Continuous Time Finance

Black-Scholes

(Ch 6-7)

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Start of lecture 1

slides 42-97.

Contents

1.

Introduction

European Call Option

The holder of this paper has the right

to buy

1 ACME INC

on the date

2025

June 30, ~~2017~~

(a future date)

at the price

$100

Financial Derivative

- A financial asset which is defined **in terms of** some **underlying** asset.
- Future stochastic claim.

Examples

- European calls and puts
- American options
- Forward rate agreements
- Convertibles
- Futures
- Bond options
- Caps & Floors
- Interest rate swaps
- CDO:s
- CDS:s

Main problems

- What is a "reasonable" price for a derivative?
- How do you hedge yourself against a derivative.

Natural Answers

Consider a random cash payment $\mathcal{Z}$ at time T.

What is a reasonable price $\Pi_0 [\mathcal{Z}]$ at time 0?

Natural answers:

1. Price = Discounted present value of future payouts.

$$\Pi_0 [\mathcal{Z}] = e^{-rT} E [\mathcal{Z}]$$

interest rate is r !

2. The question is meaningless.

Both answers are incorrect!

- Given some assumptions we **can** really talk about "the correct price" of an option.

- The correct pricing formula is **not** the one on the previous slide.

Philosophy

- The derivative is **defined in terms of** underlying.
- The derivative can be **priced in terms of** underlying price.
- **Consistent** pricing.
- **Relative** pricing.

Before we can go on further we need some simple portfolio theory

2.

Portfolio Theory

Portfolios

We consider a market with N assets.

$$S_t^i = \text{price at } t \text{, of asset No } i. \qquad i = 1, \ldots, N$$

A **portfolio** strategy is an adapted vector process

$$h_t = (h_t^1, \cdots, h_t^N)$$

where

$$\begin{aligned} h_t^i &= \text{number of units of asset } i, \\ V_t &= \text{market value of the portfolio} \end{aligned}$$

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

The portfolio is typically of the form

$$h_t = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

(sometimes also on prices from the past)

Self financing portfolios

We want to study self financing portfolio strategies, i.e. portfolios where purchase of a "new" asset must be financed through sale of an "old" asset.

How is this formalized?

Definition:
The strategy h is **self financing** if

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

Interpret!

See Appendix B for details. (p. 95)
and motivation from discrete time

Relative weights

Definition:

ω_t^i = relative portfolio weight on asset No i.

We have

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t}$$

Insert this into the self financing condition

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

We obtain

Portfolio dynamics:

$$dV_t = V_t \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$$

$$\frac{dV_t}{V_t} = \sum \omega_t^i \frac{dS_t^i}{S_t}$$

Interpret!

(also p. 94)

3.

Deriving the Black-Scholes PDE

Back to Financial Derivatives

Consider the Black-Scholes model

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

bank account
r= interest rate

We want to price a European call with strike price K and exercise time T. This is a stochastic claim on the future. The future pay-out (at T) is a stochastic variable, $\mathcal{Z}$, given by

$$\mathcal{Z} = \max[S_T - K, 0]$$

More general:

$$\mathcal{Z} = \Phi(S_T)$$

for some contract function Φ.

Main problem: What is a “reasonable” price, $\Pi_t[\mathcal{Z}]$, for $\mathcal{Z}$ at t?

Main Idea

- We demand **consistent** pricing between derivative and underlying.
- No **mispricing** between derivative and underlying.
- No $\underbrace{\textbf{arbitrage possibilities}}_{\text{viable market}}$ on the market (B, S, Π)

Arbitrage

The portfolio ω is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T > 0$ with probability one.

(or, weaker, $V_T \geq 0$ w.p. 1, and $P(V_T > 0) > 0$)

See later

Moral:

- **Arbitrage = Free Lunch**
- **No arbitrage possibilities in an efficient market.**

arbitrage possibility only in a
market with "wrong" prices

Arbitrage test
(fundamental idea)

Suppose that a portfolio ω is self financing whith dynamics

$$dV_t = kV_t dt$$

- No driving Wiener process
- Risk free rate of return.
- "Synthetic bank" with rate of return k.

