Continuous Time Finance

Completeness and Hedging

(Ch 8-9)

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Problems around Standard Black-Scholes

"over the counter"

- We **assumed** that the derivative was traded. How do we price OTC products?

- Why is the option price independent of the expected rate of return α of the underlying stock?

  previous week: μ

- Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with **completeness**.

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

Definition:
We say that a T-claim X can be **replicated**, alternatively that it is **reachable** or **hedgeable**, if there exists a self financing portfolio h such that

$$V_T^h = X, \quad P - a.s.$$

$V_t^h = V_t =$
$= h_t S_t$

In this case we say that h is a **hedge** against X. Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**

Basic Idea: If X can be replicated by a portfolio h then the arbitrage free price for X is given by

$$\Pi_t[X] = V_t^h.$$

(law of one price for reachable claim)

Trading Strategy

Consider a replicable claim X which we want to sell at $t = 0$..

- Compute the price $\Pi_0[X]$ and sell X at a slightly (well) higher price.
- Buy the hedging portfolio and invest the surplus in the bank.
- Wait until expiration date T.
- The liabilities stemming from X is exactly matched by V_T^h, and we have our surplus in the bank.

Completeness of Black-Scholes

Theorem: The Black-Scholes model is complete.

Proof. Fix a claim $X = \Phi(S_T)$. We want to find processes V, u^B and u^S such that

relative portfolios!

self financing condition:

$$dV_t = V_t \left\{ u_t^B \frac{dB_t}{B_t} + u_t^S \frac{dS_t}{S_t} \right\}$$

$$V_T = \Phi(S_T).$$

i.e. (recall $dB_t = rB_t dt$, $dS_t = \alpha S_t dt + \sigma S_t dW_t$)

$$dV_t = V_t \left\{ u_t^B r + u_t^S \alpha \right\} dt + V_t u_t^S \sigma dW_t,$$

$$V_T = \Phi(S_T).$$

$V_t = h_t^B B_t + h_t^S S_t \Rightarrow$

$u_t^B = \frac{h_t^B B_t}{V_t}$

Heuristics:
Let us **assume** that X is replicated by $h = (u^B, u^S)$ with value process V.

Ansatz: (reasonable, based on $X = \Phi(S_T)$ and Markov)

$$V_t = F(t, S_t)$$

Ito gives us

$$dV = \left\{ F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss} \right\} dt + \sigma S F_s dW,$$

Write this as

$$dV = V \left\{ \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{S F_s}{V} \sigma dW.$$

Compare with

$$dV = V \left\{ u^B r + u^S \alpha \right\} dt + V u^S \sigma dW$$

Define u^S by

$$u_t^S = \frac{S_t F_s(t, S_t)}{F(t, S_t)},$$

This gives us the eqn

$$dV = V \left\{ \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^S \alpha \right\} dt + V u^S \sigma dW.$$

Again

Compare with

$$dV = V \left\{ u^B r + u^S \alpha \right\} dt + V u^S \sigma dW$$

Natural choice for u^B is given by

$$u^B = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF},$$

The relation $u^B + u^S = 1$ gives us the Black-Scholes PDE

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0.$$

The condition

$$V_T = \Phi(S_T)$$

gives us the boundary condition

$$F(T, s) = \Phi(s)$$

Moral: The model is complete and we have explicit formulas for the replicating portfolio.

u^B and u^S

Main Result

Theorem: Define F as the solution to the boundary value problem

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

Then X can be replicated by the relative portfolio

$$u_t^B = \frac{F(t,S_t) - S_t F_s(t,S_t)}{F(t,S_t)},$$

$$u_t^S = \frac{S_t F_s(t,S_t)}{F(t,S_t)}.$$

The corresponding absolute portfolio is given by

$h_t^B = u_t^B \frac{V_t}{B_t}$, $V_t = F(t,S_t)$

$$h_t^B = \frac{F(t,S_t) - S_t F_s(t,S_t)}{B_t},$$

$$h_t^S = F_s(t,S_t),$$

and the value process V^h is given by

$$V_t^h = F(t,S_t).$$

(see also book Lemma 8.4).

Notes

- Completeness explains unique price - the claim is superfluous! nothing "new" compared to S_t and B_t in the market

- Replicating the claim $P - a.s. \Longleftrightarrow$ Replicating the claim $Q - a.s.$ for any $Q \sim P$. Thus the price only depends on the support of P.

determined by P

- Thus (Girsanov) it will not depend on the drift α of the state equation.

- The completeness theorem is a nice theoretical result, but the replicating portfolio is **continuously rebalanced**. Thus we are facing very high transaction costs.

- Proof only given for claims of the type $\Phi(S_T)$ and under the Ansatz $V_t = F(t, S_t)$

end of lecture 2a.

