Proof of the verification theorem

Consider arbitrary control law $u = u$ and

$$dX^u = \mu^u \, dt + \sigma^u \, dW \quad \text{(abbreviated notation)}$$

Apply Itô to $H(t, X^u_t)$ for $s \in [t,T]$

$$H(T, X^u_T) = H(t,x) + \int_t^T \left(\frac{\partial H}{\partial t} + A^u H \right) ds + \int_t^T H_x \sigma^u \, dW$$

H solves HJB:

$$H_t + \sup_u \{ F + A^u H \} = 0 \Rightarrow H_t + F + A^u H \leq 0$$

$$\Rightarrow \Phi(X^u_T) \overset{!}{=} H(T, X^u_T) \leq H(t,x) + \int_t^T -F \, ds + \int_t^T H_x \sigma^u \, dW_u$$

Take Expectation: Itô integral should vanish

$$\mathbb{E}\,\Phi(X^u_T) \le H(t,x) - \mathbb{E}\int_t^T F\,ds, \text{ so}$$

$$H(t,x) \geq \mathbb{E}\,\Phi(X^u_T) + \mathbb{E}\int_t^T F\,ds \overset{!}{=} J(t,x,u)$$

Now take supremum over u: $\sup_u J(t,x,u) = V(t,x)$

$$\Rightarrow H(t,x) \geq V(t,x)$$

Now for the other inequality, $H(t,x) \leq V(t,x)$

choose $u(t,x) = g(t,x)$ from the assertion

As g is the maximizer in HJB:

$$H_t(t,x) + F(t,x,g) + A^g H = 0,$$ which (analogously)

gives

$$H(t,x) \le E \int_t^T F^g \, ds + \Phi(X_T^g) \stackrel{!}{=} J(t,x,g)$$

optimality of V gives $J(t,x,g) \le V(t,x)$

and so $\quad H(t,x) \le V(t,x)$. $\quad$ QED

NB: In fact $\quad V(t,x) = J(t,x,g)$!