If the market is free of arbitrage we must have:

$$k = r,$$

because, otherwise, ...

Main Idea of Black-Scholes

- Since the derivative is defined in terms of the underlying, the derivative price should be highly correlated with the underlying price.

- We should be able to balance dervative against underlying in our portfolio, so as to cancel the randomness.

- Thus we will obtain a riskless rate of return k on our portfolio.

- Absence of arbitrage must imply

$$k = r$$

[End of lecture 1a]

Two Approaches

The program above can be formally carried out in two slightly different ways:

- The way Black-Scholes did it in the original paper. This leads to some logical problems. (?)
- A more conceptually satisfying way, first presented by Merton.

Here we use the Merton method. You will find the original BS method in Appendix C at the end of this lecture. [p. 95]

Formalized program a la Merton (outline)

- Assume that the derivative price is of the form

$$\Pi_t[\mathcal{Z}] = f(t, S_t).$$

- Form a portfolio based on the underlying S and the derivative f, with portfolio dynamics

$$dV_t = V_t \left\{ \omega_t^S \cdot \frac{dS_t}{S_t} + \omega_t^f \cdot \frac{df}{f} \right\}$$

relative weights

- Choose ω^S and ω^f such that the dW-term is wiped out. This gives us

$$dV_t = V_t \cdot k dt$$

- Absence of arbitrage implies

$$k = r$$

- This relation will say something about f.

Back to Black-Scholes

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ \Pi_t[\mathcal{Z}] &= f(t, S_t) \end{aligned}$$

Itô's formula gives us the f dynamics as

$$\begin{aligned} df &= \left\{ \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\ &+ \sigma S \frac{\partial f}{\partial s} dW \end{aligned}$$

Write this as

$$df = \mu_f \cdot f dt + \sigma_f \cdot f dW$$

where

$$\begin{aligned} \mu_f &= \frac{\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}}{f} \\ \sigma_f &= \frac{\sigma S \frac{\partial f}{\partial s}}{f} \end{aligned}$$

both in terms of partial derivatives of f.

Recall:

$$df = \mu_f \cdot f dt + \sigma_f \cdot f dW$$

$$\begin{aligned} dV &= V\left\{\omega^S \cdot \frac{dS}{S} + \omega^f \cdot \frac{df}{f}\right\} \\ &= V\left\{\omega^S(\mu dt + \sigma dW) + \omega^f(\mu_f dt + \sigma_f dW)\right\} \end{aligned}$$

$$dV = V\left\{\omega^S \mu + \omega^f \mu_f\right\} dt + V\left\{\omega^S \sigma + \omega^f \sigma_f\right\} dW$$

Now we kill the dW-term!

Choose (ω^S, ω^f) such that

$$\begin{aligned} \omega^S \sigma + \omega^f \sigma_f &= 0 \\ \omega^S + \omega^f &= 1 \end{aligned}$$

Linear system with solution (if you don't divide by zero!)

$$\omega^S = \frac{\sigma_f}{\sigma_f - \sigma}, \quad \omega^f = \frac{-\sigma}{\sigma_f - \sigma}$$

Plug into dV!

We obtain

$$dV = V\left\{\omega^S \mu + \omega^f \mu_f\right\} dt$$

This is a risk free “synthetic bank” with short rate

$$\left\{\omega^S \mu + \omega^f \mu_F\right\}$$

.

Absence of arbitrage implies

$$\left\{\omega^S \mu + \omega^f \mu_f\right\} = r$$

Plug in the expressions for ω^S, ω^f, μ_f and simplify. This will give us the following result.

that involve partial derivatives !

Black-Schole's PDE

The price is given by

$$\Pi_t\left[\mathcal{Z}\right] = f\left(t, S_t\right)$$

where the pricing function f satisfies the PDE (partial differential equation)

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,s) + rs\dfrac{\partial f}{\partial s}(t,s) + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 f}{\partial s^2}(t,s) - rf(t,s) &= 0 \\ f(T,s) &= \Phi(s) \end{cases}$$

There is a unique solution to the PDE so there is a unique arbitrage free price process for the contract.

