownian motions (under $\mathbb{Q}$). The relevant filtration is denoted $\mathcal{F}_t, t \in [0,T]$. We let $R$ be the ratio process of the two assets, $R_t = \frac{S_t^2}{S_t^1}$.

(a) Show that the dynamics of $R$ are given by

$$\mathrm{d}R_t = \sigma_1^2 R_t\,\mathrm{d}t + R_t(\sigma_2\,\mathrm{d}W_t^2 - \sigma_1\,\mathrm{d}W_t^1).$$

**Answer.** It is convenient to derive first the dynamics of $\frac{1}{S}$ for some $S$ satisfying $\mathrm{d}S = rS\,\mathrm{d}t + \sigma S\,\mathrm{d}W$. Use the Itô formula for $f(S)$ with $f(x) = \frac{1}{x}$. We need $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$. The Itô formula gives

$$\begin{aligned} \mathrm{d}\frac{1}{S} &= f'(S)\,\mathrm{d}S + \frac{1}{2}f''(S)(\mathrm{d}S)^2 \\ &= -\frac{1}{S^2}\,\mathrm{d}S + \frac{1}{S^3}(\mathrm{d}S)^2 \\ &= -\frac{1}{S^2}(rS\,\mathrm{d}t + \sigma S\,\mathrm{d}W) + \frac{1}{S^3}\sigma^2 S^2\,\mathrm{d}t \\ &= -\frac{1}{S}(r\,\mathrm{d}t + \sigma\,\mathrm{d}W) + \frac{1}{S}\sigma^2\,\mathrm{d}t \\ &= \frac{1}{S}\big((-r+\sigma^2)\,\mathrm{d}t - \sigma\,\mathrm{d}W\big). \end{aligned}$$

The next step is to substitute $S$ with $S^1$ and to compute $\mathrm{d}\frac{S^2}{S^1} = \mathrm{d}(S^2 \times \frac{1}{S^1})$. The product has the two factors with each of them having its "own" Wiener process. By independence of these, the cross product $(\mathrm{d}S^2)(\mathrm{d}\frac{1}{S^1})$ is zero, and we have the ordinary product rule

$$\begin{aligned} \mathrm{d}(S^2 \times \frac{1}{S^1}) &= \frac{\mathrm{d}S^2}{S^1} + S^2\,\mathrm{d}\frac{1}{S^1} \\ &= \frac{S^2 r\,\mathrm{d}t + \sigma_2 S^2\,\mathrm{d}W^2}{S^1} + S^2\frac{1}{S^1}\big((-r+\sigma_1^2)\,\mathrm{d}t - \sigma_1\,\mathrm{d}W^1\big) \\ &= R\big(r\,\mathrm{d}t + \sigma_2\,\mathrm{d}W^2\big) + R(-r+\sigma_1^2)\big(\mathrm{d}t - \sigma_1\,\mathrm{d}W^1\big) \\ &= R\big(\sigma_2\,\mathrm{d}W^2 + \sigma_1^2\,\mathrm{d}t - \sigma_1\,\mathrm{d}W^1\big). \end{aligned}$$

The next objective is to price the minimum option $M = \min\{S_T^1, S_T^2\}$. We will see that this can be accomplished using a Girsanov transformation and taking the process $S^1$ as a numéraire. Consider the likelihood ratio process $\tilde{L}$ with $\mathrm{d}\tilde{L}_t = \sigma_1 \tilde{L}_t\,\mathrm{d}W_t^1$, with $\tilde{L}_0 = 1$. Let $\tilde{\mathbb{Q}}$ be the probability measure on $\mathcal{F}_T$ with $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \tilde{L}_T$. It follows from general theory that under $\tilde{\mathbb{Q}}$ the processes $\tilde{W}^1$ satisfying $\mathrm{d}\tilde{W}_t^1 = \mathrm{d}W_t^1 - \sigma_1\,\mathrm{d}t$ and (the original) $W^2$ are independent Brownian motions.