Completeness vs No Arbitrage

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$
\begin{aligned}
R &= \text{number of random sources} \\
N &= \text{number of risky assets}
\end{aligned}
$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

for instance to annihilate (the) random sources

Meta-Theorem

(Compare to solve $Ax=b$, with $A \in \mathbb{R}^{m\times n}$ etc. when (unique) solution? $m \le n$ (if you ignore "rank" conditions)

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \le R$$

- The market is complete if and only if

$$N \ge R$$

Example:
The Black-Scholes model. R=N=1. Arbitrage free and complete.

Parity Relations

Let Φ and Ψ be contract functions for the T-claims $Z = \Phi(S_T)$ and $Y = \Psi(S_T)$. Then for any real numbers α and β we have the following price relation.

$$\Pi_t[\alpha\Phi + \beta\Psi] = \alpha\Pi_t[\Phi] + \beta\Pi_t[\Psi].$$

see Feynman-Kac, p. 76

Proof. Linearity of mathematical $\boxed{\text{expectation}}$.

Consider the following "basic" contract functions.

$$\begin{aligned} \Phi_S(x) &= x, \\ \Phi_B(x) &\equiv 1, \\ \Phi_{C,K}(x) &= \max[x - K, 0]. \end{aligned}$$

Prices:

$$\begin{aligned} \Pi_t[\Phi_S] &= S_t, \\ \Pi_t[\Phi_B] &= e^{-r(T-t)}, \\ \Pi_t[\Phi_{C,K}] &= c(t, S_t; K, T). \end{aligned}$$

think a bit!

If we have for more options, with strike K_i

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i},$$

then

$$\Pi_t[\Phi] = \alpha\Pi_t[\Phi_S] + \beta\Pi_t[\Phi_B] + \sum_{i=1}^{n} \gamma_i \Pi_t[\Phi_{C,K_i}]$$

We may replicate the claim Φ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a **buy-and hold** portfolio:

the α, β, γ_i

- α shares of the underlying stock,
- β zero coupon T-bonds with face value $\$1$,:
- γ_i European call options with strike price K_i, all maturing at T.

pay off at time T is $\$1$

value at time $t<T$: $e^{-r(T-t)}$ (in $)

Put-Call Parity

Consider a European put contract

$$\Phi_{P,K}(s) = \max\left[K - s, 0\right]$$

It is easy to see (draw a figure) that
(or simple algebra)

$$\begin{aligned}\Phi_{P,K}(x) &= \Phi_{C,K}(x) - s + K \\ &= \Phi_{\underset{C}{\cancel{P}},K}(x) - \Phi_S(x) + \Phi_B(x) \quad K\end{aligned}$$

We immediately get

Put-call parity:

$$p(t, s; K) = c(t, s; K) - s + K\underline{e^{-r(T-t)}}$$

Thus you can construct a synthetic put option, using a buy-and-hold portfolio. (with a call option)
(see Prop. 9.3 in the book).

Delta Hedging

name has to do with the "Greeks", see further down

Consider a fixed claim

$$X = \Phi(S_T)$$

with pricing function

$$F(t, s).$$ (or $F(t, S_t)$ with $S_t = s$)

Setup:
We are at time t, and have a short (interpret!) position in the contract. ("debt" in the contract)

Goal:
Offset the risk in the derivative by buying (or selling) the (highly correlated) underlying.

Definition:
A position in the underlying (S_t) is a **delta hedge** against the derivative if the portfolio (underlying + derivative) is immune against small changes in the underlying price.

calls for differentiation, derivatives in the sense of calculus

Formal Analysis

minus because of the short position

$$
\begin{aligned}
-1 &= \text{number of units of the derivative product} \\
x &= \text{number of units of the underlying} \\
s &= \text{today's stock price} \\
t &= \text{today's date} \qquad , \; S_t = s
\end{aligned}
$$

Value of the portfolio:

$$V = -1 \cdot F(t, s) + x \cdot s$$

A delta hedge is characterized by the property that

$$\frac{\partial V}{\partial s} = 0.$$

(insensitive to changes in s)

We obtain

$$-\frac{\partial F}{\partial s} + x = 0$$

Solve for x!

see also lect s on p. 106

Result:
We should have

$$\hat{x} = \frac{\partial F}{\partial s}$$

shares of the underlying in the delta hedged portfolio.

Definition:
For any contract, its "delta" is defined by

$$\Delta = \frac{\partial F}{\partial s}.$$ (F = pricing function)

Result:
We should have

$$\hat{x} = \Delta$$

shares of the underlying in the delta hedged portfolio.

Warning:
The delta hedge must be rebalanced over time. (why?)

Black Scholes

For a European Call in the Black-Scholes model we have

$$\Delta = N[d_1] = P(N(0,1) \le d_1)$$

NB This is **not** a trivial result! But see p. 71, Black-Scholes case.

From put call parity it follows (how?) that Δ for a European Put is given by

$$\Delta = N[d_1] - 1$$
$$= -P(N(0,1) > d_1)$$

Check signs and interpret!