Black-Scholes' PDE ct'd

$$\begin{cases} \dfrac{\partial f}{\partial t} + rs\dfrac{\partial f}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 f}{\partial s^2} - rf & = & 0 \\ f(T,s) & = & \Phi(s) \end{cases}$$

- The price of **all** derivative contracts have to satisfy the **same** PDE

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} - rf = 0$$

otherwise there will be an arbitrage opportunity.

- The only difference between different contracts is in the boundary value condition

$$f(T,s) = \Phi(s)$$

Data needed

- The contract function Φ.
- Today's date t.
- Today's stock price S.
- Short rate r.
- Volatility σ.

Note: The pricing formula does **not** involve the mean rate of return μ!

miracle **??**

Black-Scholes Basic Assumptions

Assumptions:

- The stock price is Geometric Brownian Motion
- Continuous trading.
- Frictionless efficient market.
- Short positions are allowed.
- Constant volatility σ.
- Constant short rate r.
- Flat yield curve.

Black-Scholes' Formula
European Call

T=date of expiration,
t=today's date,
K=strike price,
r=short rate,
s=today's stock price,
σ=volatility.

$$f(t,s) = sN\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right].$$

$N[\cdot]$=cdf for $N(0,1)$-distribution.

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\},$$

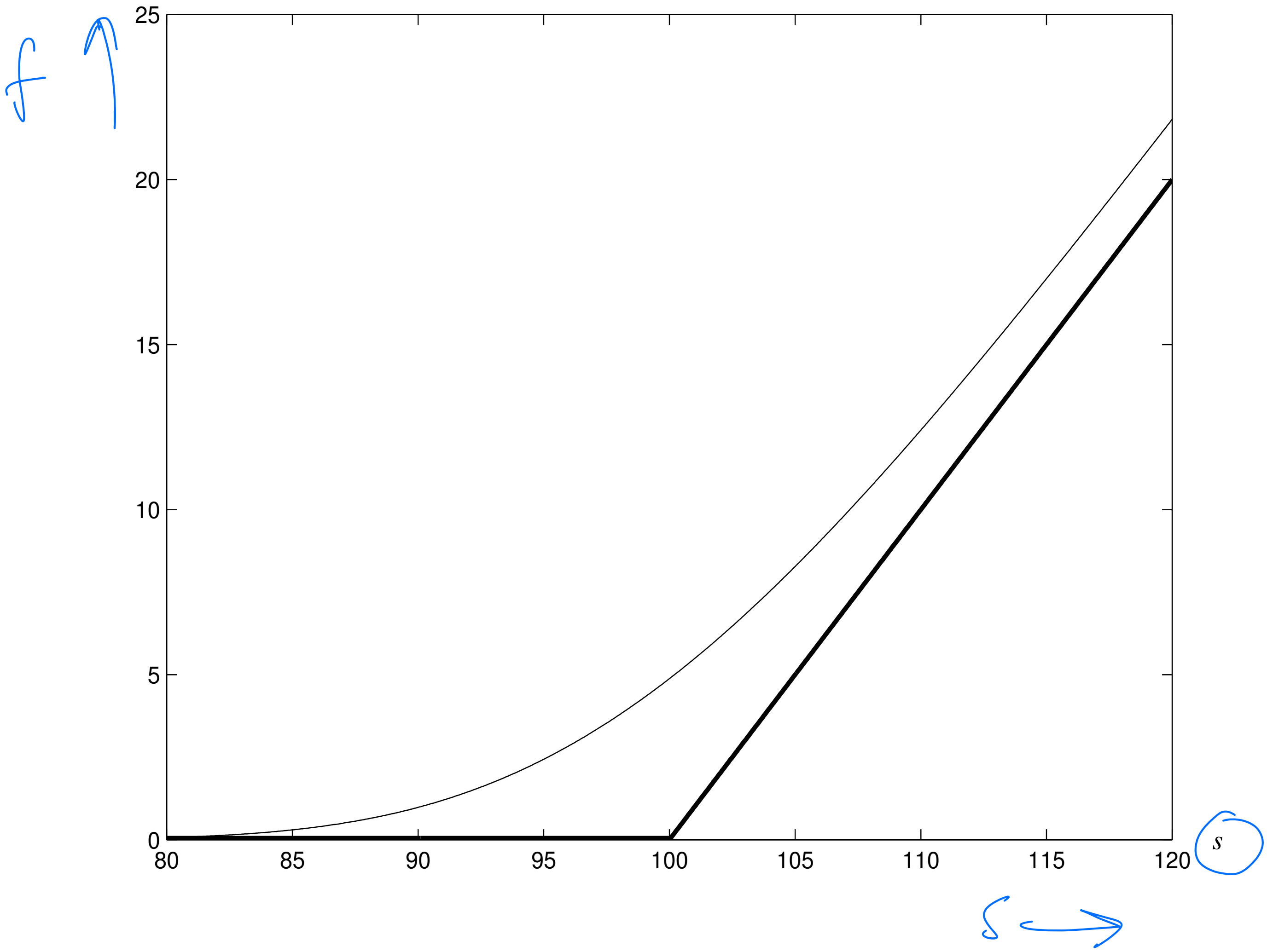
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Comes out of the blue for the time being; but this f solves the Black-Scholes PDE (check!)