(b) Show that $\tilde{L}_t = \frac{S_t^1}{S_0^1} e^{-rt}$. [You may want to use explicit expressions for $S_t^1$ and $\tilde{L}_t$.]
**Answer.** One can solve the SDEs for $S^1$ and $\tilde{L}$. The two solutions are (use that $\tilde{L}_0 = 1$)

$$S_t^1 = S_0^1 \exp(\sigma_1 W_t^1 + (r - \frac{1}{2}\sigma_1^2)t),$$
$$\tilde{L}_t = \exp(\sigma_1 W_t^1 - \frac{1}{2}\sigma_1^2 t).$$

We see that $e^{-rt}\frac{S_t^1}{S_0^1} = \exp(\sigma_1 W_t^1 - \frac{1}{2}\sigma_1^2 t)$, which is precisely $\tilde{L}_t$.

(c) Give the dynamics of $S^1$ and $R$ under the measure $\tilde{\mathbb{Q}}$, use a Brownian motion under $\tilde{\mathbb{Q}}$.
**Answer.** From $\mathrm{d}\tilde{W}^1 = \mathrm{d}W^1 - \sigma_1 \,\mathrm{d}t$ we get $\mathrm{d}S_t^1 = (r + \sigma_1^2)\,\mathrm{d}t + \sigma_1 S_t^1 \,\mathrm{d}\tilde{W}_t^1$ and $\mathrm{d}R_t = R_t\big(\sigma_2 \,\mathrm{d}W_t^2 - \sigma_1 \,\mathrm{d}\tilde{W}_t^1\big)$, a martingale under $\tilde{\mathbb{Q}}$!

(d) Show, using the abstract Bayes formula for conditional expectations, that the fair price $\Pi_t$ of $M$ at time $t$ can be computed as $\Pi_t = S_t^1\, \mathbb{E}^{\tilde{\mathbb{Q}}}[\frac{M}{S_T^1}|\mathcal{F}_t]$ (involving a conditional expectation under $\tilde{\mathbb{Q}}$ as an alternative to using a conditional expectation under $\mathbb{Q}$).
**Answer.** The usual expression is $\Pi_t = \frac{B_t}{B_T}\,\mathbb{E}^{\mathbb{Q}}[M|\mathcal{F}_t]$. With the measure change in the opposite order, i.e., from $\tilde{\mathbb{Q}}$ to $\mathbb{Q}$ and $\hat{L}_t$, $\hat{L}_T$ the R-N derivatives $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\tilde{\mathbb{Q}}}$ on $\mathcal{F}_t$ and $\mathcal{F}_T$, one gets

$$\Pi_t = \frac{B_t}{B_T}\,\frac{\mathbb{E}^{\tilde{\mathbb{Q}}}[M\hat{L}_T|\mathcal{F}_t]}{\hat{L}_t}.$$

But $\hat{L}_T = \frac{1}{\tilde{L}_T}$ and $\hat{L}_t = \frac{1}{\tilde{L}_t}$, so this becomes

$$\Pi_t = \frac{B_t}{B_T}\,\mathbb{E}^{\tilde{\mathbb{Q}}}[\frac{M}{\tilde{L}_T}|\mathcal{F}_t]\tilde{L}_t.$$

From (b) we know $\tilde{L}_t = \frac{S_t^1}{S_0^1 B_t}$, so we obtain

$$\begin{aligned}\Pi_t &= \frac{B_t}{B_T}\,\mathbb{E}^{\tilde{\mathbb{Q}}}[M\frac{S_0^1 B_T}{S_T^1}|\mathcal{F}_t]\frac{S_t^1}{S_0^1 B_t}\\ &= \mathbb{E}^{\tilde{\mathbb{Q}}}[\frac{M}{S_T^1}|\mathcal{F}_t]S_t^1.\end{aligned}$$