Rebalanced Delta Hedge

- Sell one call option a time $t = 0$ at the B-S price F.
- Compute Δ and by Δ shares. (Use the income from the sale of the option, and borrow money if necessary.)
- Wait one day (week, minute, second..). The stock price has now changed.
- Compute the new value of Δ, and borrow money in order to adjust your stock holdings.
- Repeat this procedure until $t = T$. Then the value of your portfolio (B+S) will match the value of the option almost exactly.

better if you rebalance more frequently

- Lack of perfection comes from discrete, instead of continuous, trading.

- You have created a "synthetic" option. (Replicating portfolio).

Formal result:
The relative weights in the replicating portfolio are

$$u_S = \frac{S \cdot \Delta}{F},$$

$$u_B = \frac{F - S \cdot \Delta}{F}$$

(See p. 106, with $\Delta = F_s(t, S_t)$)

end of lecture 2b

Portfolio Delta

Assume that you have a portfolio consisting of derivatives

$$\Phi_i(S_{T_i}), \quad i = 1, \cdots, n$$

all **written on the same underlying** stock S.

$$\begin{aligned} F_i(t,s) &= \text{pricing function for i:th derivative } (S_t = s) \\ \Delta_i &= \frac{\partial F_i}{\partial s} \\ h_i &= \text{units of i:th derivative} \end{aligned}$$

Portfolio value:

$$\Pi = \sum_{i=1}^{n} h_i F_i$$

Portfolio delta:

$$\Delta_\Pi = \sum_{i=1}^{n} h_i \Delta_i$$

Gamma

A problem with discrete delta-hedging is.

- As time goes by, S will change.
- This will cause $\Delta = \frac{\partial F}{\partial s}$ to change.
- Thus you are sitting with the "wrong" value of delta. (at a later time instant)

Moral:

- If delta is sensitive to changes in S, then you have to rebalance often.
- If delta is insensitive to changes in S you do not need to rebalance so often.

Definition:
Let Π be the value of a derivative (or portfolio). **Gamma** (Γ) is defined as

$$\Gamma = \frac{\partial \Delta}{\partial s}$$

i.e.

$$\Gamma = \frac{\partial^2 \Pi}{\partial s^2}$$

Gamma is a measure of the sensitivity of Δ to changes in S.

Result: For a European Call in a Black-Scholes model, Γ can be calculated as

$$\Gamma = \frac{N'[d_1]}{S\sigma\sqrt{T-t}}$$ (Exercise!)

Important fact:
For a position in the underlying stock itself we have

$$\Gamma = 0$$ (trivial!)

Gamma Neutrality

A portfolio Π is said to be **gamma neutral** if its gamma equals zero, i.e.

$$\Gamma_\Pi = 0$$

- Since $\Gamma = 0$ for a stock you can not gamma-hedge using only stocks. Typically you use some derivative to obtain gamma neutrality.

General procedure

Given a portfolio Π with underlying S. Consider two derivatives with pricing functions F and G.

$$\begin{aligned} x_F &= \text{number of units of } F \\ x_G &= \text{number of units of } G \end{aligned}$$

Problem:
Choose x_F and x_G such that the entire portfolio is delta- and gamma-neutral.

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

repeat:

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

We get the equations

$$\frac{\partial V}{\partial s} = 0, \quad \text{(delta neutral)}$$

$$\frac{\partial^2 V}{\partial s^2} = 0. \quad \text{(gamma neutral)}$$

i.e.

$$\Delta_\Pi + x_F \Delta_F + x_G \Delta_G = 0,$$

$$\Gamma_\Pi + x_F \Gamma_F + x_G \Gamma_G = 0$$

Solve for x_F and x_G! (linear system, has a unique solution?)

in general yes, if G is sufficiently different from F

Particular Case

- In many practical cases the original portfolio Π is already delta neutral.
- Then it is natural to use a derivative to obtain gamma-neutrality.
- This will destroy the delta-neutrality. for the new portfolio
- Therefore we use the underlying stock (with zero gamma!) to delta hedge in the end: next page

additional derivative

Formally:

$$V = \Pi + x_F \cdot F + x_S \cdot S$$

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S \Delta_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F + x_S \Gamma_S &= 0 \end{aligned}$$

We have

$$\begin{aligned} \Delta_\Pi &= 0, \\ \Delta_S &= 1 \\ \Gamma_S &= 0. \end{aligned}$$

(only if π is Δ neutral)

i.e.

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F &= 0 \end{aligned}$$

$$\begin{aligned} x_F &= -\frac{\Gamma_\Pi}{\Gamma_F} \\ x_S &= \frac{\Delta_F \Gamma_\Pi}{\Gamma_F} - \Delta_\Pi \end{aligned}$$

Further Greeks

$$\Theta = \frac{\partial \Pi}{\partial t},$$

$$V = \frac{\partial \Pi}{\partial \sigma},$$

$$\rho = \frac{\partial \Pi}{\partial r}$$

V is pronounced "Vega".

NB!

- A delta hedge is a hedge against the movements in the underlying stock, given a **fixed model**.
- A Vega-hedge is not a hedge against movements of the underlying asset. It is a hedge against a **change of the model itself**: σ is a model parameter.

end of lecture 2c.