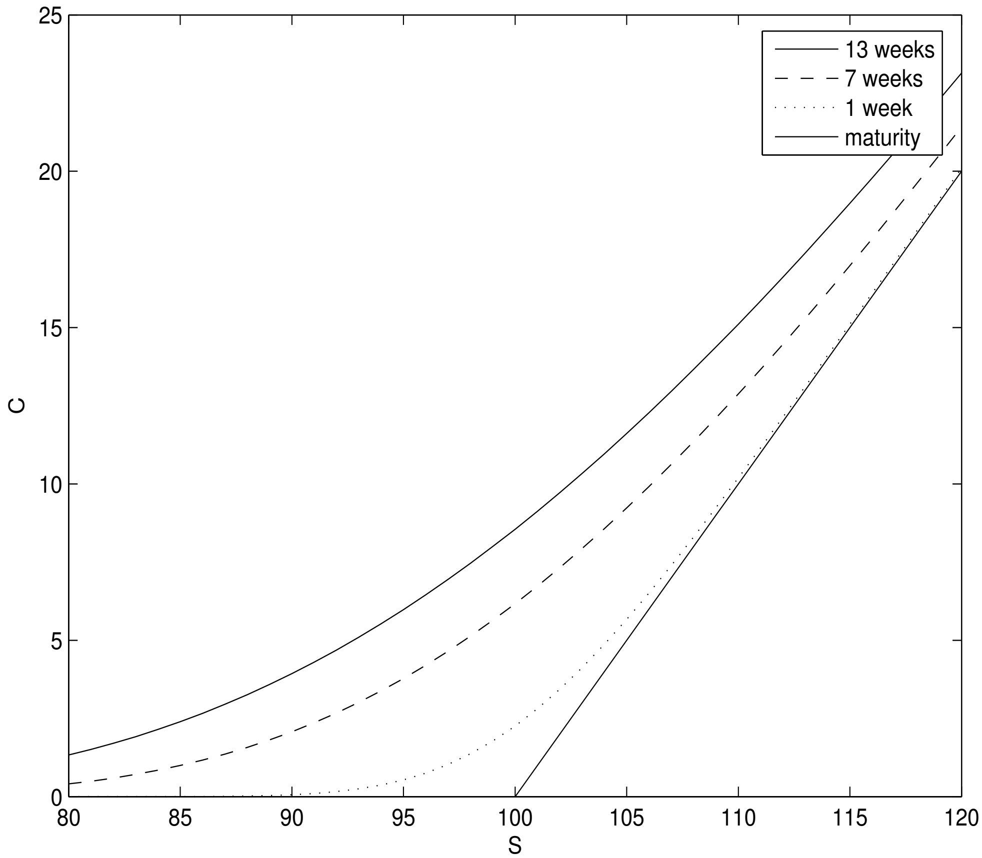
Black-Scholes

European Call,

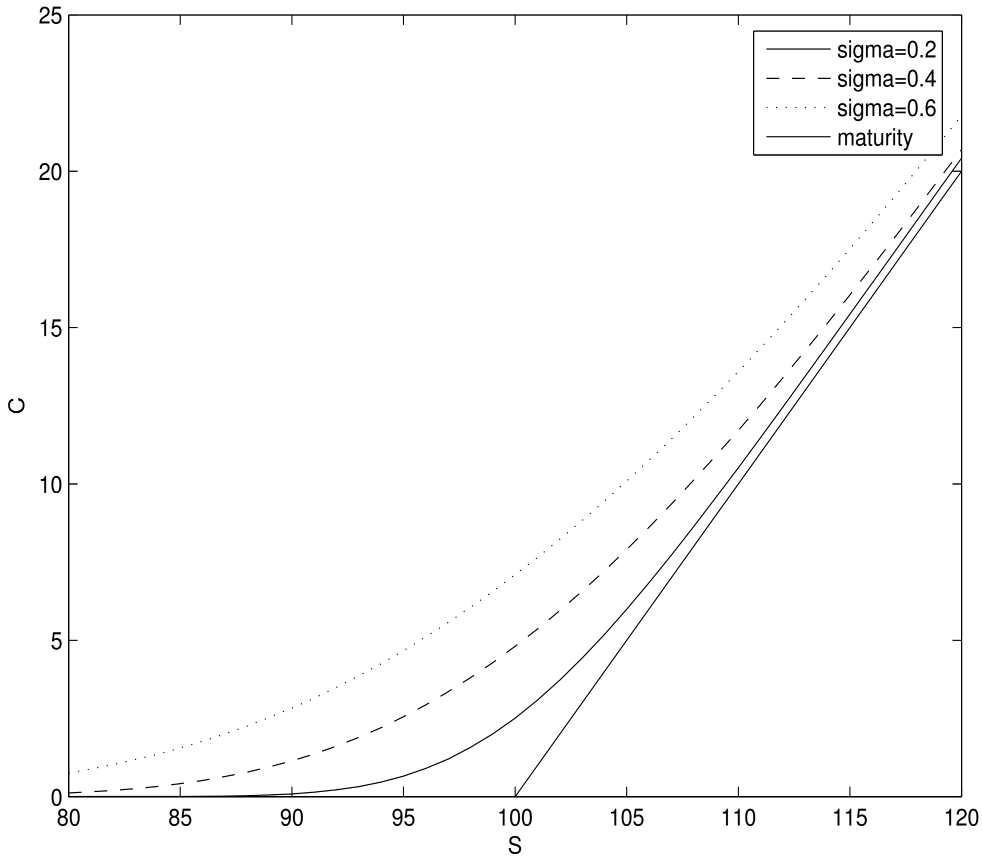
$$K = 100, \quad \sigma = 20\%, \quad r = 7\%, \quad T - t = 1/4$$



Dependence on Time to Maturity



Dependence on Volatility



4.

Risk Neutral Valuation

Risk neutral valuation

Appplying $\boxed{\text{Feynman-Kac}}$ to the Black-Scholes PDE we obtain

$$\Pi[t; X] = e^{-r(T-t)} \underline{E^Q_{t,s}} [X],$$

notation:
conditional expectation at time t, with $S_t = s$

Q**-dynamics:**

$$\begin{cases} dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\ dB_t &= rB_t dt. \end{cases}$$

- Price = Expected discounted value of future payments.
- The expectation shall **not** be taken under the "objective" probability measure P, but under the "risk adjusted" measure ("martingale measure") Q.