(e) Show that $\Pi_t$ can be written as $\Pi_t = S_t^1\,\mathbb{E}^{\tilde{\mathbb{Q}}}[R_T - \max\{R_T - 1, 0\}|\mathcal{F}_t]$.
**Answer.** Write $M = S_T^1 \min\{1, R_T\} = S_T^1(1 + \min\{0, R_T - 1\}) = S_T^1(R_T - \max\{0, R_T - 1\})$. This gives

$$\Pi_t = \mathbb{E}^{\tilde{\mathbb{Q}}}[R_T - \max\{0, R_T - 1\}|\mathcal{F}_t]S_t^1.$$

To further compute $\Pi_t$ one can use the price of a standard call option on a stock $S_T$, i.e. $\max\{S_T - K, 0\}$. In a standard Black-Scholes model with short rate $r$ and volatility $\sigma$, the time $t$ price of such a call option is denoted $c_t(S_t, \sigma, r, K)$.

(f) By selecting the right parameters, express $\Pi_t$ further in terms of the function $c_t(\cdot,\cdot,\cdot,\cdot)$.
**Answer.** Observe that $\max\{0, R_T - 1\}$ is the payoff of a call option on $R_T$ with strike $K = 1$. Under $\tilde{\mathbb{Q}}$, $R$ is a martingale (see above). We can also write $\mathrm{d}R = \sigma R\,\mathrm{d}\tilde{W}$, where $\sigma = \sqrt{\sigma_1^2 + \sigma_s^2}$ and $\tilde{W} = \frac{\sigma_2 W^2 - \sigma_1 W^1}{\sigma}$, a Wiener process under $\tilde{\mathbb{Q}}$. It follows from the martingale property of $R$ and the Black-Scholes pricing formula for a call that $\Pi_t = S_t^1\big(R_t - c_t(R_t, \sigma, 0, 1)\big)$.

3. Consider the usual Black-Scholes market (which is known to be complete), with an asset price $S$ evolving under the physical probability measure $\mathbb{P}$ as

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t,$$

where $W$ is a Wiener process under $\mathbb{P}$. The bank account is given by $B_t = e^{rt}$. We assume that the time horizon is $T$, and we will use the notation $X_t$ to denote the wealth at time $t \le T$, resulting from investing in a self-financing portfolio process and an initial capital $x > 0$.

The problem is to maximize the expected utility of terminal wealth $\mathbb{E}^{\mathbb{P}}\Phi(X_T)$, where we make the *special choice* $\Phi(z) = 2\sqrt{z}$, where $z \ge 0$. So the problem becomes the maximization of

$$\mathbb{E}^{\mathbb{P}}[2\sqrt{X_T}],$$

under the budget constraint implied by $x$. The terminal wealth $X_T$, as well as the intermediate wealths $X_t$ will be obtained by investing in self-financing portfolios, at time $t$ denoted $(h_t^S, h_t^B)$, so $X_t = h_t^S S_t + h_t^B B_t$. The relative investment in the stock is denoted $w_t$, so $w_t = h_t^S S_t / X_t$.

(a) Show that $\mathrm{d}X_t = X_t\big(w_t\mu + r(1-w_t)\big)\,\mathrm{d}t + \sigma X_t w_t\,\mathrm{d}W_t$.
**Answer.** First we use that the portfolio is self financing, $\mathrm{d}X = h^S\,\mathrm{d}S + h^B\,\mathrm{d}B = h^S(\mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}W) + h^B r B\,\mathrm{d}t$. We also know $h_t^S S_t = w_t X_t$ and $h_t^B B_t = (1-w_t)X_t$. This results in $\mathrm{d}X = \mu w X\,\mathrm{d}t + \sigma w X\,\mathrm{d}W + (1-w)X\,\mathrm{d}t$, which we easily rewrite as $\mathrm{d}X_t = X_t\big(w_t\mu + r(1-w_t)\big)\,\mathrm{d}t + \sigma X_t w_t\,\mathrm{d}W_t$.