Note: $P \sim Q$ (Girsanov)

Concrete formulas

$t=0$:
$$\Pi[0;\Phi] = e^{-rT}\int_{-\infty}^{\infty}\Phi(se^{z})f(z)dz$$

$$f(z) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left\{-\frac{\left[z-(r-\frac{1}{2}\sigma^2)T\right]^2}{2\sigma^2 T}\right\}$$

density of $N\left((r-\frac{1}{2}\sigma^2)T,\ \sigma^2 T\right)$

variance

Note: $S_t = S_0 \exp\left((r-\frac{1}{2}\sigma^2)t + \sigma W_t^Q\right)$

Interpretation of the risk adjusted measure

- **Assume** a risk neutral world.
- Then the following must hold

$$s = S_0 = e^{-rt} E^{Q}[S_t]$$

- In our model this means that

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

- The risk adjusted probabilities can be intrepreted as probabilities in a fictuous risk neutral economy.

Moral

- When we compute prices, we can compute **as if** we live in a risk neutral world.
- This does **not** mean that we live (or think that we live) in a risk neutral world.
- The formulas above hold regardless of the investor's attitude to risk, as long as he/she prefers more to less.
- The valuation formulas are therefore called "preference free valuation formulas".

Properties of Q

- $P \sim Q$ (Girsanov)

- For the price pricess π of any traded asset, derivative or underlying, the process

$$Z_t = \frac{\pi_t}{B_t}$$

  is a Q-martingale.

- Under Q, the price pricess π of any traded asset, derivative or underlying, has r as its local rate of return:

$$d\pi_t = r\pi_t dt + \sigma_\pi \pi_t dW_t^Q$$

- The volatility of π is the same under Q as under P.

end of lecture 16 (or after next slide)

A Preview of Martingale Measures

Consider a market, under an objective probability measure P, with underlying assets

$$B, S^1, \ldots, S^N$$

Definition: A probability measure Q is called a **martingale measure** if

- $P \sim Q$
- For every i, the process

$$Z_t^i = \frac{S_t^i}{B_t}$$

is a Q-martingale.

Theorem: The market is arbitrage free **iff** there exists a martingale measure. FTAP 1:

1st fundamental theorem of asset pricing

5.

Appendices

Appendix A: Black-Scholes vs Binomial

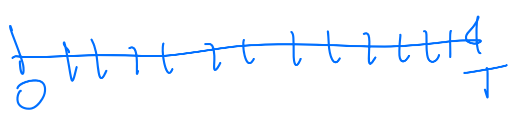
if you know this

Consider a binomial model for an option with a fixed time to maturity T and a fixed strike price K.

- Build a binomial model with n periods for each $n = 1, 2, ...$
- Use the standard formulas for scaling the jumps:

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}} \quad \Delta t = T/n$$

- For a large n, the stock **price** at time T will then be a **product** of a large number of i.i.d. random variables.
- More precisely



n divisions of $[0,T]$

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

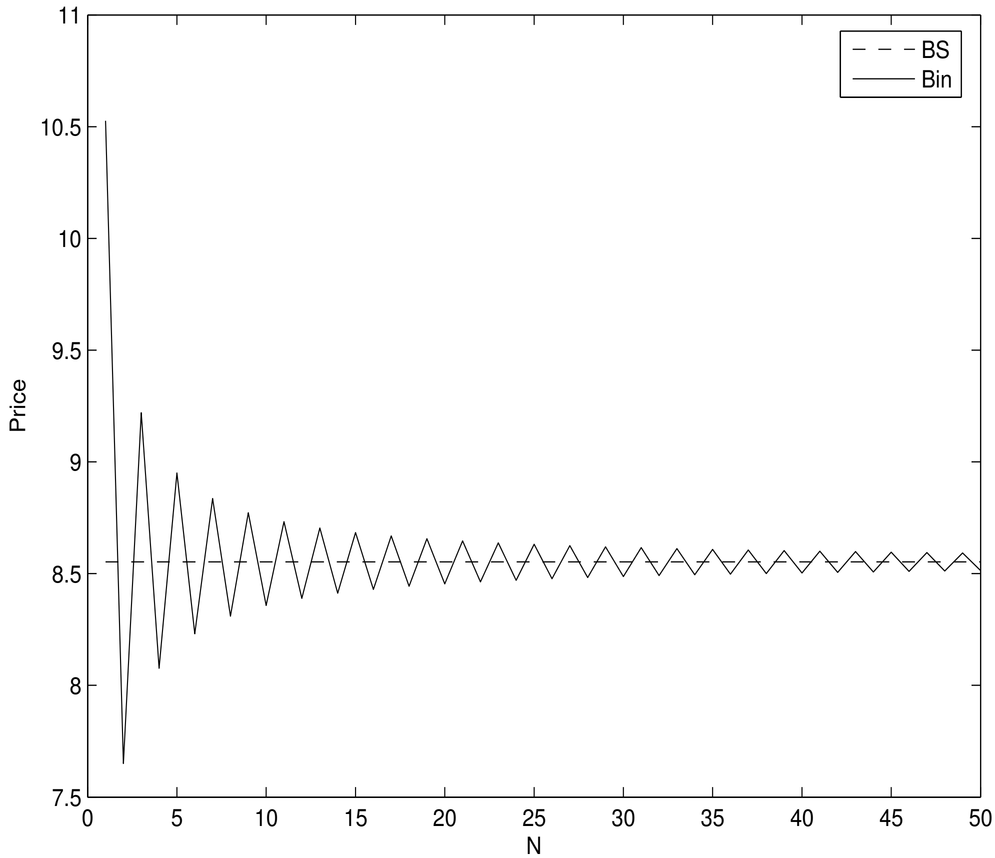
where n is the number of periods in the binomial model and $Z_i = u, d$; In S_T number of u's and d's matters only, not the order ⇒ looks like Binomials

Recall (this is the Cox-Ross-Rubinstein model)

$$S_T = S_0 Z_1 Z_2 \cdots Z_n,$$

- The stock **price** at time T will be a **product** of a large number of i.i.d. random variables.
- The log- **return** will be a large **sum** of i.i.d. variables.
  $\log S_T = \log S_0 + \sum_{i=1}^{n} \log Z_i$
- The Central Limit Theorem will kick in.
- In the limit, **returns** will be **normally** distributed.
- Stock **prices** will be **lognormally** distributed.
- We are in the Black-Scholes model.
- The binomial price will converge to the Black-Scholes price.

Binomial convergence to Black-Scholes



Binomial $\sim$ Black-Scholes

The intuition from the Binomial model carries over to Black-Scholes.

- The B-S model is "just" a binomial model where we rebalance the portfolio infinitely often.
- The B-S model is "thus" complete. (notion comes later)
- Completeness explains the unique prices for options in the B-S model.
- The B-S price for a derivative is the limit of the binomial price when the number of periods is very large.

Appendix B: Portfolio theory

We consider a market with N assets.

$$S_t^i = \text{price at } t\text{, of asset No } i.$$

A **portfolio** strategy is an adapted vector process

$$h_t = (h_t^1, \cdots, h_t^N)$$

where

$$\begin{aligned} h_t^i &= \text{number of units of asset } i, \\ V_t &= \text{market value of the portfolio} \end{aligned}$$

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

The portfolio is typically of the form

$$h_t = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

Self financing portfolios

We want to study **self financing** portfolio strategies, i.e. portfolios where

- There is now external infusion and/or withdrawal of money to/from the portfolio.
- Purchase of a “new” asset must be financed through sale of an “old” asset.

How is this formalized?

Problem: Derive an expression for dV_t for a self financing portfolio.

We analyze in discrete time, and then go to the continuous time limit.

Discrete time portfolios

We trade at discrete points in time $t = 0, 1, 2, \ldots$.

Price vector process:

$$S_n = (S_n^1, \cdots, S_n^N), \quad n = 0, 1, 2, \ldots$$

Portfolio process:

$$h_n = (h_n^1, \cdots, h_n^N), \quad n = 0, 1, 2, \ldots$$

Interpretation: At time n we buy the portfolio h_n at the price S_n, and keep it until time $n + 1$.