(b) Argue that $X_t$ is positive.
**Answer.** The equation for $X$ is like the one that is known from the Black-Scholes theory, with the only difference that there are time varying coefficients, due to the $w_t$. But, still, the solution is of the type $X_t = X_0\exp(\ldots)$ and exponentials are positive.

(c) Derive the HJB equation for the value function $V(t,x)$ for this problem from the general theory of HJB equations. I.e., show that (suppressing dependence on $t$ and $x$)

$$V_t + \sup_w\{x(w\mu + r(1-w))V_x + \frac{1}{2}\sigma^2 x^2 w^2 V_{xx}\} = 0.$$

**Answer.** The general form of the HJB equation is, in abbreviated form, $V_t + \sup_w\{bV_x + \frac{1}{2}\sigma^2 V_{xx}\} = 0$, when $\mathrm{d}X = b\,\mathrm{d}t + \sigma\,\mathrm{d}W$. Here we identify $b = x(w\mu + r(1-w))$ and $\sigma$ as $\sigma x w$. The result follows

(d) Show that the optimal weight $w_t$ is constant in time. [Use the educated guess $V(t,x) = a_t\sqrt{x} + b_t$.]
**Answer.** With the educated guess one finds $V_t(t,x) = \dot{a}_t\sqrt{x} + \dot{b}_t$, $V_x(t,x) = \frac{1}{2}a_t x^{-1/2}$, $V_{xx}(t,x) = -\frac{1}{4}a_t x^{-3/2}$. It follows that $\frac{V_x}{V_{xx}} = -2x$. The maximization over $w$ gives the first order derivative $x(\mu - r)V_x + \sigma^2 x^2 w V_{xx} = 0$, so the optimal $w$, which indeed yields a maximum since $V_{xx} < 0$, is given by $\hat{w} = -\frac{x(\mu-r)V_x}{\sigma^2 x^2 V_{xx}}$, which becomes $\hat{w} = \frac{2(\mu-r)}{\sigma^2}$, independent of $t$.

(e) Find the optimal value function and compute from this the final result, the maximal $2\mathbb{E}^{\mathbb{P}}[\sqrt{X_T}]$.
**Answer.** Here comes the tedious part, substituting the optimal $w$ into the HJB equation. This gives $V_t + x(\hat{w}(\mu - r) + r))V_x + \frac{1}{2}\sigma^2 x^2 \hat{w}^2 V_{xx} = 0$, and substitute then also the $V_t$ etc. This results in $\dot{a}_t\sqrt{x} + \dot{b}_t + x(\hat{w}(\mu - r) + r))\frac{1}{2}a_t x^{-1/2} + \frac{1}{2}\sigma^2 x^2 \hat{w}^2 \times -\frac{1}{4}a_t x^{-3/2} = 0$. Use the expression for $\hat{w}$ to rewrite this as

$$\dot{a}_t\sqrt{x} + \dot{b}_t + x(\frac{2(\mu-r)}{\sigma^2}(\mu - r) + r))\frac{1}{2}a_t x^{-1/2} + \frac{1}{2}\sigma^2 x^2(\frac{2(\mu-r)}{\sigma^2})^2 \times -\frac{1}{4}a_t x^{-3/2} = 0,$$

which becomes after separating the terms with $\sqrt{x}$ and the terms without $x$

$$\sqrt{x}\big(\dot{a}_t + (\frac{2(\mu-r)}{\sigma^2}(\mu-r)+r))\frac{1}{2}a_t + \frac{2(\mu-r)^2}{\sigma^2}\times -\frac{1}{4}a_t\big) + \dot{b}_t = 0,$$

further simplified to

$$\sqrt{x}\big(\dot{a}_t + (\frac{(\mu-r)^2}{2\sigma^2}+\frac{1}{2}r)\frac{1}{2}a_t\big) + \dot{b}_t = 0.$$

This equation has to hold for all $t$ and $x$. This results in two ODEs,

$$\dot{a}_t + (\frac{(\mu-r)^2}{2\sigma^2}+\frac{1}{2}r)a_t = 0,$$

$$\dot{b}_t = 0.$$

The latter one implies that $b_t$ is a constant, $b_t = b_T = 0$ because of the boundary condition $V(T,x) = 2\sqrt{x}$. The equation for $a$ is a linear ODE of the type $\dot{a}_t = -ca_t$, with $c = \frac{(\mu-r)^2}{2\sigma^2}+\frac{1}{2}r$. The solution is known to be $a_t = c_0\exp(-ct)$ for some $c_0$, which can be computed from the boundary condition $a(T) = 2$, so $c_0 = 2\exp(cT)$. This all results in $a_t = 2\exp(c(T-t))$ and $V(t,x) = 2\exp(c(T-t))\sqrt{x}$.
The maximal $2\mathbb{E}^{\mathbb{P}}[\sqrt{X_T}]$ is nothing else but $V(0,x) = 2e^{cT}\sqrt{x}$.

4. Consider the process $Y$ given by $\mathrm{d}Y_t = \frac{1}{2}(w_t(\mu-r)+r-\tau^2w_t^2)Y_t\,\mathrm{d}t + \tau w_t Y_t\,\mathrm{d}W_t$ and $Y_0 = y(0) > 0$. The aim is to maximize $\mathbb{E}Y_T$ by selecting the $w_t$ optimally. The function $w$ is assumed to be non-random. Let $y(t) = \mathbb{E}Y_t$.

   (a) Show that $\mathrm{d}y(t) = \frac{1}{2}(w_t(\mu-r)+r-\tau^2w_t^2)y(t)\,\mathrm{d}t$, and that $y(t) = y(0)\exp(\frac{1}{2}\int_0^t w_s(\mu-r)+r-\tau^2w_s^2\,\mathrm{d}s)$.
   **Answer.** As the function $w$ is deterministic we can simply take expectation in the equation for $Y_t$, knowing that the Itô integral vanishes, and thus replacing $Y_t$ there with $y(t)$ to arrive at the ODE. It is a linear ODE of the type $\dot{y}(t) = a(t)y(t)$, whose solution is known to be $y(t) = y(0)\exp(\int_0^t a(s)\,\mathrm{d}s)$. Fill in the expression for $a(s)$,

   (b) Cleary, $y(t)$ depends on the function $w$, write $y^w(t)$ instead. Explain that the maximization of $y^w(T)$ is equivalent to maximizing $w_s(\mu-r)-\tau^2w_s^2$ for every $s$.
   **Answer.** The maximization of the exponential expression is equivalent to the maximization of its argument, because exp is an increasing function. The integral in the argument can simply be maximized by maximizing the integrand (for every $s$).

   (c) Compute the optimal $w_s$, call them $\hat{w}_s$. Show that $y^{\hat{w}}(T) = y(0)\exp\big(\frac{1}{2}(\frac{(\mu-r)^2}{4\tau^2}+r)T\big)$.
   **Answer.** Differentiate $w_s(\mu-r)-\tau^2w_s^2$ w.r.t. $w_s$ and set the derivative equal to zero. It results $(\mu-r)-2\tau^2w_s = 0$ and hence $\hat{w}_s = \frac{\mu-r}{2\tau^2}$, independent of $t$! This gives by integration the desired expression for $y^{\hat{w}}(T)$.

   (d) Exercise in Itô calculus. Let $X_t = \frac{1}{4}Y_t^2$. Show that $\mathrm{d}X_t = X_t\big(w_t\mu+r(1-w_t)\big)\,\mathrm{d}t + 2\tau X_t w_t\,\mathrm{d}W_t$. Compare your answers to those of Exercise 3.
   **Answer.** Left to you. Observe the similarities between the answers.