Value process:

$$V_n = \sum_{i=1}^{N} h_n^i S_n^i = h_n S_n$$

The self financing condition

- At time $n-1$ we buy the portfolio h_{n-1} at the price S_{n-1}.
- At time n this portfolio is worth $h_{n-1}S_n$.
- At time n we buy the new portfolio h_n at the price S_n.
- The cost of this new portfolio is h_nS_n.
- The self financing condition is the **budget constraint**

$$h_{n-1}S_n = h_nS_n$$

The self financing condition

Recall:

$$V_n = h_n S_n$$

Definition: For any sequence $x_1, x_2, \ldots$ we define the sequence Δx_n by

$$\Delta x_n = x_n - x_{n-1}$$

Problem: Derive an expression for ΔV_n for a self financing portfolio.

Lemma: For any pair of sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ we have the relation

$$\Delta(xy)_n = x_{n-1}\Delta y_n + y_n \Delta x_n$$

(Abel's summation formula:

$x_n y_n - x_0 y_0 = \sum x_{i-1} \Delta y_i + \sum y_i \Delta x_i$)

Proof: Do it yourself.

like integration by parts !

Recall

$$V_n = h_n S_n$$

From the Lemma we have

$$\Delta V_n = \Delta(hS)_n = h_{n-1}\Delta S_n + S_n \Delta h_n$$

Recall the self financing condition

$$h_{n-1} S_n = h_n S_n$$

which we can write as

$$S_n \Delta h_n = 0$$

Inserting this into the expression for ΔV_n gives us.

Proposition: The dynamics of a self financing portfolio are given by

$$\Delta V_n = h_{n-1} \Delta S_n$$

Note the forward increments!

Portfolios in continuous time

Price process:

$$S_t^i = \text{price at } t, \text{ of asset No } i.$$

Portfolio:

$$h_t = (h_t^1, \cdots, h_t^N)$$

Value process

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

From the self financing condition in discrete time

$$\Delta V_n = h_{n-1} \Delta S_n$$

we are led to the following definition. (by analogy!)

Definition: The portfolio h is self financing if and only if

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

need that the S^i are "semimartingales" (or diffusions)

Relative weights

Definition:

$\omega_t^i =$ relative portfolio weight on asset No i.

We have

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t}$$

Insert this into the self financing condition

$$dV_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

We obtain

Portfolio dynamics:

$$dV_t = V_t \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$$

Interpret!

Appendix C: The original Black-Scholes PDE argument

Consider the following portfolio.

- Short one unit of the derivative, with pricing function $f(t, s)$.
- Hold x units of the underlying S. (or x_t at time t)

The portfolio value is given by

$$V = -f(t, S_T) + \underset{t}{x} S_t$$

The object is to choose x such that the portfolio is risk free for an infinitesimal interval of length dt.

self financing

! We have $\underline{dV = -df + xdS}$ and from Itô we obtain

$$\begin{aligned} dV &= -\left\{ \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial s} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\ &- \sigma S \frac{\partial f}{\partial s} dW + x\mu S dt + x\sigma S dW \end{aligned}$$

$$
\begin{aligned}
dV &= \left\{ x\mu S - \frac{\partial f}{\partial t} - \mu S \frac{\partial f}{\partial s} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt \\
&+ \sigma S \left\{ x - \frac{\partial f}{\partial s} \right\} dW
\end{aligned}
$$

We obtain a risk free portfolio if we choose x as

$$
x = \frac{\partial f}{\partial s}
$$

and then we have, after simplification,

$$
dV = \left\{ -\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2} \right\} dt
$$

Using $V = -f + xS$ and x as above, the return dV/V is thus given by

$$
\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}}{-f + S \frac{\partial f}{\partial s}} dt
$$

We had (previous page)

$$\frac{dV}{V} = \frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}}dt$$

This portfolio is risk free, so absence of arbitrage implies that

$$\frac{-\frac{\partial f}{\partial t} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial s^2}}{-f + S\frac{\partial f}{\partial s}} \overset{!}{=} r$$

Simplifying this expression gives us the Black-Scholes PDE.

$$\begin{aligned} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2} - rf &= 0, \\ f(T,s) &= \Phi(s). \end{aligned}$$

end of lecture 1C